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On some flag-transitive non-classical $c.C_2$-geometries II

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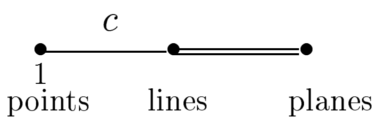
Abstract. The flag-transitive automorphism groups of the Tits generalized quadrangle $T_2^*(O_4)$ are classified up to conjugacy in $Aut(T_2^*(O_4))$. Basing on the classification, it is shown that there is no flag-transitive circular extension of $T_2^*(O_4)$ or of its dual other than those constructed in [6]. This result completes the classification of flag-transitive extensions of $T_2^*(O_4)$ and its dual.

1. Introduction

In this paper, the complete classification is given for the flag-transitive circular extensions of the Tits generalized quadrangle $T_2^*(O_4)$ as well as its dual for a hyperoval O_4 in $PG(2,4)$. See corollary 4.5 for the precise statement. Such geometries are classified in [6] under two assumptions; a restriction on the structure of automorphism groups and the so called (LL) property. The latter turns out to be superfluous. In this paper, we remove the former assumption by giving a complete classification of flag-transitive automorphism groups of $T_2^*(O_4)$. It will be shown that no new example arises.

Given a generalized quadrangle $\mathcal{Q}$, *(circular) extension* of $\mathcal{Q}$ is a residually connected geometry of rank 3 belonging to the following diagram with point-residues isomorphic to $\mathcal{Q}$:

$(c.C_2)$



Extensions of generalized quadrangles are interesting objects, especially when they admit flag-transitive automorphism groups, because some of them arise from sporadic groups. See

[2, 1.1] for the known examples. Note that two infinite families of extensions of $T_2^*(O)$ and its dual for regular hyperovals O in $PG(2, 2^e)$ for any $e \geq 1$ have recently been obtained in [2] and [7]. They are flag-transitive only when $e = 1, 2$.

The structure of a generalized quadrangle $\mathcal{Q}$ is rather restricted, if an extension of $\mathcal{Q}$ exists. For example, if $\mathcal{Q}$ admits a flag-transitive extension, $\mathcal{Q}$ is also flag-transitive. There are only four known examples of such quadrangles besides the classical generalized quadrangles: the Tits quadrangle $T_2^*(O)$ and its dual for certain hyperoval O in $PG(2, q)$ ($q = 4, 16$). For each of these quadrangles, their flag-transitive extensions were investigated: Those with $\mathcal{Q}$ classical are classified ([1],[5]). Those with $\mathcal{Q} = T_2^*(O_4)$ or its dual for a hyperoval O_4 in $PG(2, 4)$ are classified under some assumptions ([6]), which will be removed in this paper. When O is a hyperoval in $PG(2, 16)$, it is easy to see there is no flag-transitive extension of the dual of $T_2^*(O)$, whereas flag-transitive extensions of $T_2^*(O)$ have been studied by Nakagawa. He obtained the possible amalgams of stabilizers [4], though their completions are not yet determined.

It should also be mentioned that under geometric assumptions of (LL) and the triangular properties, Makhnev obtained the distribution diagram of the collinearity graph of an extension of $T_2^*(O_4)$ (but not that of the dual of $T_2^*(O_4)$) [3] without assuming any conditions on automorphisms.

2. The quadrangle $T_2^*(O_4)$ and its automorphisms

In this section, we review some properties of the Tits quadrangle $T_2^*(O_4)$.

For any prime power q and a non-negative integer n, $PG(n, q)$ denotes a desarguesian n-dimensional projective space associated with the row vector space $PG(q)^{n+1}$. Each point of $PG(n, q)$ is a 1-dimensional subspace of $PG(q)^{n+1}$ generated by a vector $(x_0, \dots, x_n)$, which we denote by $[x_0, \dots, x_n]$ throughout the paper.

Recall that a *hyperoval* O in a projective plane $PG(2, q)$ ($q = 2^e$) is a set of $q+2$ projective points such that there is no line through three distinct points of O. For example, the set O_4 consisting of the following six points P_i ($i = 1, \dots, 6$) is a hyperoval in $PG(2, 4)$, where ω is a generator of the multiplicative group of $GF(4)^\#$ and $\bar{\omega} = \omega^2$:

$$P_1 := [1, 0, 0], P_2 := [0, 1, 0], P_3 := [0, 0, 1], P_4 := [1, 1, 1], P_5 := [1, \omega, \bar{\omega}], P_6 := [1, \bar{\omega}, \omega].$$

The set O_4 consists of the zeros $[x_1, x_2, x_3]$ of a quadratic form $X_2^2 + X_1X_3$ together with its nucleus $[1, 0, 0]$, the unique point lying on all its tangents. It is known that any hyperoval O in $PG(2, 4)$ can be transformed to O_4 by an element of $Aut(PG(2, 4))$. Throughout this paper we use the symbol O_4 to denote the above hyperoval in $PG(2, 4)$.

The group $Aut(O_4)$ of automorphisms of $PG(2, 4)$ preserving O_4 is a permutation group on the six points of O_4. We can verify that $Aut(O_4) \cong S_6$, in which the field automorphism corresponds to the transposition (P_5, P_6) and the matrices of $GL_3(4)$ preserving O_4 induces A_6 with the kernel of order 3 (scalar multiplication).

For any hyperoval O in $PG(2, q)$ with q even, a generalized quadrangle $T_2^*(O)$, called the *Tits quadrangle*, can be constructed as follows. Take a hyperplane H of the projective space $PG(3, q)$ given by

$$H := \{[0, x_1, x_2, x_3] | x_i \in GF(q) (i = 1, 2, 3)\}$$

and identify H with $PG(2,q)$ via the map sending $[0,x_1,x_2,x_3]$ to $[x_1,x_2,x_3]$. Define $\mathcal{P}$ to be the set of projective points of $PG(3,q)$ not contained in H, and define $\mathcal{L}$ to be the set of projective lines l of $PG(3,q)$ with the property that l intersects H at a point on the hyperoval O. The geometry $(\mathcal{P},\mathcal{L};*)$ with incidence $*$ given by inclusion turns out to be a generalized quadrangle, which is denoted $T_2^*(O)$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called POINTs and LINEs of $T_2^*(O)$ respectively.

Now we consider $T_2^*(O_4)$ for the above hyperoval O_4 in $PG(2,4)$. It follows from the construction that every automorphism of $PG(3,4)$ stabilizing O_4 (and so H) and every translation by a vector in H (recognized as a subspace of $GF(4)^4$) are automorphisms of $T_2^*(O_4)$. We use the following conventions to denote these automorphisms:

σ : the field automorphism sending $[x_0,x_1,x_2,x_3]$ to $[x_0^2,x_1^2,x_2^2,x_3^2]$.

$L(X)$ for each matrix X of $GL_3(4)$ preserving O_4:
the automorphism sending $[x_0,x_1,x_2,x_3]$ to $[x_0,(x_1,x_2,x_3)X]$.

$t(\mathbf{v})$ for each vector $\mathbf{v}=(0,v_1,v_2,v_3)$ in $H\cong GF(4)^3$:
the translation by $\mathbf{v}$ sending $[x_0,x_1,x_2,x_3]$ to $[x_0,x_1+v_1,x_2+v_2,x_3+v_3]$.

S : the subgroup of $Aut(T_2^*(O_4))$ generated by $L(X)$ for all $X\in GL(3,4)$ preserving O_4 together with the filed automorphism σ.

N : the subgroup of $Aut(T_2^*(O_4))$ generated by $t(\mathbf{v})$ for all $\mathbf{v}\in H$.

By the remark on $Aut(O_4)$ above, the group S induces S_6 on the six LINEs through a POINT $\alpha := [1,0,0,0]$ with kernel $\langle L(\omega I_3)\rangle$ of order 3, because they correspond to the six points P_i $(i=1,\dots,6)$ on O_4 in $H=PG(2,4)$. We set

$$d := L(\omega I_3), \text{ a generator of the kernel of } S \text{ on the points on } O_4.$$

The group $N=\{t(\mathbf{v})|\mathbf{v}\in H\}$ of translations is isomorphic to the additive group of $H\cong GF(4)^3$, and hence N is an elementary abelian group of order 2^6. Clearly the group S acts on N.

The actions of elements of S on N can be easily calculated, since we have $t(\mathbf{v})^{L(X)}=t(\mathbf{v}X)$ and $t(\mathbf{v})^{L(X)\sigma}=t(\overline{\mathbf{v}X})$ for $\mathbf{v}\in H=GF(4)^3$ and $X\in GL(3,4)$, where $\mathbf{v}X$ is the matrix product of a vector $\mathbf{v}$ with X and $\overline{\mathbf{x}}$ means the vector obtained by replacing each entry of $\mathbf{v}$ by its square. Throughout this paper, we write a vector $(0,v_1,v_2,v_3)$ of H by (v_1,v_2,v_3) in short. Moreover in §3, for a subset W of H we slightly abuse the notation by using the same symbol W to denote the subset $\{t(\mathbf{w})|\mathbf{w}\in W\}$ of $N\cong 2^6$ for short. In particular, we write $t(\mathbf{w})$ by $\mathbf{w}$ in short. Moreover, in some proofs of §3 we understand the extended matrix multiplication of S with H (written as $\mathbf{w}L(X)\sigma$) to be the vector of H corresponding to the conjugation $t(\mathbf{w})^{L(X)\sigma}$.

Lemma 2.1. [6, 2.2] *The automorphism group $Aut(T_2^*(O_4))$ is generated by N and S. Thus $Aut(T_2^*(O_4))=N:S\cong 2^6:(3S_6)$. Moreover, $Aut(T_2^*(O_4))$ acts flag-transitively on $T_2^*(O_4)$ with the following stabilizers:*

The stabilizer of a POINT $\alpha=[1,0,0,0]$ is S, which induces S_6 on the residue at α with kernel $\langle d\rangle$;

The stabilizer of a LINE $\beta=\langle[1,0,0,0],[0,1,0,0]\rangle$ is a group isomorphic to $(A_4\times A_5)2$ generated by $t(1,0,0)$, d, $L(X)$ for $X\in GL(3,4)$ fixing O_4 and the point $P_1=[1,0,0]$, *and σ, in which $\langle t(1,0,0), d, \sigma\rangle$ induces S_4 on the four POINTs of β and the kernel on the residue of β is a group isomorphic to A_5 generated by $L(X)$ for $X \in GL(3,4)$ fixing O_4 and $P_1 = [1,0,0]$.*

2.1. Transitive subgroups of S

For later purposes, we need to know all subgroups of S acting transitively on the six LINEs through the POINT $\alpha = [1,0,0,0]$. Note that through the natural projection S to $S/\langle d\rangle \cong S_6$ an element $L(X)$ of S (with $X \in GL(3,4)$ preserving O_4) corresponds to the element $X mod(\omega I)$ of $PGL(3,4)$ preserving O_4. Thus the images of such subgroups of S under the natural projection correspond to the subgroups of $Aut(O_4) \cong S_6$ acting transitively on the 6 points P_i $(i = 1,\ldots,6)$ on the hyperoval O_4. The latter subgroups of S_6 can be easily classified up to conjugacy under S_6. The result is, for example, given in [6, Lemma 2.4, Figure 1].

Remark. Before determining the subgroups of S in question, let me correct several typographic errors found in [6]: First, a line joining F_7 and F_{16} is missing in [6, Figure 1]. The permutation induced by f should be (2536)(14), not (2356)(14) (see the Table in [6, Lemma 2.4] and [6, p.64, line 7]). The entry $\bar{\omega}$ in the matrix q in [6, p.64] should be ω, and the upper $\mathcal{F}(n)$ should be $\mathcal{F}(m)$ in the relations collected in [6, p.64]. A sloppy argument found in [6, 3.6], which does not affect the conclusions of that paragraph, is also mentioned at the end of this paper. Finally, the geometric construction given in [6, 4.4] should be modified, though it is totally independent of the rest of [6] and of the present paper (See [7] for the detail).

To describe the result, we set several explicit automorphisms of $T_2^*(O_4)$: the same notation as in [6, 2.5] will be used. We give both u and d^2u explicitly, and give d^2l, instead of l.

$$u := L\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \bar{\omega} & 0 & 0 \end{pmatrix}, \quad d^2u := L\begin{pmatrix} 0 & \bar{\omega} & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}, \quad p := L\begin{pmatrix} 1 & 0 & 0 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix};$$

$$m := L\begin{pmatrix} \omega & \omega & \omega \\ \omega & \bar{\omega} & 1 \\ \omega & 1 & \bar{\omega} \end{pmatrix} \cdot \sigma, \quad d^2l := L\begin{pmatrix} \omega & \omega & \omega \\ \omega & 1 & \bar{\omega} \\ \omega & \bar{\omega} & 1 \end{pmatrix} \cdot \sigma,$$

From now on throughout this paper, the letters u, p, m, l as well as d and σ denote the above automorphisms, unless otherwise stated. The other automorphims in [6, 2.5] can be obtained as follows: $a := (duu^\sigma)^2$, $b := a^{-1}du$, $n := md^2l$, $q := anb^a$ and $f := d^2m\sigma$. These symbols are used only in Lemma 2.2 and in the proof of Lemma 3.3.

Lemma 2.2. *Assume that a subgroup G of S is transitive on the six LINEs of $T_2^*(O_4)$ through the POINT $\alpha = [1,0,0,0]$. Then G is conjugate under S to one of the following groups under the above notation:*

$i=1$			$H_1 := S \cong 3S_6$
$i=2$			$H_2 := S' \cong 3A_6$
$i=3$	$G_3 := \langle u,p,q,m\rangle$		$H_3 := \langle d,u,p,q,m\rangle \cong 3S_5$
$i=4$	$G_4 := \langle u,p,q\rangle$		$H_4 := \langle d,u,p,q\rangle \cong (3\times A_5)$
$i=5$	$G_5 := \langle d^2u,p,m,d^2l\rangle$		$H_5 := \langle d,u,p,m,l\rangle \cong 3(S_4\times 2)$
$i=6$	$G_6 := \langle d^2u,p,n\rangle$		$H_6 := \langle d,u,p,n\rangle \cong 3S_4$
$i=7$	$G_7 := \langle d^2u,p,d^2l\rangle$		$H_7 := \langle d,u,p,l\rangle \cong 3(A_4\times 2)$
$i=8$	$G_{8-1} := \langle u,p,m\rangle$	$G_{8-2} := \langle d^2u,p,m\rangle$	$H_8 := \langle d,u,p,m\rangle \cong 3S_4$
$i=9$	$G_{9-1} := \langle u,p\rangle$	$G_{9-2} := \langle d^2u,p\rangle$	$H_9 := \langle d,u,p\rangle \cong 3A_4$
$i=10$			$H_{10} := \langle a,b,f,ana^{-1}\sigma\rangle \cong 3_+^{1+2}D_8$
$i=11$			$H_{11} := \langle a,b,f\rangle \cong 3_+^{1+2}4$
$i=12$			$H_{12} := \langle a,b,m,l\rangle \cong 3_+^{1+2}2^2$
$i=13$			$H_{13} := \langle a,b,l\rangle \cong 3_+^{1+2}2$
$i=14$	$G_{14} := \langle d^2u,m,d^2l\rangle$		$H_{14} := \langle d,u,m,l\rangle \cong 3(S_3\times 2)$
$i=15$	$G_{15} := \langle d^2u,m\rangle$		$H_{15} := \langle d,u,m\rangle \cong 3S_3$
$i=16$	$G_{16} := \langle d^2u,d^2l\rangle$		$H_{16} := \langle d,u,l\rangle \cong 3.6$

Proof. As $G\langle d\rangle/\langle d\rangle$ is transitive on the six points on O_4 (via the correspondence at the beginning of 2.1), it follows from [6, Lemma 2.4] that it is conjugate under $S/\langle d\rangle$ to exactly one of the subgroups $F_i = H_i/\langle d\rangle$ of $S/\langle d\rangle \cong S_6$ ($i=1,\ldots,16$). (Note that every H_i above contains the kernel $\langle d\rangle$ on the residue at α.) Thus we may assume that $G\langle d\rangle = H_i$. Then we have either $G = H_i$ or $G\cap\langle d\rangle = 1$. We may assume that the latter case holds. In particular, the extension $H_i/\langle d\rangle$ splits.

If F_i contained a Sylow 3-subgroup of S_6, G with $G\langle d\rangle/\langle d\rangle = F_i$ would contain d, as a Sylow 3-subgroup of S is an extraspecial group 3_+^{1+2}, and hence it does not split over its center. This eliminates the possibilities for $i = 1,2,10,11,12,13$.

If F_i does not contain a Sylow 3-subgroup of S_6, a Sylow 3-subgroup of G is a subgroup of 3^2, and hence the extension $H_i/\langle d\rangle$ in fact splits. Explicit complements are given as G_i (or G_{i-1} and G_{i-2} for $i=8,9$) in the statement of the Lemma. We can verify this fact using the relations in [6, 2.5]. Since G is also a complement for $H_i/\langle d\rangle$, it follows from Sylow's theorem that we may assume G contains a Sylow 2-subgroup of G_i (or G_{i-1} for $i=8,9$).

If $i=3$ or 4, H_i'' is the unique subgroup of H_i isomorphic to A_5, as $H_i/\langle d\rangle$ splits. Since G is generated by H_i'' and a Sylow 2-subgroup of G_i, it coincides with G_i.

For $i=5,7,14,16$, G contains an involution d^2l in its center. Since $\langle d,u\rangle$ is a d^2l-invariant Sylow 3-subgroup of H_i and $C(d^2l)\cap\langle d,u\rangle = \langle d^2u\rangle$, G is generated by a Sylow 2-subgroup of G_i and d^2u, and hence $G = G_i$.

In the remaining cases, G is generated by a Sylow 2-subgroup of G_i (or G_{i-1} for $i=8,9$) and an element of order 3 of shape d^su for some $s=0,1,2$. For $i=15$, G is generated by $\langle m\rangle$ and d^su. Since an element $g := d^{-1}ab^{-1}$ of order 3 satisfies $u^g = du$, $(du)^g = d^2u$ and $[m,g]=1$, G is conjugate to G_{15} under $\langle g\rangle$. For $i=6$, G contains $(d^su)(d^su)^n = d^{2s}uu^n = d^{-s+2}$, and hence $s=2$ and $G=G_6$.

Since $u^{d^2l} = du$, $[p,d^2l] = [m,d^2l] = [n,d^2l] = 1$, we may take $G = G_{i-1}$ or G_{i-2} in each of the remaining cases $i=8,9$. Since the centralizer in $S/\langle d\rangle \cong S_6$ of the image of G_{i-j} ($j=1,2$) is a group of order 2 generated by the image of d^2l, $C_S(G_{i-j}) = 1$ or $\langle d^2l\rangle$. Thus

$C_S(G_{i-1}) = 1$ and $C_S(G_{i-2}) = \langle d^2 l \rangle$ for each $i = 8, 9$. In particular, G_{i-1} is not conjugate to G_{i-2} under S. $\square$

3. Flag-transitive automorphism groups on $T_2^*(O_4)$

In this section, we will classify all flag-transitive subgroups of $Aut(T_2^*(O_4))$ up to conjugacy. We use the notation in §2, and ρ denotes the homomorphism of the semidirect product $Aut(T_2^*(O_4)) = N : S$ onto S sending an element ns ($n \in N$, $s \in S$) to s. Let F be a subgroup of $Aut(T_2^*(O_4)) = N : S$ acting flag-transitively on $T_2^*(O_4)$, and let $\rho(F)$ be the image of F by ρ. In other words,

$$\rho(F) := \{s \in S | ns \in F \text{ for some } n \in N\}.$$

The group $\rho(F)$ is a subgroup of S containing $F \cap S$ and normalizing $F \cap N$, as N is abelian. Moreover, ρ induces an isomorphism of $F/(F \cap N)$ with $\rho(F)$.

Since F is flag-transitive on the generalized quadrangle $T_2^*(O_4)$ of order $(3, 5)$, the stabilizer $F \cap S$ in F of a POINT $\alpha := [1, 0, 0, 0]$ is of index $2^6 = |N|$ in F and it is transitive on the six LINEs through α. Then $[\rho(F) : S \cap F] = |F|/|F \cap N||S \cap F| = |N|/|F \cap N|$.

Summarizing, we observed the following, which is a starting point of the classification.

Lemma 3.1. *Let F be a flag-transitive automorphism group on $T_2^*(O_4)$. Then $\rho(F)$ is a subgroup of S containing $S \cap F$ of index $[N : F \cap N]$ and it normalizes $F \cap N$. Moreover, $S \cap F$ is transitive on the six LINEs through α.*

Note that $\rho(F)\langle d \rangle/\langle d \rangle$ is also a subgroup of $S/\langle d \rangle \cong S_6$ transitive on the six points of O_4 corresponding to the six LINEs through α (see the remark in 2.1). Thus we may assume that under the conjugacy by S the group $\rho(F)$ is one of the groups listed in Lemma 2.2. It has a subgroup $S \cap F$ with index $[N : N \cap F]$, which is also a conjugate of one of the groups given in Lemma 2.2.

If $[N : N \cap F] = 1$, or equivalently F contains N, we have $\rho(F) = S \cap F$, and the transitivity of $S \cap F$ on the six LINEs through α is equivalent for F to be flag-transitive on $T_2^*(O_4)$. Thus, as we saw in [6],

Proposition 3.2. *For F a subgroup of $Aut(T_2^*(O_4))$ containing N, F is flag-transitive group on $T_2^*(O_4)$ if and only if F is conjugate under S to NG_i, NG_{i-j} or NH_i ($i = 1, \ldots, 16$; $j = 1, 2$), the semi-direct products of N with the subgroups of S in Lemma 2.2.*

Hence from now on we may assume that a flag-transitive group F does not contain N. Examining the inclusion relations among subgroups in Lemma 2.2 up to conjugacy (see also [6, Figure 1]), we conclude that the all possible 2-power indices between these groups are 2, 2^2 and 2^3. Moreover, we can list all the pairs of such subgroups with given 2-power index.

We also observe the following facts: The group $\rho(F)$ contains d iff $S \cap F$ contains d, since $[\rho(F) : F \cap S] = [N : N \cap F]$ is a power of 2. If $\rho(F)$ contains d, the dimension of $N \cap F$ over $GF(2)$ is even, since d acts fixed point freely on N. A subgroup of G_5 conjugate to G_{9-j} ($j = 1, 2$) should be conjugate to G_{9-2}, as it is centralized by an involution $d^2 l$. By the same reason, a subgroup of G_{8-i} conjugate to G_{9-j} for some $j = 1, 2$ is conjugate to G_{9-i}.

The above considerations yield the following conclusion:

Lemma 3.3. *If a flag-transitive subgroup F of $Aut(T_2^*(O_4))$ does not contain F, the triple $(F \cap N, \rho(F), S \cap F)$ of subgroups of N and S is one of the following table, up to conjugacy in $Aut(T_2^*(O_4)) = NS$. Here, the column for $S \cap F$ is determined up to conjugacy: that is, if $(\rho(F), S \cap F) = (X_i, X_j^g)$ for some $g \in S$ with $X_j^g \le X_i$, we just put X_i and X_j in the columns for $\rho(F)$ and $S \cap F$ respectively.*

$\|F \cap N\|$	$\rho(F)$		$S \cap F$	
2^5	G_3	$(\cong S_5)$	G_4	$(\cong A_5)$
	G_5	$(\cong S_4 \times 2)$	G_6	$(\cong S_4)$
	G_5		G_7	$(\cong A_4 \times 2)$
	G_5		G_8	$(\cong S_4)$
	G_6	$(\cong S_4)$	G_{9-i}	$i = 1, 2$ $(\cong A_4)$
	G_7	$(\cong A_4 \times 2)$	G_{9-i}	$i = 1, 2$
	G_{8-i}	$i = 1, 2$ $(\cong S_4)$	G_{9-i}	$i = 1, 2$
	G_{14}	$(\cong S_3 \times 2)$	G_{15}	$(\cong S_3)$
	G_{14}		G_{16}	$(\cong Z_6)$
2^4	G_5	$(\cong S_4 \times 2)$	G_{9-2}	$(\cong A_4)$
	H_5	$(\cong 3.(S_4 \times 2))$	H_9	$(\cong 3.A_4)$
	G_7	$(\cong A_4 \times 2)$	G_{16}	$(\cong Z_6)$
	H_7	$(\cong 3(A_4 \times 2))$	H_{16}	$(\cong 3.Z_6)$
	G_{8-i}	$i = 1, 2$ $(\cong S_4)$	G_{15}	$(\cong S_3)$
	H_8	$(\cong 3.S_4)$	H_{15}	$(\cong 3.S_3)$
2^3	G_5	$(\cong S_4 \times 2)$	G_{16}	$(\cong Z_6)$

To each candidate for $\rho(F)$ (that is, G_i (or G_{i-j}) of H_i in the table of Lemma 3.3), we now examine the subspaces of N invariant under $\rho(F)$ in order to determine $N \cap F$. The actions of elements of S on N can be easily computed as we observed in the paragraphs previous to Lemma 2.1. Note also that for a subset W of H we also denote the subset $\{t(\mathbf{w}) | \mathbf{w} \in W\}$ of N by the same symbol W for short. In particular, $t(\mathbf{w})$ is also denoted by $\mathbf{w}$ in short, if there is no risk of confusion. Furthermore, a vector $(0, v_1, v_2, v_3)$ of H is denoted simply by (v_1, v_2, v_3).

We first treat the subgroup $\langle d^2u, p\rangle$ of S, because it is contained in many candidates for $\rho(F)$ in Lemma 3.3. The $\langle d^2u, p\rangle$-orbits on $N^\#$ can be easily determined as follows. (Note that the commutativity of $\langle d^2u, p\rangle$ with d implies that the multiples by ω and $\bar{\omega}$ of each $\langle d^2u, p\rangle$-orbit are also such orbits.)

Lemma 3.4. *The group $\langle d^2u, p\rangle \cong A_4$ has the following* 11 *orbits $\mathcal{O}_i$ $(i = 1, \dots, 11)$ on $N^\#$ of the indicated lengths: In particular, there are exactly* 3 (*resp.* 6 *and* 2) *orbits of length* 1 (*resp.* 6 *and* 12).

$\mathcal{O}_1$	1	$\{(1, \bar{\omega}, \bar{\omega})\}$
$\mathcal{O}_2 = \omega\mathcal{O}_1$	1	$\{(\omega, 1, 1)\}$
$\mathcal{O}_3 = \bar{\omega}\mathcal{O}_1$	1	$\{(\bar{\omega}, \omega, \omega)\}$
$\mathcal{O}_4$	6	$\{(1, 1, \bar{\omega}), (1, \bar{\omega}, 1), (\omega, \bar{\omega}, \bar{\omega}), (0, \omega, \omega), (\bar{\omega}, 0, \omega), (\bar{\omega}, \omega, 0)\}$
$\mathcal{O}_5 = \omega\mathcal{O}_4$	6	$\{(\omega, \omega, 1), (\omega, 1, \omega), (\bar{\omega}, 1, 1), (0, \bar{\omega}, \bar{\omega}), (1, 0, \bar{\omega}), (1, \bar{\omega}, 0)\}$
$\mathcal{O}_6 = \bar{\omega}\mathcal{O}_4$	6	$\{(\bar{\omega}, \bar{\omega}, \omega), (\bar{\omega}, \omega, \bar{\omega}), (1, \omega, \omega), (0, 1, 1), (\omega, 0, 1), (\omega, 1, 0)\}$

$\mathcal{O}_7$	6	$\{(1,1,1),(\omega,\bar{\omega},1),(\omega,1,\bar{\omega}),(0,0,\omega),(\bar{\omega},0,0),(0,\omega,0)\}$
$\mathcal{O}_8=\omega\mathcal{O}_7$	6	$\{(\omega,\omega,\omega),(\bar{\omega},1,\omega),(\bar{\omega},\omega,1),(0,0,\bar{\omega}),(1,0,0),(0,\bar{\omega},0)\}$
$\mathcal{O}_9=\bar{\omega}\mathcal{O}_7$	6	$\{(\bar{\omega},\bar{\omega},\bar{\omega}),(1,\omega,\bar{\omega}),(1,\bar{\omega},\omega),(0,0,1),(\omega,0,0),(0,1,0)\}$
$\mathcal{O}_{10}$	12	$\{(1,\omega,1),(\omega,\bar{\omega},\omega),(\bar{\omega},1,\bar{\omega}),(0,1,\omega),(\bar{\omega},0,1),(\omega,\omega,0),$ $(0,\omega,\bar{\omega}),(1,0,\omega),(\bar{\omega},\bar{\omega},0),(0,\bar{\omega},1),(\omega,0,\bar{\omega}),(1,1,0)\}$
$\mathcal{O}_{11}$	12	$\{(1,1,\omega),(\omega,\omega,\bar{\omega}),(\bar{\omega},\bar{\omega},1),(0,1,\bar{\omega}),(1,0,1),(\omega,\bar{\omega},0),$ $(0,\omega,1),(\omega,0,\omega),(\bar{\omega},1,0),(0,\bar{\omega},\omega),(\bar{\omega},0,\bar{\omega}),(1,\omega,0)\}$

The above lemma eliminates many candidates for $\rho(F)$ when $|N\cap F|=2^5$.

Lemma 3.5. *There is no subgroup of N of order 2^5 invariant under $\langle d^2u,p\rangle$. In particular, if $N\cap F\cong 2^5$, then $\rho(F)$ is conjugate under S to G_{8-1} or G_{14}.*

Proof. Let M be a subgroup of N of order 2^5 invariant under $\langle d^2u,p\rangle$. Since $M^\#$ is a disjoint union of some $\langle d^2u,p\rangle$-orbits $\mathcal{O}_i$ $(i=1,\dots,11)$ in Lemma 3.4, there are integers a,b,c satisfying

$$a+6b+12c=31,\quad 0\le a\le 3,\ 0\le b\le 6,\ 0\le c\le 2.$$

In particular, a is an odd integer prime to 3. Thus $a=1$ and $5=b+2c$.

We can easily verify that $\mathcal{O}_1\cup\mathcal{O}_2\cup\mathcal{O}_3\subseteq\langle\mathcal{O}_j\rangle$ for $j=7,10,11$ (see also Lemma 3.6(1)). This implies that $M\cap\mathcal{O}_j\neq\emptyset$ for every $j=7,\dots,11$, for otherwise we would have $a=3$. Thus $c=0$ and $b\le 3$, but this contradicts the equation $5=b+2c$ above. Hence there is no such subgroup M of N.

The candidates $\rho(F)$ with $|N\cap F|=2^5$ in the table in Lemma 3.3 contain $\langle d^2u,p\rangle$ up to conjugacy in S, except G_{8-1} and G_{14}. (Note that $G_3\cong S_5$ contains a subgroup conjugate to $G_6\cong S_4$.) Hence the latter claim follows from the non-existence of a $\langle d^2u,p\rangle$-invariant subgroup of order 2^5 of N. □

By the lemma above and Lemma 3.3, $\rho(F)$ is conjugate to one of the following subgroups of S: G_{14}, G_{8-i} $(i=1,2)$, G_5, G_7, H_5, H_7 and H_8. In order to examine the subgroups of N invariant under one of these subgroups of S, we need the lemmas 3.6, 3.7, 3.8 below. Using the remarks on the actions of elements of S on N in the paragraphs previous to Lemma 2.1, these results can be verified by taking multiplications of matrices and vectors as well as applying the field automorphism.

Lemma 3.6. (1) *Under the notation in Lemma 3.4, the group $G_{8-2}=\langle d^2u,p,m\rangle$ $(\cong S_4)$ has the following 8 orbits on $N^\#$:*

one orbit $\mathcal{O}_2$ of length 1,
one orbit $\mathcal{O}_1\cup\mathcal{O}_3$ of length 2,
two orbits $\mathcal{O}_4$ and $\mathcal{O}_7$ of length 6;
four orbits $\mathcal{O}_{10}$, $\mathcal{O}_{11}$, $\mathcal{O}_5\cup\mathcal{O}_6$, and $\mathcal{O}_8\cup\mathcal{O}_9$ of length 12.

Moreover they generate the following subspaces of N:

$\langle \mathcal{O}_2 \rangle \cong 2,\quad \langle \mathcal{O}_1 \cup \mathcal{O}_3 \rangle = \{\mathbf{0}\} \cup \mathcal{O}_2 \cup (\mathcal{O}_1 \cup \mathcal{O}_3) \cong 2^2;$
$\langle \mathcal{O}_4 \rangle = \{\mathbf{0}\} \cup \mathcal{O}_2 \cup \mathcal{O}_4 \cong 2^3;$
$\langle \mathcal{O}_7 \rangle = \{\mathbf{0}\} \cup \mathcal{O}_2 \cup (\mathcal{O}_1 \cup \mathcal{O}_3) \cup \mathcal{O}_4 \cup \mathcal{O}_7 \cong 2^4;$
$\langle \mathcal{O}_{10} \rangle = \{\mathbf{0}\} \cup \mathcal{O}_2 \cup (\mathcal{O}_1 \cup \mathcal{O}_3) \cup \mathcal{O}_{10} \cong 2^4,$
$\langle \mathcal{O}_{11} \rangle = \{\mathbf{0}\} \cup \mathcal{O}_2 \cup (\mathcal{O}_1 \cup \mathcal{O}_3) \cup \mathcal{O}_{11} \cong 2^4;$
$\langle \mathcal{O}_5 \cup \mathcal{O}_6 \rangle = \langle \mathcal{O}_8 \cup \mathcal{O}_9 \rangle = N.$

(2) *The involution d^2l preserves each of the above G_{8-2}-orbits, except $\mathcal{O}_{10}$ and $\mathcal{O}_{11}$ which are fused by d^2l. Then the following 7 subsets are orbits on $N^\#$ under $G_7 = \langle d^2u, p, d^2l \rangle$ $(\cong A_4 \times 2)$ as well as $G_5 = \langle G_7, m \rangle = \langle G_{8-2}, d^2l \rangle \cong S_4 \times 2$:*

$\mathcal{O}_2,\ \mathcal{O}_1 \cup \mathcal{O}_3,\ \mathcal{O}_4,\ \mathcal{O}_7,\ \mathcal{O}_5 \cup \mathcal{O}_6,\ \mathcal{O}_8 \cup \mathcal{O}_9;$
and $\mathcal{O}_{10} \cup \mathcal{O}_{11}$ *of length* 24.

Lemma 3.7. (1) *The group $G_{14} = \langle d^2u, m, d^2l \rangle \cong S_3 \times 2$ has the following 12 orbits $\mathcal{A}_i$ $(i = 1, \dots, 12)$ on $N^\#$ of the indicated lengths.*

$\mathcal{A}_1$	1	$\{(\omega,1,1)\}$
$\mathcal{A}_2$	2	$\{(1,\bar\omega,\bar\omega),(\bar\omega,\omega,\omega)\}$
$\mathcal{A}_3$	3	$\{(1,1,\bar\omega),(1,\bar\omega,1),(\omega,\bar\omega,\bar\omega)\}$
$\mathcal{A}_4$	3	$\{(0,\omega,\omega),(\bar\omega,0,\omega),(\bar\omega,\omega,0)\}$
$\mathcal{A}_5$	6	$\{(1,\omega,1),(\omega,\bar\omega,\omega),(\bar\omega,1,\bar\omega),(1,1,\omega),(\omega,om,\bar\omega),(\bar\omega,\bar\omega,1)\}$
$\mathcal{A}_6$	6	$\{(\omega,\omega,1),(\omega,1,\omega),(\bar\omega,1,1),(\bar\omega,\bar\omega,\omega),(\bar\omega,\omega,\bar\omega),(1,\omega,\omega)\}$
$\mathcal{A}_7$	6	$\{(1,1,1),(\omega,\bar\omega,1),(\omega,1,\bar\omega),(0,0,\omega),(\bar\omega,0,0),(0,\omega,0)\}$
$\mathcal{A}_8$	6	$\{(\omega,\omega,\omega),(\bar\omega,1,\omega),(\bar\omega,\omega,1),(0,0,1),(\omega,0,0),(0,1,0)\}$
$\mathcal{A}_9$	6	$\{(\bar\omega,\bar\omega,\bar\omega),(1,\omega,\bar\omega),(1,\bar\omega,\omega),(0,0,\bar\omega),(1,0,0),(0,\bar\omega,0)\}$
$\mathcal{A}_{10}$	6	$\{(0,1,1),(\omega,0,1),(\omega,1,0),(0,\bar\omega,\bar\omega),(1,0,\bar\omega),(1,\bar\omega,0)\}$
$\mathcal{A}_{11}$	6	$\{(0,\omega,\bar\omega),(1,0,\omega),(\bar\omega,\bar\omega,0),(0,\bar\omega,\omega),(\bar\omega,0,\bar\omega),(1,\omega,0)\}$
$\mathcal{A}_{12}$	12	$\{(0,1,\omega),(\bar\omega,0,1),(\omega,\omega,0),(0,\bar\omega,1),(\omega,0,\bar\omega),(1,1,0),$ $(0,1,\bar\omega),(1,0,1),(\omega,\bar\omega,0),(0,\omega,1),(\omega,0,\omega),(\bar\omega,1,0)\}$

(2) *The subspaces of N generated by $\mathcal{A}_i$ $(i = 1, \dots, 12)$ are as follows:*

$\langle \mathcal{A}_1 \rangle \cong 2,\quad \langle \mathcal{A}_2 \rangle = \{\mathbf{0}\} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cong 2^2;$
$\langle \mathcal{A}_3 \rangle = \langle \mathcal{A}_4 \rangle = \{\mathbf{0}\} \cup \mathcal{A}_1 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cong 2^3;$
$\langle \mathcal{A}_7 \rangle = \{\mathbf{0}\} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_7 \cong 2^4;$
$\langle \mathcal{A}_i \rangle = \{\mathbf{0}\} \cup \mathcal{A}_1 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5 \cup \mathcal{A}_8 \cup \mathcal{A}_{10} \cup \mathcal{A}_{11} \cong 2^5$ *for* $i = 5, 8, 10, 11;$
$\langle \mathcal{A}_6 \rangle = \langle \mathcal{A}_9 \rangle = \langle \mathcal{A}_{12} \rangle = N.$

Lemma 3.8. (1) *The group $G_{8-1} = \langle u, p, m \rangle \cong S_4$ has the following 8 orbits $\mathcal{B}_i$ $(i = 1, \dots, 8)$ on $N^\#$ of the indicated lengths.*

$\mathcal{B}_1$	3	$\{(\omega,1,1),(1,\bar\omega,\bar\omega),(\bar\omega,\omega,\omega)\}$
$\mathcal{B}_2$	4	$\{(0,1,\bar\omega),(\omega,0,\omega),(1,\omega,0),(\bar\omega,\bar\omega,1)\}$
$\mathcal{B}_3$	6	$\{(0,\omega,\omega),(1,0,\bar\omega),(\omega,1,0),(\omega,\omega,1),(\bar\omega,\omega,\bar\omega),(\omega,\bar\omega,\bar\omega)\}$
$\mathcal{B}_4$	6	$\{(0,0,1),(\bar\omega,0,0),(1,\bar\omega,\omega),(\bar\omega,1,\omega),(0,\bar\omega,0),(1,1,1)\}$
$\mathcal{B}_5$	8	$\{(0,\omega,1),(\bar\omega,0,\bar\omega),(1,0,1),(\omega,\bar\omega,0),(0,\bar\omega,\omega),(\bar\omega,1,0),(1,1,\bar\omega),(\omega,\omega,\bar\omega)\}$
$\mathcal{B}_6$	12	$\{(0,\omega,\bar\omega),(\omega,0,\bar\omega),(1,1,0),(\bar\omega,\bar\omega,0),(\omega,\omega,0),(1,\omega,1),$ $(\bar\omega,0,1),(0,1,\omega),(0,\bar\omega,1),(\bar\omega,1,\bar\omega),(\omega,\bar\omega,\omega),(1,0,\omega)\}$

$\mathcal{B}_7$	12	$\{(0,0,\omega),(1,0,0),(\omega,1,\bar{\omega}),(\omega,\bar{\omega},1),(0,1,0),(\bar{\omega},\bar{\omega},\bar{\omega}),$ $(\omega,\omega,\omega),(0,\omega,0),(\bar{\omega},\omega,1),(1,\omega,\bar{\omega}),(\omega,0,0),(0,0,\bar{\omega})\}$
$\mathcal{B}_8$	12	$\{(0,1,1),(\bar{\omega},0,\omega),(0,\bar{\omega},\bar{\omega}),(1,\bar{\omega},0),(1,1,\bar{\omega}),(\bar{\omega},\omega,0),$ $(\omega,0,1),(\omega,1,\omega),(1,\bar{\omega},1),(\bar{\omega},\bar{\omega},\omega),(1,\omega,\omega),(\bar{\omega},1,1)\}$

(2) *The subspaces of N generated by $\mathcal{B}_i$ $(i=1,\dots,8)$ are as follows:*

$\langle\mathcal{B}_1\rangle \cong \{\mathbf{0}\}\cup\mathcal{B}_1 \cong 2^2$; $\langle\mathcal{B}_2\rangle = \{\mathbf{0}\}\cup\mathcal{B}_1\cup\mathcal{B}_2 \cong 2^3$;
$\langle\mathcal{B}_6\rangle = \{\mathbf{0}\}\cup\mathcal{B}_1\cup\mathcal{B}_6 \cong 2^4$, $\langle\mathcal{B}_5\rangle = \{\mathbf{0}\}\cup\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_5 \cong 2^4$;
$\langle\mathcal{B}_3\rangle = \langle\mathcal{B}_4\rangle = \{\mathbf{0}\}\cup\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{B}_4\cup\mathcal{B}_6 \cong 2^5$; $\langle\mathcal{B}_7\rangle = \langle\mathcal{B}_8\rangle = N$.

Basing on the informations in Lemma 3.4, 3.6, 3.7 and 3.8 above, we can obtain a list of the possible triples $(N\cap F, S\cap F, \rho(F))$.

Proposition 3.9. *Let F be a flag-transitive automorphism group of $T_2^*(O_4)$ not containing $N = O_2(Aut(T_2^*(O_4)))$. Then, up to conjugacy by S, the triple $(F\cap N, F\cap S, \rho(F))$ is one of the following.*

(1) *If $F\cap N \cong 2^3$, $(\langle\mathcal{O}_4\rangle, G_{16}, G_5)$.*

(2) *If $F\cap N \cong 2^4$, $(\langle\mathcal{O}_7\rangle, G_{9-2}, G_5)$, $(\langle\mathcal{O}_7\rangle, G_{16}, G_7)$; $(\langle\mathcal{B}_5\rangle, \langle u,m\rangle, G_{8-1})$, $(\langle\mathcal{B}_6\rangle, \langle u,m\rangle, G_{8-1})$; $(\langle\mathcal{O}_7\rangle, G_{15}, G_{8-2})$, $(\langle\mathcal{O}_{10}\rangle, G_{15}, G_{8-2})$; or $(\langle\mathcal{O}_{10}\rangle, H_{15}, H_8)$.*

(3) *If $F\cap N \cong 2^5$, $(\langle\mathcal{A}_5\rangle, G_{16}, G_{14})$, $(\langle\mathcal{A}_5\rangle, G_{15}, G_{14})$, $(\langle\mathcal{A}_5\rangle, \langle d^2u, md^2l\rangle, G_{14})$ or $(\langle\mathcal{B}_3\rangle, G_{9-1}, G_{8-1})$.*

Proof. By Lemma 3.3, if $N\cap F \cong 2^3$, we may assume that $\rho(F) = G_5$ and $S\cap F$ is a conjugate of G_{16}. As the subgroups of $G_5 \cong S_4\times 2$ isomorphic to $G_{16}\cong Z_6$ are conjugate to each other, we may also take $G_{16} = S\cap F$. By Lemma 3.6(1)(2), the unique G_5-invariant subspace of N of order 2^3 is $\langle\mathcal{O}_4\rangle$. Thus $N\cap F = \langle\mathcal{O}_4\rangle$ by Lemma 3.1, which establishes the claim (1).

If $N\cap F\cong 2^5$, it follows from Lemma 3.5 that $\rho(F) = G_{8-1}$ (or G_{14}) and $S\cap F$ is a conjugate of G_{9-1} (resp. G_{15} or G_{16}). If $\rho(F) = G_{8-1}\cong S_4$, the subgroup $S\cap F\cong A_4$ of $\rho(F)$ is uniquely determined as the derived subgroup of $\rho(F)$, and hence $S\cap F = G_{9-1}$. Moreover, $\langle\mathcal{B}_3\rangle$ is the unique G_{8-1}-invariant subspace of N of order 2^5 by Lemma 3.8, and so $N\cap F = \langle\mathcal{B}_3\rangle$. If $\rho(F) = G_{14} = \langle d^2u, m, d^2l\rangle \cong S_3\times 2$, $\langle\mathcal{A}_5\rangle$ is the unique G_{14}-invariant subspace of order 2^5 by Lemma 3.7. The group G_{16} is the unique subgroup of G_{14} isomorphic to Z_6, and so $S\cap F = G_{16}$ if $S\cap F$ is conjugate to G_{16}. If $S\cap F$ is conjugate to $G_{15}\cong S_3$, we have $S\cap F = \langle d^2u, m\rangle$ or $\langle d^2u, md^2l\rangle$. Since G_{14} is self-normalizing in S and the above two subgroups are not conjugate in $\rho(F) = G_{14}$, there is no element $g\in NS$ such that $F^g = F_1$ satisfies $(N\cap F_1, S\cap F_1, \rho(F_1)) = (\langle\mathcal{A}_5\rangle, \langle d^2u, md^2l\rangle, G_{14})$ for F with $(N\cap F, S\cap F, \rho(F)) = (\langle\mathcal{A}_5\rangle, G_{15}, G_{14})$. The claim (3) is verified.

Now assume that $N\cap F\cong 2^4$. If $\rho(F)$ contains d, $\rho(F) = H_5$, H_7 or H_8. It follows from Lemma 3.6 that there is no subgroup of N of order 2^4 invariant under H_5 or H_7, while $\langle\mathcal{O}_{10}\rangle$ and $\langle\mathcal{O}_{11}\rangle$ are the set of all H_8-invariant subspaces of N of order 2^4. Since d^2l interchanges $\mathcal{O}_{10}$ and $\mathcal{O}_{11}$ and normalizes H_8, we may assume $N\cap F = \langle\mathcal{O}_{10}\rangle$. Moreover, as $S\cap F$ is a subgroup of $H_8\cong 3.S_4$ isomorphic to $H_{15}\cong 3.S_3$, it is a Sylow 2-normalizer, and hence we may take $S\cap F = H_{15}$. By the same reasoning, if $\rho(F) = G_{8-2}$, we may take $S\cap F = G_{15}$. There are three subgroups $\langle\mathcal{O}_7\rangle$, $\langle\mathcal{O}_{10}\rangle$ and $\langle\mathcal{O}_{11}\rangle$ of N of order 2^4 invariant under G_{8-2}, among which the latter two are interchanged by the involution d^2l normalizing G_{8-2}. The claim (2) is verified if $\rho(F) = H_i$ $(i = 5, 7, 8)$ or G_{8-2}.

For $\rho(F) = G_5$ or G_7, $\langle \mathcal{O}_7 \rangle$ is the unique invariant subgroup of N of order 2^4 by Lemma 3.6. If $\rho(F) = G_7 \cong A_4 \times 2$, $S \cap F \cong Z_6$ is a Sylow 3-normalizer, and we may take $S \cap F = G_{16}$. If $\rho(F) \cong G_5 \cong S_4 \times 2$, $S \cap F$ is a conjugate of $G_{9-2} \cong A_4$ contained in $\rho(F)$. If $S \cap F$ contains a product of an involution of G_{9-2} with d^2l, it induces a permutation of type 2^3 on the set of six LINEs through α, while G_{9-2} induces even permutations only. Thus $S \cap F$ coincides with $\langle d^2u, p \rangle = G_{9-2}$. This verifies the claim (2) for $\rho(G) = G_5$ and G_7.

The remaining case $\rho(F) = G_{8-1}$ is easy to treat: we may take $S \cap F = \langle u, m \rangle$ as it is a Sylow 3-normalizer of $\rho(F)$. By Lemma 3.8, $\langle \mathcal{B}_5 \rangle$ and $\langle \mathcal{B}_6 \rangle$ are all G_{8-1}-invariant subspaces of N of order 2^4. □

We will now examine for each possible triple in Proposition 3.9 whether or not there exists a flag-transitive subgroup F of $NS = Aut(T_2^*(O_4))$ having that triple. The outline is as follows: Take a generator of $\rho(F)$, say $s_1, \ldots, s_m$, outside from $S \cap F$. By the definition of $\rho(F)$, there are some elements $n_i \in N$ with $n_i s_i \in F$ $(i = 1, \ldots, m)$. Note that n_i is not contained in $F \cap N$, so that we may take n_i as a representative of $(N/N \cap F)^{\#}$. Since F is generated by $N \cap F$, $S \cap F$ and $n_i s_i$ $(i = 1, \ldots, m)$, we will first examine for each non-trivial representative n_i of $N/(N \cap F)$ whether the intersection of $\langle N \cap F, S \cap F, n_i s_i | i = 1, \ldots, m \rangle$ with S in fact coincides with $S \cap F$. For representatives n_i satisfying this condition, F is an extension of $F \cap N$ by $\rho(F)$. Observing the stabilizer of the flag (α, β) (see Lemma 2.1), we can check the flag-transitivity of F.

Proposition 3.10. *Under the notation in §2, we have the following.*

(1) *There is no flag-transitive subgroup F of $Aut(T_2^*(O_4))$ with triple $(\langle \mathcal{O}_7 \rangle, G_{9-2}, G_5)$, $(\langle \mathcal{A}_5 \rangle, \langle d^2u, md^2l \rangle, G_{14})$, $(\langle \mathcal{B}_6 \rangle, \langle u, m \rangle, G_{8-1})$, $(\langle \mathcal{O}_7 \rangle, G_{15}, G_{8-2})$, $(\langle \mathcal{O}_{10} \rangle, G_{15}, G_{8-2})$, or $(\langle \mathcal{O}_{10} \rangle, H_{15}, H_8)$.*

(2) *The group $F_1 = \langle \langle \mathcal{O}_4 \rangle, G_{16}, t(1, \omega, \omega)p, t(1, \bar{\omega}, \bar{\omega})m \rangle$ $(\cong 2^3(S_4 \times 2))$ is a unique flag-transitive group with triple $(\langle \mathcal{O}_4 \rangle, G_{16}, G_5)$. The stabilizers in F_1 of a POINT α and a LINE β are $G_{16} \cong Z_6$ and $\langle t(0, \omega, \omega) \cdot t(1, \omega, \omega)p, t(\omega, \bar{\omega}, \bar{\omega}) \cdot t(1, \bar{\omega}, \bar{\omega})m \cdot d^2l \rangle \cong 2^2$ respectively.*

(3) *The group $F_2 = \langle \langle \mathcal{B}_5 \rangle, \langle u, m \rangle, t(0, 1, 0)p \rangle$ $(\cong 2^4 S_4)$ is a unique flag-transitive group with triple $(\langle \mathcal{B}_5 \rangle, \langle u, m \rangle, G_{8-1})$. The stabilizers in F_2 of a POINT α and a LINE β are $\langle u, m \rangle \cong S_3$ and $\langle t(\omega, 0, \omega) \cdot (t(0, 1, 0)p)^u \cdot m \rangle \cong Z_4$ respectively.*

(4) *There are exactly two flag-transitive groups with triple $(\langle \mathcal{O}_7 \rangle, G_{16}, G_7)$. They are $F_3 := \langle \langle \mathcal{O}_7 \rangle, G_{16}, t(1, 0, 0)p \rangle$ $(\cong 2^4(A_4 \times 2))$ and $F_4 := \langle \langle \mathcal{O}_7 \rangle, G_{16}, t(0, 1, 0)p \rangle$ $(\cong 2^4(A_4 \times 2))$. The stabilizers in F_3 (resp. F_4) of a POINT α and a LINE β are $G_{16} \cong Z_6$ and $\langle t(\bar{\omega}, 0, 0), t(0, 1, 0)p \rangle \cong 2^2$ (resp. $\langle t(\omega, 1, \bar{\omega}) \cdot (t(0, 1, 0)p)^{d^2u} \cdot d^2l \cdot t(1, 0, 0)p \rangle \cong Z_4$) respectively.*

(5) *The group $F_5 = \langle \langle \mathcal{A}_5 \rangle, G_{16}, t(1, 0, 0)m \rangle$ $(\cong 2^5(S_3 \times 2))$ is a unique flag-transitive group with triple $(\langle \mathcal{A}_5 \rangle, G_{16}, G_{14})$. The stabilizers in F_5 of a POINT α and a LINE β are $G_{16} \cong Z_6$ and $\langle t(\omega, 0, 0), t(1, 0, 0)m \cdot d^2l \rangle \cong 2^2$ respectively.*

(6) *The group $F_6 = \langle \langle \mathcal{A}_5 \rangle, G_{15}, t(1, 0, 0)d^2l \rangle$ $(\cong 2^5(S_3 \times 2))$ is a unique flag-transitive group with triple $(\langle \mathcal{A}_5 \rangle, G_{15}, G_{14})$. The stabilizers in F_6 of a POINT α and a LINE β are $G_{15} \cong S_3$ and $\langle t(\omega, 0, 0), t(\omega, \bar{\omega}, \bar{\omega}) \cdot m \cdot t(1, 0, 0)d^2l \rangle \cong 2^2$ respectively.*

(7) *The group* $F_7 = \langle\langle\mathcal{B}_3\rangle, G_{9-1}, t(1,0,0)m\rangle$ $(\cong 2^5S_4)$ *is a unique flag-transitive group with triple* $(\langle\mathcal{B}_3\rangle, G_{9-1}, G_{8-1})$. *The stabilizers in* F_7 *of a POINT* α *and a LINE* β *are* $G_{9-1} \cong A_4$ *and* $\langle t(\bar\omega,0,0), t(1,0,0)m\cdot p^u\rangle \cong D_8$ *respectively.*

Proof. We first describe the above procedure in detail for the triple $(N\cap F, S\cap F, \rho(F)) = (\langle\mathcal{O}_4\rangle, G_{16}, G_5)$, and establish the claim (2). Note that the set $\langle\mathcal{O}_6\rangle^\# = \mathcal{O}_1\cup\mathcal{O}_6$ consisting of $(\omega,1,1)$, $(\bar\omega,\bar\omega,\omega)$, $(\bar\omega,\omega,\bar\omega)$, $(1,\omega,\omega)$, $(0,1,1)$, $(\omega,0,1)$ and $(\omega,1,0)$ intersects trivially with $\langle\mathcal{O}_4\rangle$ (see Lemma 3.6), and hence $\langle\mathcal{O}_6\rangle \cong 2^3$ is a set of representatives for $N/\langle\mathcal{O}_4\rangle$. As $\rho(F) = G_5 = \langle G_{16}, p, m\rangle$, we may assume $t(x)p$ and $t(y)m$ are elements of F for some $x,y\in\langle\mathcal{O}_6\rangle^\#$ (identified with a subset of $H\cong GF(4)^3$).

Now the condition $F/(F\cap N)\cong\rho(F) = G_5\cong S_4\times Z_2$ is equivalent to saying that the following elements are contained in $F\cap N$: $(t(x)p)^2$, $(t(y)m)^2$, $(t(x)p\cdot d^2u)^3$, $(d^2u\cdot t(y)m)^2$, $(d^2l\cdot t(x)p)^2$, and $(d^2l\cdot t(y)m)^2$. As $p^2 = m^2 = (d^2u)^3 = 1$, $[d^2u, p] = 1$ and $(d^2u)^m = (d^2u)^{-1}$, we have $(t(x)p)^2 = t(x)\cdot t(x)^p$, $(t(y)m)^2 = t(y)\cdot t(y)^m$, $(t(x)p\cdot d^2u)^3 = t(x)\cdot t(x)^{(d^2u)^2}\cdot t(x)^{(d^2u)}$, and $(d^2u\cdot t(y)m)^2 = t(y)^{(d^2u)^2}t(y)^m$.

As we saw in the paragraphs previous to Lemma 2.1, we can easily compute these elements by matrix multiplication and the field automorphism for each $x,y\in\langle\mathcal{O}_6\rangle^\#$. For example, for $y = (\bar\omega,\bar\omega,\omega)$, we have $t(y)\cdot t(y)^m = t(z)$, where under the notation for the extended matrix multiplication in the paragraphs previous to Lemma 2.1

$$z = y + ym = (\bar\omega,\bar\omega,\omega) + (\bar\omega,\bar\omega,\omega)\begin{pmatrix}\omega & \omega & \omega\\ \omega & \bar\omega & 1\\ \omega & 1 & \bar\omega\end{pmatrix}\sigma =$$
$$(\bar\omega,\bar\omega,\bar\omega) + \overline{(\bar\omega,1,\bar\omega)} = (\bar\omega,\bar\omega,\omega) + (\omega,1,\omega) = (1,\omega,0)$$

As $(1,\omega,0)\in\mathcal{O}_{11}\not\subseteq\langle\mathcal{O}_4\rangle$, we conclude that $y\neq(\bar\omega,\bar\omega,\omega)$. Similarly, we can verify that the conditions $x + xp\in\langle\mathcal{O}_4\rangle$ and $y + ym\in\langle\mathcal{O}_4\rangle$ for $x,y\in\langle\mathcal{O}_6\rangle^\#$ imply $x,y\in\{(1,\bar\omega,\bar\omega),(1,\omega,\omega),(0,1,1)\}$. Then the conditions $x + x(d^2u) + x(d^2u)^2\in\langle\mathcal{O}_4\rangle$ (equivalent to $t(x)\cdot t(x)^{(d^2u)^2}\cdot t(x)^{(d^2u)}\in N\cap F$) and $y(d^2u)^2 + ym\in\langle\mathcal{O}_4\rangle$ (equivalent to $t(y)^{(d^2u)^2}\cdot t(y)^m\in N\cap F$) implies $x = (1,\omega,\omega)$ and $y = (1,\bar\omega,\bar\omega)$. These x,y satisfy the remaining conditions for F to be an extension of $N\cap F$ by $\rho(F)$.

Thus F is uniquely determined to be F_1 in this case. In particular, $|F_1| = 2^3|S_4|2 = 2^73$. As the stablizer of a POINT α in F_1 is $F_1\cap S = G_{16}\cong Z_6$, the stabilizer of a flag (α,β) is trivial. (Recall that the description of the stabilizer of β in Lemma 2.1.) Thus F_1 acts flag-transitively on the $2^73 = |F_1|/1$ flags of $T_2^*(O_4)$ with the stabilizer of β of order $2^73/2^53 = 4$. As $t(0,\omega,\omega), t(\omega,\bar\omega,\bar\omega)\in\langle\mathcal{O}_4\rangle$, we can see that $t(0,\omega,\omega)\cdot t(1,\omega,\omega)p$ and $t(\omega,\bar\omega,\bar\omega)\cdot t(1,\bar\omega,\bar\omega)m\cdot d^2l$ are mutually commuting elements of the stabilizer of β in F_1. Then the stabilizer is the four group. The claim (2) is verified.

Since the remaining claims are similarly verified, we just sketch the proofs.

Note that $\{(1,0,0),(0,1,0),(1,1,0)\}$ is a set of representatives for $(N/\langle\mathcal{O}_7\rangle)^\#$. For a triple $(F\cap N, S\cap F, \rho(F)) = (\langle\mathcal{O}_7\rangle, G_{9-2}, G_5)$, there are some $x\in\{(1,0,0),(0,1,0),(1,1,0)\}$ with $t(x)d^2l\in F$. As d^2l centralizes $d^2u\in S\cap F$, we have $[(t(x)d^2l)^{-1},(d^2u)^{-1}] = t(x)\cdot t(x)^{(d^2u)^{-1}}\in F\cap N = \langle\mathcal{O}_7\rangle$. However, none of $(1,0,0)+(1,0,0)(u^2d) = (1,0,\bar\omega)$, $(0,1,0)+(0,1,0)(u^2d) = (\omega,1,0)$ and $(1,1,0)+(1,1,0)(u^2d) = (\bar\omega,1,\bar\omega)$ is contained in $\langle\mathcal{O}_7\rangle$. Similarly, there is no F with triple $(\langle\mathcal{O}_7\rangle, G_{15}, G_{8-2})$, since $(t((1,0,0))p\cdot u)^3 = t((1,0,0)+(1,0,0)pu+(1,0,0)(pu)^2) =$

$t((\bar{\omega},0,\bar{\omega})) \notin \langle \mathcal{O}_7 \rangle$, $(t((0,1,0)p \cdot m)^2 = t((0,1,0) + (0,1,0)pm) = t((0,1,\bar{\omega})) \notin \langle \mathcal{O}_7 \rangle$, and $(t((1,1,0)p \cdot m)^2 = t((1,1,0) + (1,1,0)pm) = t((\omega,\omega,0)) \notin \langle \mathcal{O}_7 \rangle$. This verifies the claim (1) for F with $F \cap N = \langle \mathcal{O}_7 \rangle$.

On the other hand, for a triple $(F \cap N, S \cap F, \rho(F)) = (\langle \mathcal{O}_7 \rangle, G_{16}, G_7)$, we may verify that any $x \in \{(1,0,0),(0,1,0),(1,1,0)\}$ satisfies the condition for $F = \langle \langle \mathcal{O}_7 \rangle, G_{16}, t(x)p \rangle$ to be an extension of $\langle \mathcal{O}_7 \rangle$ by G_7. Since $(t((0,1,0))p)^m = t((\bar{\omega},\bar{\omega},\bar{\omega}))p = t((\omega,\bar{\omega},1)) \cdot t((1,1,0))p$ with $(\omega,\bar{\omega},1) \in \langle \mathcal{O}_7 \rangle$, we have $\langle \langle \mathcal{O}_7 \rangle, G_{16}, t((0,1,0))p \rangle^m = \langle \langle \mathcal{O}_7 \rangle, G_{16}, t((1,1,0))p \rangle$. Thus we have two possible F up to conjugacy in this case. As above, we can verify that both of them are flag-transitive and determine the stabilizes. As the stabilizers of β in them are not isomorphic, they are not conjugate. This proves the claim (4).

Note that $\{(0,\omega,0),(0,0,\omega),(0,\omega,\omega)\}$ is a set of non-trivial representatives of $N/\langle \mathcal{O}_{10} \rangle$. There is no F with triple $(\langle \mathcal{O}_{10} \rangle, G_{15}, G_{8-2})$, since for $x = (0,\omega,0)$, $(0,0,\omega)$ and $(0,\omega,\omega)$, we have $x + xpu + x(pu)^2 = (1,0,1)$, $(\bar{\omega},0,\bar{\omega})$ and $(\omega,0,\omega)$ respectively, none of which lies in $\langle \mathcal{O}_{10} \rangle$, and hence the relation $(t(x)p \cdot u)^3 = 1$ does not hold. This also proves the non-existence of F with triple $(\langle \mathcal{O}_{10} \rangle, H_{15}, H_8)$.

There is no F with triple $(\langle \mathcal{B}_6 \rangle, \langle u, m \rangle, G_{8-2})$, since for representatives $x = (1,0,0)$, $(0,0,1)$ and $(1,0,1)$ of $(N/\langle \mathcal{B}_6 \rangle)^\#$, we have $(1,0,0)p + (1,0,0)m = (\omega,\bar{\omega},\bar{\omega})$, $(0,0,1) + (0,0,1)pu + (0,0,1)(pu)^2 = (\omega,0,\omega)$ and $(1,0,1)p + (1,0,1)m = (0,1,\bar{\omega})$ respectively, none of which lies in $\langle \mathcal{O}_{10} \rangle$, and hence one of the relations $(m \cdot t(x)p)^2 \in N \cap F$ and $(t(x)p \cdot u)^3 \in N \cap F$ does not hold.

It is easy to see that there is no F with triple $(\langle \mathcal{A}_5 \rangle, \langle d^2u, md^2l \rangle, G_{14})$, since md^2l fixes a flag (α,β) and so the group F of order $2^5|G_{14}| = 2^7 3$ is not flag-transitive. Hence the claim (1) is established.

For the case (3), the group $(N/\langle \mathcal{B}_5 \rangle)^\#$ has representatives $x = (1,0,0)$,$(0,1,0)$,$(1,1,0)$, among which $(0,1,0)$ is a unique element satisfying the condition $x^p + x^m \in \langle \mathcal{B}_5 \rangle$. For the remaining possible triples, we have $F \cap N \cong 2^5$, and the non-trivial representative of $N/(N \cap F)$ is uniquely determined ($(1,0,0)$ for the claims (5) and (6); and $(0,1,0)$ for the claim (7)). It is now routine to verify that the unique possible F is flag-transitive with the stabilizers in each claim. □

By Proposition 3.9 and 3.10 we complete the classification of flag-transitive subgroups of $Aut(T_2^*(O_4))$.

Theorem 3.11. *A flag-transitive subgroup F of $Aut(T_2^*(O_4)) = NS$ not containing N is conjugate under S to exactly one of the seven groups F_i $(i = 1,\ldots,7)$ in Proposition 3.10.*

4. Flag-transitive extensions of $T_2^*(O_4)$ and its dual

In this section, we complete the classification of flag-transitive $c.C_2$-geometries with point residues isomorphic to $T_2^*(O_4)$ or its dual. Let $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2; *)$ be such a geometry and G a flag-transitive subgroup of $Aut(\mathcal{G})$. Then the stabilizer G_P of a point $P \in \mathcal{G}_0$ acts on the residue $Res(P)$ at P, which is a generalized quadrangle $T_2^*(O_4)$ or its dual. First note that the kernel K_P, the normal subgroup of G_P acting trivially on $Res(P)$, is trivial by a standard argument given in [6, 3.2]. Thus G_P is isomorphic to a flag-transitive subgroup of

$Aut(T_2^*(O_4)) = NS$, and hence, identifying G_P with a subgroup of NS, either it contains N or it is conjugate to one of the seven groups F_i $(i = 1, \dots, 7)$ in Proposition 3.10.

In the former case, G is determined in [6] under the assumption of (LL) property. However, we did not essentially use this assumption in [6]. This was used to identify a line l with the set of two points, say P, Q on it, so that $G_l = G_{\{P,Q\}}$, the setwise stabilizer of $\{P, Q\}$. However, this information is not needed in [6]: though in [6, 3.6,4.2] the group $G_{P,Q}$ was examined, every argument there goes through if we replace $G_{P,Q}$ by $G_{P,l}$. Hence the classification of [6] was done without using the (LL)-property.

Thus we may assume that G_P is one of the flag-transitive subgroup F_i $(i = 1, \dots, 7)$ in Proposition 3.10. We will treat this case by the same method adopted in [6], namely generators and relations. For that purpose, we first determine presentations of some F_i in terms of their generators of POINT and LINE stabilizers. To describe the result, besides the notation in §3 we fix some more notation.

Notation 4.1. We take the following basis of $F_i \cap N$ for $i = 1, \dots, 5$.

For $F_2 \cap N = \langle \mathcal{B}_5 \rangle \cong 2^4$:
$b_1 := t(\omega, 1, \omega)$, $b_2 := t(1, 0, 1)$, $b_3 := t(0, \omega, 1)$, $b_4 := t(\omega, 1, 1)$;

For $F_3 \cap N = F_4 \cap N = \langle \mathcal{O}_7 \rangle \cong 2^4$:
$v_1 := t(\bar{\omega}, 0, 0)$, $v_2 := t(0, \omega, 0)$, $v_3 := t(0, 0, \omega)$, $v_4 := t(1, 1, 1)$;

For $F_1 \cap N = \langle \mathcal{O}_4 \rangle \cong 2^3$: $c_i := v_i + v_4$ $(i = 1, 2, 3)$;

For $F_5 \cap N = \langle \mathcal{A}_5 \rangle \cong 2^5$:
$w_1 := t(\bar{\omega}, 0, 0)$, $w_2 := t(0, 1, 0)$, $w_3 := t(0, 0, 1)$,
$w_4 := t(\omega, \omega, \omega)$, $w_5 := t(\bar{\omega}, \bar{\omega}, 0)$.

Consider also the following elements of the stabilizer of a LINE β in $Aut(T_2^*(O_4))$:

$$k := t(\omega, 0, \omega) \cdot (t(0, 1, 0)p)^u \cdot m = t(\omega, 0, 0) \cdot \begin{pmatrix} \bar{\omega} & 0 & 0 \\ \omega & \bar{\omega} & 1 \\ 0 & \omega & 0 \end{pmatrix} \cdot \sigma,$$

$$h := t(\omega, 1, \bar{\omega}) \cdot (t(0, 1, 0)p)^{d^2 u} \cdot d^2 l \cdot t(0, 1, 0)p = t(1, 0, 0) \cdot \begin{pmatrix} \bar{\omega} & 0 & 0 \\ 0 & \omega & 0 \\ \omega & \bar{\omega} & 1 \end{pmatrix} \cdot \sigma,$$

$$x := t(0, \omega, \omega) \cdot t(1, \omega, \omega)p = t(1, 0, 0)p = t(1, 0, 0) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix} \cdot \sigma,$$

$$y := t(\omega, \bar{\omega}, \bar{\omega}) \cdot t(1, \bar{\omega}, \bar{\omega}) \cdot (m \cdot d^2 l) = t(\bar{\omega}, 0, 0) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \sigma, \text{ and } z := t(1, 0, 0) \cdot (m \cdot d^2 l).$$

As we stated in Proposition 3.10, $N \cap F_i$, the stabilizers of the POINT $\alpha = [1, 0, 0, 0]$ and the LINE $\beta = \langle [1, 0, 0, 0], [0, 1, 0, 0] \rangle$ in F_i for $i = 1, 2, 3, 4$ and 5 are given as follows:

	$N \cap F_i$	POINT stabilizer	LINE stabilizer
F_1	$\langle \mathcal{O}_4 \rangle = \langle c_1, c_2, c_3 \rangle \cong 2^3$	$G_{16} = \langle d^2u \cdot d^2l \rangle \cong Z_6$	$\langle x, y \rangle \cong 2^2$
F_2	$\langle \mathcal{B}_5 \rangle = \langle b_i \vert i = 1, \ldots, 4 \rangle \cong 2^4$	$\langle u, m \rangle \cong S_3$	$\langle k \rangle \cong Z_4$
F_3	$\langle \mathcal{O}_7 \rangle = \langle v_i \vert i = 1, \ldots, 4 \rangle \cong 2^4$	$G_{16} = \langle d^2u \cdot d^2l \rangle \cong Z_6$	$\langle t(\bar{\omega}, 0, 0), x \rangle \cong 2^2$
F_4	$\langle \mathcal{O}_7 \rangle = \langle v_i \vert i = 1, \ldots, 4 \rangle \cong 2^4$	$G_{16} = \langle d^2u \cdot d^2l \rangle \cong Z_6$	$\langle h \rangle \cong Z_4$
F_5	$\langle \mathcal{A}_5 \rangle = \langle w_i \vert i = 1, \ldots, 5 \rangle \cong 2^5$	$G_{16} = \langle d^2u \cdot d^2l \rangle \cong Z_6$	$\langle t(\omega, 0, 0), z \rangle \cong 2^2$

Since F_i is flag-transitive, each F_i is generated by the stabilizers of α and β. Thus we do not need basis of $N \cap F_i$ as generators for F_i. However, in the above we include them to explicitly exhibit the actions of generators of stabilizers on $N \cap F_i$.

Lemma 4.2. *For each i in $\{1, 2, 3, 4, 5\}$, the above generators of F_i (see 4.1) satisfies the following set $\mathcal{R}(F_i)$ of relations. Furthermore, as an abstract group, F_i can be presented by the set $\mathcal{R}(F_i)$ of relations.*

$\mathcal{R}(F_1)$ *Generators:* c_i $(i = 1, 2, 3)$, d^2u, d^2l, x, y:
Relations: $c_i^2 = [c_i, c_j] = 1$ $(1 \le i, j \le 3)$, $(d^2u)^3 = (d^2l)^2 = [d^2u, d^2l] = x^2 = y^2 = 1$,
$c_1^{d^2u} = c_1$, $c_2^{d^2u} = c_1c_2c_3$, $c_3^{d^2u} = c_2$,
$c_1^{d^2l} = c_1$, $c_2^{d^2l} = c_2$, $c_3^{d^2l} = c_3$,
$c_1^x = c_1$, $c_2^x = c_1c_2$, $c_3^x = c_1c_3$,
$c_1^y = c_1$, $c_2^y = c_3$, $c_3^y = c_2$,
$(x \cdot d^2u)^3 = c_1$, $[x, d^2l] = c_1c_2c_3$, $(y \cdot d^2u)^3 = c_1c_2$, $[y, d^2l] = c_1$, *and* $[x, y] = 1$.

$\mathcal{R}(F_2)$ *Generators:* b_i $(i = 1, \ldots, 4)$, u, m, k:
Relations: $b_i^2 = [b_i, b_j] = 1$ $(1 \le i, j \le 4)$, $u^3 = m^2 = (u \cdot m)^2 = k^4 = 1$,
$b_1^u = b_2b_3$, $b_2^u = b_2b_4$, $b_3^u = b_1b_2$, $b_4^u = b_1b_2b_3$,
$b_1^m = b_1b_4$, $b_2^m = b_3$, $b_3^m = b_2$, $b_4^m = b_4$
$b_1^k = b_2b_3$, $b_2^k = b_3b_4$, $b_3^k = b_1b_2$, $b_4^k = b_4$
$(k\dot{u})^4 = 1$, $(k \cdot m)^2 = b_1b_2b_3b_4$, $(k^2 \cdot u)^3 = b_1$, $(k^2m)^2 = b_4$, $(k^{-1}u)^2 = b_2b_3$,
and $(k \cdot m \cdot u)^3 = b_2b_4$.

$\mathcal{R}(F_3)]$ *Generators:* v_i $(i = 1, \ldots, 4)$, d^2u, d^2l, x:
Relations: $v_i^2 = [v_i, v_j] = 1$ $(1 \le i, j \le 4)$, $d^2u^3 = d^2l^2 = [d^2u, d^2l] = x^2 = 1$,
$v_1^{d^2u} = v_2$, $v_2^{d^2u} = v_3$, $v_3^{d^2u} = v_1$, $v_4^{d^2u} = v_1v_2v_4$,
$v_1^{d^2l} = v_4$, $v_2^{d^2l} = v_1v_2v_4$, $v_3^{d^2l} = v_1v_3v_4$, $v_4^{d^2l} = v_1$,
$v_1^x = v_1$, $v_2^x = v_1v_2v_4$, $v_3^x = v_1v_3v_4$, $v_4^x = v_4$,
$(x \cdot d^2u)^3 = v_1v_4$ *and* $[x, d^2l] = v_1v_2v_3v_4$.

$\mathcal{R}(F_4)$ *Generators:* v_i $(i = 1, \ldots, 4)$, d^2u, d^2l, h:
Relations: $v_i^2 = [v_i, v_j] = 1$ $(1 \le i, j \le 4)$, $d^2u^3 = d^2l^2 = [d^2u, d^2l] = 1$, $h^2 = v_1$,
$v_1^{d^2u} = v_2$, $v_2^{d^2u} = v_3$, $v_3^{d^2u} = v_1$, $v_4^{d^2u} = v_1v_2v_4$,
$v_1^{d^2l} = v_4$, $v_2^{d^2l} = v_1v_2v_4$, $v_3^{d^2l} = v_1v_3v_4$, $v_4^{d^2l} = v_1$,
$v_1^h = v_1$, $v_2^h = v_2$, $v_3^h = v_1v_3v_4$, $v_4^h = v_4$,
$(h \cdot d^2u \cdot d^2l)^3 = v_1v_4$ *and* $[h, d^2l] = v_2v_3$.

$\mathcal{R}(F_5)$ *Generators:* w_i $(i = 1, \ldots, 5)$, d^2u, d^2l, z:
Relations: $w_i^2 = [w_i, w_j] = 1$ $(1 \le i, j \le 4)$, $d^2u^3 = d^2l^2 = [d^2u, d^2l] = 1$, $h^2 = w_1$,
$w_1^{d^2u} = w_2$, $w_2^{d^2u} = w_3$, $w_3^{d^2u} = w_1$, $w_4^{d^2u} = w_1w_4w_5$, $w_5^{d^2u} = w_1w_3w_4$,

$w_1^{d^2l} = w_4$, $w_2^{d^2l} = w_1w_4w_5$, $w_3^{d^2l} = w_2w_3w_5$, $w_4^{d^2l} = w_1$, $w_5^{d^2u} = w_1w_2w_4$,
$w_1^z = w_1$, $w_2^z = w_3$, $w_3^z = w_2$, $w_4^z = w_4$, $w_5^z = w_1w_2w_3w_4w_5$,
$(z \cdot d^2u)^2 = w_1w_5$ *and* $[w, d^2l] = w_2w_3w_4$.

Proof. It is straightforward to verify that the above explicit elements satisfy the relations in $\mathcal{R}(F_i)$. Thus in order to see that $\mathcal{R}(F_i)$ presents F_i, it suffices to verify that the group X_i presented by $\mathcal{R}(F_i)$ is of order at most $|F_i|$ ($= 2^7 3$ for all $i = 1, \dots, 5$). This can be done as follows.

For example, take $i = 1$. Then X_1 normalizes an elemntary abelian group $N_1 = \langle c_1, c_2, c_3 \rangle$. The factor group X_1/N_1 is generated by the image of d^2u, d^2l, x and y, which are d^2u, d^2l, p and $m \cdot (d^2l)$, respectively, modulo N_1. The relations among the former generators, each c_i being reduced to 1, are nothing but the relations among the latter generators, which can be easily observed to present $\langle d^2u, p, m \rangle \times \langle d^2l \rangle \cong S_4 \times 2$. Hence $|X_1| \leq |N_1||X_1/N_1| \leq 2^3|S_4 \times 2| = |F_1|$. By the same argument, one can prove $|X_i| \leq |F_i|$ for the remaining cases. □

Theorem 4.3. *There is no flag-transitive extension of the dual of $T_2^*(O_4)$ admitting a flag-transitive automorphism group G in which the stabilizer G_P of a point P does not contain N.*

Proof. Let (P, l, u) be a maximal flag of $\mathcal{G}$, a flag-transitive extension of $T_2^*(O_4)$, admitting a flag-transitive automorphism group G such that $G_P \not\supseteq N$. Thus $G_P = F_i$ for some $i = 1, \dots, 7$, as we remarked above.

Since the dual of $Res(P)$ is $T_2^*(O_4)$, the stabilizer $G_{P,u}$ is the stabilizer in F_i of a POINT u of $T_2^*(O_4)$, and the six lines incident with (P, u) are identified with six LINEs of $T_2^*(O_4)$ through a POINT u. By Proposition 3.10, $G_{P,u}$ is isomorphic to Z_6, S_3 or A_4. On the other hand, $Res(u)$ is a circle geometry with $5 + 2 = 7$ points, on which the stabilizer G_u acts doubly transitively by the flag-transitivity of G. Thus $(G_u/K_u, G_{P,u}/K_u)$ is isomorphic to (S_7, S_6), (A_7, A_6), $(L_3(2), S_4)$ or (F_7^6, Z_6). Hence we have $G_{P,u} \cong Z_6$, $K_u = 1$ and $G_u \cong F_7^6$. In particular, $G_{P,u} = G_{16} = \langle d^2u, d^2l \rangle$, and G_P is F_1, F_3, F_4 or F_5. Moreover, $G_{P,l,u} = 1$ and $G_{l,u} \cong Z_2$ is generated by an involution v inducing a permutation of type 2^3 on the seven points on u.

We will determine the possible relations satisfied by generators of G. Note first that in Lemma 4.2 we have already determined presentations of G_P in terms of generators of $G_P \cap N$, $G_{P,u}$ and $G_{P,l}$.

We next examine the possible relations betweeen v and generators of $G_{P,u}$. Let $P = P_1$ and $Q = P_2$ be the two points on a line l and P_i ($i = 3, \dots, 7$) be the remaining points on u. Suitably arranging the five points on u not on l, we may assume that a generator $d^2u \cdot d^2l$ of $G_{P,u} = G_{16} \cong Z_6$ induces a 6-cycle $(P_2, P_3, P_4, P_5, P_6, P_7)$. Now v is an involution of type 2^3 on the points on u with cycle (P_1, P_2) such that $G_u = \langle G_{P,u}, G_{l,u} \rangle = \langle d^2u \cdot d^2l, v \rangle \cong F_7^6$. We may verify that this is satisfied exactly when v induces either $(P_1, P_2)(P_3, P_7)(P_5, P_6)$ or $(P_1, P_2)(P_3, P_7)(P_4, P_5)$. In the former (resp. latter) case we can check that v and d^2ud^2l satisfy the following set $\mathcal{R}(4)$ (resp. $\mathcal{R}(6)$) of relations:

Setting $s := d^2u \cdot d^2l$,
$\mathcal{R}(4)$: $v^2 = 1$, $s^6 = 1$, $(vs)^3 = 1$, $[s, v]^7 = 1$, $v^s = s^3$;
$\mathcal{R}(6)$: $v^2 = 1$, $s^6 = 1$, $(vs)^3 = 1$, $[s, v]^7 = 1$, $v^{s^2} = s^3$.

By coset enumerations (available e.g. in GAP), we can verify that both $\mathcal{R}(4)$ and $\mathcal{R}(6)$ define a group of order 42, and hence they present the Frobenius group F_7^6.

Finally we examine the possible relations betweeen v and generators of $G_{P,l}$. Here we use the notation 4.1 for elements of $G_{P,l}$ freely. The involution $v \in G_l$ acts on the subgroup $G_{P,l}$ of G_l of index 2. Note that $G_{P,l}$ is either Z_4 or E_4, as $G_P = F_i$ for $i = 1, 3, 4$, or 5. If $i = 4$, then $G_{P,l} = \langle h \rangle \cong Z_4$, and hence

$\mathcal{R}_4(\varepsilon)$: $h^v = h^\varepsilon$ for $\varepsilon = \pm 1$

In the remaining cases, $G_{P,l}$ is the four group generated by two involutions, say a and b. Under the notation 4.1, they are explicitly given as

$a = x$, $b = y$ if $G_P = F_1$,
$a = t(\bar{\omega}, 0, 0)$, $b = x$ if $G_P = F_3$,
$a = t(\omega, 0, 0)$, $b = z$ if $G_P = F_5$.

Since v induces on $G_{P,l}$ either the identity element or one of the three involutions of $GL(2, 2)$, we have four possible relations:

$\mathcal{R}_{a,b}(A)$: $a^v = a$, $b^v = b$; $\mathcal{R}_{a,b}(B)$: $a^v = b$, $b^v = a$
$\mathcal{R}_{a,b}(C)$: $a^v = a$, $b^v = ab$; $\mathcal{R}_{a,b}(D)$: $a^v = ab$, $b^v = b$

The flag-transitivity implies that G is generated by $G_{P,l}$, $G_{P,u}$ and $G_{l,u}$, and hence $G = \langle G_P, v \rangle$. In the above paragraphs, we have determined possible sets of relations satisfied by some generators of G. Explicitly, G satisfies one of the following sets of relations for some $j = 4$ or 6 and some $X = A, B, C, D$ (or $\varepsilon = \pm 1$), where $\mathcal{R}(F_i)$ means the set of relations for $G_P = F_i$ in Lemma 4.2.

(i) Generators: c_i $(i = 1, \ldots, 3)$, d^2u, d^2l, x, y and v;
Relations: $\mathcal{R}(F_1) \cup \mathcal{R}(j) \cup \mathcal{R}_{x,y}(X)$

(ii) Generators: v_i $(i = 1, \ldots, 4)$, d^2u, d^2l, x and v;
Relations: $\mathcal{R}(F_3) \cup \mathcal{R}(j) \cup \mathcal{R}_{t(\bar{\omega},0,0),x}(X)$

(iii) Generators: v_i $(i = 1, \ldots, 4)$, d^2u, d^2l, h and v;
Relations: $\mathcal{R}(F_3) \cup \mathcal{R}(j) \cup \mathcal{R}_4(\varepsilon)$

(iv) Generators: w_i $(i = 1, \ldots, 5)$, d^2u, d^2l, z and v;
Relations: $\mathcal{R}(F_3) \cup \mathcal{R}(j) \cup \mathcal{R}_{t(\omega,0,0),z}(X)$

Now coset enumeration (performed with the aid of computer) tells us the index of G_P, presented by a subset of those relations, in the group $\tilde{G}$ presented by the whole relations. It is always 1 in each of the possible cases above, which implies that $v \in G_P$, a contradiction. Thus there is no flag-transitive extension of the dual of $T_2^*(O_4)$ with G_P not containing N. □

Theorem 4.4. *If an extension $\mathcal{G}$ of $T_2^*(O_4)$ admits a flag-transitive automorphism group G in which the stabilizer G_P of a point P does not contain N, then G_P is conjugate to the group F_2 in Proposition 3.10 and $\mathcal{G}$ is the simply connected geometry $\mathcal{G}$ with* 160 *points in [6, Theorem 1.1] or its quotient on* 80 *points.*

Proof. The proof is quite similar to that for Theorem 4.3. In this case, the group $G_{P,u}/K_u$ is the stabilizer of a point in a doubly transitive group on $3+2 = 5$ points. As $G_{P,u} \cong E_4$, Z_4 or D_8 by Proposition 3.10, we have $G_u \cong F_5^4$, $G_{P,u} \cong Z_4$, $K_u = 1$ and $G_{l,u} \cong Z_2$. In particular, $(G_P, G_{P,u}) = (F_2, \langle h \rangle)$ or $(F_4, \langle k \rangle)$. We may assume that a generator $s := h$ or k of $G_{P,u}$ induces a 4-cycle (P_2, P_3, P_4, P_5) with $P = P_1$ and $P_2 = Q$ lying on l. Then the involution v

of $G_{l,u} \cong Z_2$ induces either $(P_1, P_2)(P_3, P_4)$ or $(P_1, P_2)(P_4, P_5)$ to generate F_5^4 with s. We can verify that in each of these cases, v and s satisfy the following relations, which present F_5^4.

$\mathcal{R}(5)$: If v induces $(P_1, P_2)(P_3, P_4)$,
$v^2 = s^4 = 1$, $(vs)^4 = 1$, $[s, v]^5 = 1$, $(s^2)^{[s,v]^2} = v$.

$\mathcal{R}(3)$: If v induces $(P_1, P_2)(P_4, P_5)$,
$v^2 = s^4 = 1$, $(vs)^4 = 1$, $[s, v]^5 = 1$, $(s^2)^{[s,v]} = v$.

Since $v \in G_l$ acts on $G_{P,l}$, which is $\langle u, m\rangle \cong S_3$ if F_2 and $\langle d^2u \cdot d^2l\rangle \cong Z_6$ if F_4, we have the following relations.

$\mathcal{R}_2(i, j)$: If $G_P = F_2$, $u^v = u^i$, $m^v = mu^j$ for some $i = 1, 2$ and $j = 0, 1, 2$.

$\mathcal{R}_4(i)$: If $G_P = F_4$, $(d^2u)^v = (d^2u)^i$, $(d^2l)^v = d^2l$ for some $i = 1, 2$.

The relation $v^2 = 1$ restricts (i, j) in $\mathcal{R}_2(i, j)$, because it implies $m = m^{v^2} = mu^{j(1+i)}$, and so $(i, j) = (1, 0)$, $(2, 0)$, $(2, 1)$, $(2, 2)$. Now $G = \langle G_P, v\rangle$ satisfies one of the following sets of relations:

Generators: b_i $(i = 1, 2, 3, 4)$, u, m, k, v:
Relations: $\mathcal{R}(F_2) \cup \mathcal{R}_2(i, j) \cup \mathcal{R}(q)$ for some $(i, j) = (1, 0)$, $(2, 0)$, $(2, 1)$, $(2, 2)$ and $q = 5, 3$.
Generators: v_i $(i = 1, 2, 3, 4)$, d^2u, d^2l, h, v:
Relations: $\mathcal{R}(F_4) \cup \mathcal{R}_4(i) \cup \mathcal{R}(q)$ for some $i = 1, 2$ and $q = 5, 3$.

The coset enumeration shows that the index of G_P in the group $\tilde{G}$ presented by one of these sets of relations is 1 except when $\tilde{G}$ is presented by $\mathcal{R}(F_2) \cup \mathcal{R}(2, 2) \cup \mathcal{R}(5)$. In this case, the index is 160. Since the group F_2 is contained in the group $N\langle d, p, q, m\rangle$ of type F_3 in [6, 3.7], the resulting geometry coincides with that for the group of type F_3 on 160 points, denoted by $\mathcal{G}^{(0)}$ in [6]. □

Remark. Our F_2 is also contained in the group $N\langle d, p, m\rangle$ of type F_{15} in [6, 3.6]. Thus a geometry on 160 points should arise from the consideration of group of type F_{15} in [6, 3.6]. This possibility was overlooked there, because in [6] (Page 68, line 9 from the bottom), I carelessly concluded that $m^v = md^i$ for some i, but in fact $m^v = md^iu^j$ for some i, j, from which the overlooked case arises. Fortunately, the overlooked group is conjugate under $\langle b\rangle$ into the group arising from the flag-transitive group of type F_3, and hence there is no mistake in conclusion.

Together with [6], Theorem 4.4 completes the classification of flag-transitive extensions of $T_2^*(O_4)$ and its dual. For completeness, we summarize the results.

Corollary 4.5.

(1) *A flag-transitive extension of $T_2^*(O_4)$ is isomorphic to one of the following two geometries: (their geometric constructions are given in [2]).*

  (*i*) *the simply connected geometry $\mathcal{G}$ on* 160 *points and* 3072 *planes with $Aut(\mathcal{G}) \cong 2_+^{1+8} \cdot (A_5 \times A_5)2$ found in [6, Theorem 1.1]*

  (*ii*) *the half quotient $\bar{\mathcal{G}}$ of $\mathcal{G}$ on* 80 *points and* 1536 *planes with $Aut(\bar{\mathcal{G}}) \cong 2^8 \cdot (A_5 \times A_5)2$*

(2) *A flag-transitive extension of the dual of* $T_2^*(O_4)$ *is isomorphic to one of the following three geometries:*

(i) *the simply connected geometry* $\mathcal{G}^{(0)}$ *on* 896 *points and* 8192 *planes with* $Aut(\mathcal{G}^{(0)}) \cong 2_+^{1+12} : 3S_7$ *found in [6, Theorem 1.2]*

(ii) *the half quotient* $\bar{\mathcal{G}}^{(0)}$ *of* $\mathcal{G}^{(0)}$ *on* 448 *points and* 4096 *planes with* $Aut(\bar{\mathcal{G}}^{(0)}) \cong 2^{12} : 3S_7$ *(its geometric construction is given in [7])*

(iii) *the simply connected geometry* $\mathcal{G}^{(1)}$ *on* 448 *points and* 4096 *lines with* $Aut(\bar{\mathcal{G}}^{(1)}) \cong 2^{6+6} : L_3(2)$

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