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Some Affine Properties of the k-Symplectic Manifolds

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Abstract. We study some affine properties of the k-symplectic manifolds and also the complete locally affine k-symplectic manifolds; a familly of these spaces is given in the compact 3-dimensional case.

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1. Introduction

Mathematical and physical considerations (the local study of Pfaffian systems and Nambu's statistical mechanic) have led us to introduce the notion of k-symplectic structure [4]. By the Heisenberg group of rank k in the sense of Goze-Haraguchi [11] we see that the k-symplectic geometry is related to the k-contact systems in analogy with the well-known relationship between symplectic and contact structures [6].

The Kostant-Souriau's geometric prequantization is obtained in the context of these structures [15].

A k-symplectic structure on an $n(k+1)$-dimensional manifold M is given by an n-codimensional foliation $\mathcal{F}$ and a system of k differential closed 2-forms vanishing on the subbundle of TM defined by $\mathcal{F}$ with transversal characteristic spaces. The fundamental example is given by the natural k-symplectic structure on a fibre product of k Lagrangian fibrations, in particular we have a k-symplectic structure on the Whitney sum $T^*B \oplus \ldots \oplus T^*B$ over an n-dimensional manifold B [5].

In terms of G-structures, a k-symplectic structure on M can be defined by an integrable $Sp(k,n;\mathbb{R})$-structure, where $Sp(k,n;\mathbb{R})$ is the group of all matrices of the type

$$\begin{pmatrix} T & & 0 & S_1 \\ & \ddots & & \vdots \\ & & T & S_k \\ 0 & & & {}^t(T^{-1}) \end{pmatrix}$$

where $T \in Gl(n,\mathbb{R})$, $S_1,\cdots,\ S_k$ are $n{\times}n$ real matrices such that $T\,{}^tS_\alpha = S_\alpha\,{}^tT$.

The particular case $k = 1$ corresponds to a classical symplectic structure endowed with a Lagrangian foliation. This last structure is usually called a real polarization of a symplectic manifold or a quasi-cotangent structure in the sense of Molino, Clark and Goel. The generalization to the k-system is strongly motivated by the geometrical formalization of the statistical mechanic.

This paper is devoted to the study of foliations subordinate to a k-symplectic structure and locally affine manifolds endowed with a k-symplectic structure. We study some affine properties of k-symplectic manifolds; in particular we show that a k-symplectic manifold M is locally affine relatively to the nk-dimensional differential structure on M given by the leaves of the foliation $\mathcal{F}$. This leads us to study some properties of the complete connected locally affine k-symplectic manifolds; these manifolds are quotients $\mathbb{R}^{n(k+1)}/\Gamma$ where Γ are subgroups of the group of all affine transformations $X \longmapsto AX + B$ of $\mathbb{R}^{n(k+1)}$ such that $A \in Sp(k,n;\mathbb{R})$, acting freely and properly discontinuously on $\mathbb{R}^{n(k+1)}$, and also, we give an infinite family in the 3-dimensional compact case.

2. Affine structure of the characteristic foliations

Let M be an $n(k{+}1)$-dimensional smooth manifold. Let E be an integrable n-codimensional subbundle of TM. Let $\theta^1,\ldots,\theta^k$ be closed 2-forms on M. Let TM/E be the quotient bundle. Let ν be the canonical projection $TM \longrightarrow TM/E = \nu E$. Let ν^*E be the dual of νE. We denote by $\mathcal{F}$ the foliation defined by the subbundle E. For each α, $(\alpha = 1,\ldots,k)$ and $x \in M$, we denote by $C_x(\theta^\alpha)$ the characteristic space of θ^α at x.

All geometric objects on M are assumed to be of class C^∞.

We say that $(\theta^1,\ldots,\theta^k;\ E)$ defines a k-symplectic structure on M if for each $x \in M$, the following conditions are satisfied :

1) $C_x(\theta^1)\cap\cdots\cap C_x(\theta^k) = \{0\}$,

2) $\theta^\alpha(X,Y) = 0$ for all $X,Y \in \Gamma(E)$ and α, $(\alpha = 1,\ldots,k)$.

A k-symplectic manifold is a manifold with a k-symplectic structure.

Example 2.1. A k-symplectic structure on the cotangent bundle of k^1- *covelocities.*
Let B be an n-dimensional manifold. We denote by $T^*_{k^1}B$ the cotangent bundle of k^1- *covelocities* of B, that is the manifold of all 1-jets of mapping from B to $\mathbb{R}^k$ with target $0 \in \mathbb{R}^k$.

For each coordinate system $(x^j)_{1\le j\le n}$ on B we associate the local coordinates

$$(x^i,\, x^1_i,\ldots,x^k_i)_{1\le i\le n}$$

on $T^*_{k^1}B$ defined by

$$x^i(J_{x,0^1}f) \;=\; x^i(x)$$
$$x^\alpha_i(J_{x,0^1}f) \;=\; (\frac{\partial f^\alpha}{\partial x^i})(x),$$

where $J_{x,0^1}f$ is the 1-jet at $x \in B$ of the map $f = (f^1, \dots, f^k) : B \longrightarrow \mathbb{R}^k$ such that $f(x) = 0$.

We have $T^*_{k^1}B$ is an $n(k+1)$-dimensional vector bundle with standard fibre type $\mathbb{R}^{nk}$; the canonical projection is the map $\pi : T^*_{k^1}B \longrightarrow B$ defined by $\pi(J_{x,0^1}f) = x$.

For each α, $(\alpha = 1, \dots, k)$ we have a canonical 1-form λ^α defined by

$$\lambda^\alpha(u)(X) \;=\; u^\alpha(x)(\pi_* X)$$

where $u = (u^1, \dots, u^k) \in T^*_{k^1}B$, $X \in T_u(T^*_{k^1}B)$ and $\pi(u) = x$.

Let $\theta^\alpha = d\lambda^\alpha$ where $\alpha = 1, \dots, k$ and let $E = Ker\,\pi_*$. The $(k+1)$-tuple $(\theta^1, \dots, \theta^k; E)$ defines a k-symplectic structure on $T^*_{k^1}B$.

Let $(\theta^1, \dots, \theta^k; E)$ be a k-symplectic structure on M. Recall [4] (see also [5]) that about every point of M, we can find an open neighborhood U and local coordinates $(x^{\alpha i}, x^i)_{1\le\alpha\le k, 1\le i\le n}$ on U, called adapted coordinates, so that

$$\theta^\alpha_{|U} \;=\; \sum_{i=1}^{n} dx^{\alpha i} \wedge dx^i \qquad , \qquad E_{|U} \;=\; Ker\,dx^1 \cap \dots \cap Ker\,dx^n$$

for each α, $(\alpha = 1, \dots, k)$ (Darboux theorem).

Equivalently, there exists an atlas $\mathcal{A}$ of M, called Darboux atlas, such that the changes of coordinates are in the pseudogroup of local diffeomorphisms of $\mathbb{R}^{n(k+1)}$ leaving the canonical k-symplectic structure invariant.

It follows from the previous canonical Darboux expressions that the differential forms θ^α, $(\alpha = 1, \dots, k)$ are of class $2n$; this implies that the distribution $x \longmapsto C_x(\theta^\alpha)$ defines integrable subbundles of TM.

For each α, $(\alpha = 1, \dots, k)$ we take

$$E^\alpha \;=\; \bigcap_{\beta\neq\alpha} C(\theta^\beta).$$

In terms of adapted coordinates $(x^{\alpha i}, x^i)_{1\le\alpha\le k, 1\le i\le n}$, νE is spanned by the derivatives $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$, $\nu * E$ is spanned by the differential forms $dx^1, \dots, dx^n$ and E^α is spanned by the derivatives $\frac{\partial}{\partial x^{\alpha 1}}, \dots, \frac{\partial}{\partial x^{\alpha n}}$.

We have :

i) for each α, $(\alpha = 1, \dots, k)$, the subbundle E^α is integrable,

ii) $E \;=\; E^1 \oplus \dots \oplus E^k$ (direct sum),

iii) for each α, $(\alpha = 1, \dots, k)$ the map $X \longmapsto i(X)\theta^\alpha$ defines an isomorphism i_α of vector bundles over M of E^α onto $\nu^* E$.

Let $\mathcal{F}^\alpha$ be the foliation of M defined by the subbundle E^α, $(\alpha = 1,\ldots,k)$. We call $\mathcal{F}^1,\ldots,\mathcal{F}^k$ the characteristic foliations of the k-symplectic structure.

In terms of adapted coordinates $(x^{\alpha i}, x^i)_{1\leq\alpha\leq k, 1\leq i\leq n}$ the map i_α expresses the duality $\frac{\partial}{\partial x^{\alpha s}} \longmapsto dx^s$ between the geometry along the leaves of $\mathcal{F}^\alpha$ and the transverse geometry.

The Darboux expressions imply the following local transition formulas of the canonical coordinates

$$\bar{x}^{\alpha i} = \sum_{j=1}^{n} \frac{\partial x^j}{\partial \bar{x}^i} x^{\alpha j} + \varphi^{\alpha i}(x^1,\ldots,x^n) \quad , \quad \bar{x}^i = \bar{x}^i(x^1,\ldots,x^n)\,,$$

where $\varphi^{\alpha i}$ are some well defined functions. Indeed, these expressions are affine with respect to $x^{\alpha i}$; this proves that we have

Proposition 2.2. [1] *Each leaf of $\mathcal{F}^\alpha(\alpha = 1,\ldots,k)$ (resp. $\mathcal{F}$) is locally affine.*

Invariantly, the affine structure of the leaves under discussion has as local parallel vector fields the fields which correspond to the E-projectable 1-forms under the isomorphism i_α.

We give now a flat affine connection along the leaves. Let $\overline{\nabla}$ be the basic connection on νE [8]; it is the $\mathbb{R}$-bilinear map $\overline{\nabla} : \Gamma(E)\times\Gamma(\nu E) \longrightarrow \Gamma(\nu E)$ given by $\overline{\nabla}_X Z = \nu[X,\overline{Z}]$, for each $X\in\Gamma(E)$ and $Z\in\Gamma(\nu E)$ where $\overline{Z}\in\chi(M)$ such that $\nu(\overline{Z}) = Z$.

Recall that the curvature $\bar{k}$ of $\overline{\nabla}$ is a correspondence that associates to every pair $(X,Y)\in\Gamma(E)\times\Gamma(E)$ a mapping $\bar{k}(X,Y) : \Gamma(\nu E) \longrightarrow \Gamma(\nu E)$ given by

$$\bar{k}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \quad Z\in\Gamma(\nu E)\,.$$

We have $\bar{k}(X,Y) = 0$ for each X, $Y\in\Gamma(E)$.

Let $\alpha = 1,\ldots,k$ and let $\mathcal{L}$ be a leaf of $\mathcal{F}^\alpha$. For each X, $Y\in\chi(\mathcal{L})$ we take

$$\nabla_X Y = i_\alpha^{-1}(\overline{\nabla}^*_X(i_\alpha(Y)))$$

where $\overline{\nabla}^*$ is the map of $\Gamma(E^\alpha)\times\Gamma(\nu^*E)$ into $\Gamma(\nu^*E)$ defined by

$$(\overline{\nabla}^*_X\sigma)Z = L_X(\sigma(Z)) - \sigma(\overline{\nabla}_X Z)$$

for each $X\in\Gamma(E^\alpha)$, $\sigma\in\Gamma(\nu^*E)$ and $Z\in\Gamma(\nu E)$; L_X denotes the Lie derivative with respect to X. ∇ defines an affine connection along the leaf $\mathcal{L}$.

Let T be the torsion tensor of ∇. Let Z be a local cross section of the bundle $\nu E \longrightarrow M$ and let $\overline{Z}$ be a local cross section of the bundle $TM \longrightarrow M$ such that $\nu(\overline{Z}) = Z$. We have

$$\theta^\alpha(T(X,Y),\overline{Z}) = \theta^\alpha(\nabla_X Y - \nabla_Y X,\overline{Z}) - \theta^\alpha([X,Y],\overline{Z})$$

and

$$\begin{aligned}\theta^\alpha(\nabla_X Y - \nabla_Y X,\overline{Z}) &= \theta^\alpha(i_\alpha^{-1}(\overline{\nabla}^*_X(i_\alpha(Y))\overline{\nabla}^*_Y(i_\alpha(X)))\,,\overline{Z})\\ &= (\overline{\nabla}^*_X(i_\alpha(Y)))(Z) - (\overline{\nabla}^*_Y(i_\alpha(X)))(Z),\end{aligned}$$

[1]The proof of this proposition was indicated to me by the referee

then

$$\begin{aligned}\theta^\alpha(\nabla_X Y - \nabla_Y X, \overline{Z}) &= L_X i_\alpha(Y)(Z) - i_\alpha(Y)\overline{\nabla}_X Z - L_Y i_\alpha(X)(Z) + i_\alpha(X)\overline{\nabla}_Y Z \\ &= L_X(\theta^\alpha(Y, \overline{Z})) - \theta^\alpha(Y, [X, \overline{Z}]) + L_Y(\theta^\alpha(\overline{Z}, X)) + \theta^\alpha(X, [Y, \overline{Z}])\end{aligned}$$

and also

$$\begin{aligned}\theta^\alpha(T(X,Y), \overline{Z}) &= L_X(\theta^\alpha(Y, \overline{Z})) + L_Y(\theta^\alpha(\overline{Z}, X)) \\ &\quad + \theta^\alpha(X, [Y, \overline{Z}]) - \theta^\alpha(Y, [X, \overline{Z}]) - \theta^\alpha([X, Y], \overline{Z}) \\ &= d\theta^\alpha(X, Y, \overline{Z}).\end{aligned}$$

θ^α is closed, Z is arbitrary and i_α is an isomorphism, then $T(X,Y) = 0$.

The curvatures k, $\bar{k}$ and $\bar{k}^*$ of ∇, $\overline{\nabla}$ and $\overline{\nabla}^*$ respectively are related by

$$i_\alpha(k(X_1, X_2)Y) = \bar{k}^*(X_1, X_2)(i_\alpha(Y))$$

where X_1, X_2, $Y \in \chi(\mathcal{L})$.

We have $\bar{k} \equiv 0$, then $\bar{k}^* \equiv 0$ and $k \equiv 0$; that is ∇ is a flat affine connection.

As above, we define the flat connection along the leaves of the foliation $\mathcal{F}$ as the connection which induces the other flat connections along the leaves of the characteristic foliations.

Let M be a k-symplectic manifold. If M is compact then the fundamental group $\pi_1(M)$ of M is infinite, and if $H_1(M, \mathbb{Z}) = 0$ then the foliation $\mathcal{F}$ is defined by a nowhere zero closed n-form on M [7].

quad Recall [5] that an infinitesimal automorphism X of the subbundle E is a Hamiltonian system if the differential forms $i(X)\theta^1, \ldots, i(X)\theta^k$ are closed.

Let X be a Hamiltonian system of a k-symplectic structure $(\theta^1, \ldots, \theta^k;\, E)$. In terms of adapted local coordinates $(x^{\alpha i}, x^i)_{1 \le \alpha \le k, 1 \le i \le n}$, the vector field X takes the form

$$\sum_{i=1}^n \Big(- \sum_{\alpha=1}^k \big(\sum_{j=1}^n \frac{\partial f_j}{\partial x^i}(x^1, \ldots, x^n) x^{\alpha i} + \frac{\partial g^\alpha}{\partial x^i}(x^1, \ldots, x^n) \big) \frac{\partial}{\partial x^{\alpha i}} + f_i(x^1, \ldots, x^n) \frac{\partial}{\partial x^i} \Big)$$

where f_j and g^α are smooth functions. Then the flow generated by X respects the affine structure of the leaves of $\mathcal{F}$, i.e. X is an infinitesimal affine transformation of the space M with its differential structure of dimension nk defined by the leaves of $\mathcal{F}$.

3. Locally affine k-symplectic manifolds

Let M be an $n(k+1)$-dimensional manifold with a k-symplectic structure

$$(\theta^1, \ldots, \theta^k;\, E).$$

M is called locally affine if the Darboux atlas $\mathcal{A}$ confers to M a structure of locally affine manifold.

Let $Gp(k, n; \mathbb{R})$ be the group of all affine transformations of $\mathbb{R}^{n(k+1)}$ leaving the canonical k-symplectic structure of $\mathbb{R}^{n(k+1)}$ invariant. The group $Gp(k, n; \mathbb{R})$ is the set of all affine transformations $X \longmapsto AX + B$ of $\mathbb{R}^{n(k+1)}$ such that $A \in Sp(k, n; \mathbb{R})$.

Let M be a complete connected locally affine k-symplectic manifold of dimension $n(k+1)$. By [2] and [3] (see also [17]), the manifold M is just a quotient $\mathbb{R}^{n(k+1)}/\Gamma$ and with a fundamental group Γ, where Γ is a subgroup of $A(n(k+1))$ acting freely and properly discontinuously on $\mathbb{R}^{n(k+1)}$.

Let $q : \mathbb{R}^{n(k+1)} \longrightarrow M = \mathbb{R}^{n(k+1)}/\Gamma$ be the canonical projection (q is covering map), and let P be the subbundle of $T\mathbb{R}^{n(k+1)}$ defined by $q_* P_x = E_{q(x)}$ for all $x \in \mathbb{R}^{n(k+1)}$. The $(k+1)$-tuple $(q^*\theta^1, \ldots, q^*\theta^1; P)$ defines a Γ-invariant k-symplectic structure on $\mathbb{R}^{n(k+1)}$. The changes of coordinates of M are affine transformations of $\mathbb{R}^{n(k+1)}$ leaving the canonical k-symplectic structure invariant. By the definition of the quotient $M = \mathbb{R}^{n(k+1)}/\Gamma$, we see that Γ is contained in $Gp(k, n; \mathbb{R})$.

Let $Hp(k, n; \mathbb{R})$ be the group of all matrices

$$\begin{pmatrix} I_n & & 0 & S_1 & T_1 \\ & \ddots & & \vdots & \\ & & I_n & S_k & T_k \\ 0 & & & I_n & Q \\ 0 & & & 0 & 1 \end{pmatrix} \tag{3.1}$$

where I_n is the unit matrix of rank n, $S_1, \cdots, S_k$ are $n \times n$ real symmetric matrices and $T_1, \ldots, T_k$, Q are column vectors of rank n.

We denote by (S, Q, T) the matrices of the form (3.1) where $S = (S_1, \ldots, S_k)$, $T = (T_1, \ldots, T_k)$.

The affine group $Hp(k, n; \mathbb{R})$ admits similar properties as the Heisenberg group of rank k in the sense of Goze-Haraguchi; it is a nilpotent, connected and simply connected Lie group such that $[Hp(k, n; \mathbb{R}), Hp(k, n; \mathbb{R})]$ coincides with the center of $Hp(k, n; \mathbb{R})$.

Assume that $n = 1$. An affine transformation $g \in Gp(k, 1; \mathbb{R})$ is without a fixed point if and only if the following conditions are satisfied:

(1) $g = (S, q, T) \in Hp(k, 1; \mathbb{R})$.

(2) $T \notin \mathbb{R}.S$.

In particular we have $\Gamma \subset Hp(k, 1; \mathbb{R})$.

We now assume that $n = 1$ and $k = 2$.

Let Γ_2^{02012} be the subgroup of $Hp(2, 1, \mathbb{R})$ generated by

$$(S^0, 0, T^0) \quad , \quad (mS^0, 0, T^1) \quad , \quad (S^2, q_2, T^2)$$

where S^0, S^2, T^0, T^1, $T^2 \in \mathbb{Z}^2$, $q_2 \in I\!N^*$, $m \in \mathbb{Q}$ with $q_2 \neq 0$, $det(T^0, T^1) \neq 0$ and $\delta = mD_2$; here δ and D_2 denote the determinants $det(T^1, S^0)$ and $det(T^0, S^0)$ respectively.

Proposition 3.1. *We have:*

(i) *The subgroup Γ_2^{02012} acts freely and properly discontinuously on $\mathbb{R}^3$.*

(ii) *The quotient $M_2^{02012} = \mathbb{R}^3/\Gamma_2^{02012}$ is an orientable complete 2-symplectic compact connected locally affine manifold.*

(iii) *The manifold M_2^{02012} is homeomorphic to the torus T^3 if and only if $S^0 = 0$.*

Proof. (i) Every element g of Γ_2^{02012} can be written $g = (S, q, T)$ with

(a) $S = \Big(\sum_{i=1}^n \alpha_i + m\sum_{i=1}^n \beta_i\Big)S^0 + (\sum_{i=1}^n \gamma_i)S^2$

(b) $q = (\sum_{i=1}^n \gamma_i)q_2$

(c) $T = \Big(\sum_{p=1}^n \Big(\sum_{j=1}^p(\alpha_j + m\beta_j)\Big)\gamma_p\Big)q_2S^0 + q_2P_{(\sum_{i=1}^n \gamma_i)}S^2 + (\sum_{i=1}^n \alpha_i)T^0 + (\sum_{i=1}^n \beta_i)T^1 + (\sum_{i=1}^n \gamma_i)T^2$

where $n \in \mathbb{N}$, α_i, β_i, $\gamma_i \in \mathbb{Z}$ for all $i, j, k = 1, \ldots, n$ and $P_\gamma = \gamma(\gamma-1)/2$.

The subgroup $G = \Gamma_2^{02012}$ acts freely on $\mathbb{R}^3$. We show now that G acts properly discontinuously on $\mathbb{R}^3$. Two cases are possible.

Case 1: $S^0 \neq 0$.
Let $p_0 \in \mathbb{Z}$ and $q_0 \in I\!N^*$ such that $m = \frac{p_0}{q_0}$. Let $Q_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, let

$$\varepsilon = \min\Big(\frac{1}{2q_0}, \frac{|D_2|}{2q_0(|s_1^0| + |s_2^0|)}\Big),$$

where $S^0 = (s_1^0, s_2^0)$, and let U_0 be the open ball $B(Q_0, \varepsilon)$ with center at the point Q_0 and with radius ε (for the norm $||(u, v, w)|| = max(|u|, |v|, |w|)$).

Let $g \in G$ such that $gU \cap U \neq \emptyset$ and let $Q = (x, y, z) \in U$ such that $g(Q) = (x', y', z') \in U$. Then we have $|x - x'| < 2\varepsilon$, $|y - y'| < 2\varepsilon$ and $|z - z'| < 2\varepsilon$.

By (a), (b), (c) we have $|z - z'| = |(\sum_{i=1}^n \gamma_i)q_2| < 2\varepsilon < 1$, then $|\sum_{i=1}^n \gamma_i|$ is an integer less than 1, then $\sum_{i=1}^n \gamma_i = 0$ and $z = z'$. By the inequalities $|x-x'| < 2\varepsilon$ and $|y-y'| < 2\varepsilon$ we deduce that $|s_2^0(x' - x) - s_1^0(y' - y)| \leq 2\varepsilon(|s_1^0| + |s_2^0|)$.
And also by (a), (b), (c) we have $|\Big(\sum_{i=1}^n \alpha_i + m\sum_{i=1}^n \beta_i\Big)q_0| < 1$, then $\sum_{i=1}^n \alpha_i + m\sum_{i=1}^n \beta_i = 0$.
It follows that $q_0|x - x'|$ and $q_0|y - y'|$ are two integers less than 1, then $x = x'$, $y = y'$ and also $g = (0, 0, 0)$; in other words, we have

$$\{g \in G \,/\, gU \cap U \neq \emptyset\} = \{id_{\mathbb{R}^3}\}.$$

Consider now two points Q_1 and Q_2 of $\mathbb{R}^3$ not congruent modulo G, let K be a compact set contained in $\mathbb{R}^3$ and let $r > 0$ such that $K \subset B(Q_1, r)$, where $B(Q_1, r)$ is the open ball with center at the point Q_1 and with radius r.

The following sets

$$E_1 = \{\gamma \in \mathbb{Z} \,/\, |\gamma q_2| < r\}$$

$$E_2 = \{(\alpha + m\beta)q_0 \,/\, |\alpha + m\beta| < 2r(|s_1^0| + |s_2^0|)\ ,\ \alpha, \beta \in \mathbb{Z}\}$$

and

$$E_{(j,3)} = \{(\alpha + m\beta)z_1s_j^0 + \gamma t_j^2 + u \,/\, |(\alpha + m\beta)z_1s_j^0 + \gamma t_j^2 + u| < r, \ (\alpha + m\beta)q_0 \in E_2, \gamma \in E_1\ ,\ u \in \mathbb{Z}\}\ (j = 1, 2)$$

where $T^2 = (t_1^2, t_2^2)$, are finite. Then $G.Q_1 \cap K$ is finite; this implies that

$$\inf\{d(Q, Q_1) = ||\overrightarrow{QQ_1}|| \,/\, Q \in G.Q_1\} = \alpha > 0.$$

Let

$$\varepsilon' = \min\left(\frac{1}{q_0}, \frac{|D_2|}{4q_0(|s_1^0| + |s_2^0|)}, \frac{\alpha}{2}\right),$$

and let U_1 (resp. U_2) be the open ball with center Q_1 (resp. Q_2) and with radius ε'. We have

$$G.U_1 \cap U_2 = \emptyset.$$

Case 2: $S^0 = 0$.
We may prove the assertion for this case exactly as for the first case.
Observe that, in this case, the group Γ_2^{02012} is the free abelian group on three generators.

(ii) The fact that the vectors $(T^0, 0)$, $(T^1, 0)$ and (T^2, q_2) are linearly independent permits us to see that the quotient M_2^{02012} is a compact manifold.

The orientability of this manifold follows from the fact that each element of the group $Hp(2, 1, \mathbb{R})$ has positive determinant.

(iii) By [2] and [3], two compact connected complete locally affine manifolds M_1 and M_2 are homeomorphic if and only if the fundamental groups $\pi_1(M_1)$ and $\pi_1(M_2)$ are isomorphic; this implies that M_2^{02012} is homeomorphic to the torus T^3 if and only if the fundamental group $\pi_1(M_2^{02012}) = \Gamma_2^{02012}$ of M is an abelian group if and only if $S^0 = 0$.

References

[1] Auslander, A.: *Examples of locally affine spaces.* Ann. of Math. **64** (1964), 255–259.

[2] Auslander, A.: *The structure of complete locally affine manifolds.* Topology 3 suppl.1 (1964), 131–139.

[3] Auslander, L.; Markus, M.: *Holonomy of flat affinely connected manifolds.* Ann. of Math. Princeton **62** (1955), 139–151.

[4] Awane, A.: *Sur une généralisation des structures symplectiques.* Thèse Strasbourg (1984).

[5] Awane, A.: *k-symplectic structures.* J. Math. Phys. **33** (1992), 4046–4052.

[6] Awane, A. *G-espaces k-symplectiques homogènes.* J. Geom. Phys. **13** (1994), 139–157.

[7] Blumenthal, R. A.: *Foliated manifolds with flat basic connection.* J. Differential Geometry **16** (1981), 401–406.

[8] Bott,R.: *Lectures on characteristic classes and foliations.* Lecture Notes in Math. **279**, Springer-Verlag, New-York (1972), 1–80.

[9] Dazord, P. *Sur la géométrie des sous-fibrés et des feuilletages lagrangiens.* Ann. Ecole Normale Sup. **14** (1981), Paris, 465–480.

[10] Goze, M.: *Systèmes de Pfaff.* Rend. Sem. Fac. Sc. Univ. Cagliari. **60** (1990), 167–187.

[11] Goze, M.; Haraguchi, Y.: *Sur les r-systèmes de contact.* C. R. Acad. Sci. Paris, **294** (1982), 95–97.

[12] de Leon-Mendez-Salgado, M.: *Regular p-almost cotangent structures.* J. Korean Math. Soc. **25** (1988), 273–287.

[13] Molino, P.: *Géométrie de Polarisation.* Travaux en cours Hermann (1984), 37–53 .

[14] Molino, P.: *Géométrie globale des feuilletages riemanniens.* Proc. Konink. Nederl. Akad. Wetensch. **85** (1982), 45–76.

[15] Puta, M.: *Some Remarks on the k-symplectic manifolds.* Tensors. N.S. **47** (1988), 109–115.

[16] Sari, T.: *Sur les variétés de contact localement affines.* C. R. Acad. Sci. Paris **292** (1981), 809–812.

[17] Wolf, J. A.: *Space of constant curvature.* University of California, 1972.

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