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A Tilde Geometry for $F_4(2)$

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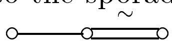
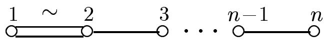
1. Introduction

The aim of this paper is to construct a geometry $\mathcal{G}$ (resp. its flag-transitive automorphism group) with diagram

$(\tilde{F}_4)$ 

where the rank-3-residues of objects of types 1 and 4 are so-called tilde geometries belonging to the group $3^7Sp_6(2)$. The question of the existence (and uniqueness) of such a geometry arises in the classification of semiclassical geometries (resp. semiclassical parabolic systems), i.e. of flag-transitive geometries in which each residue of corank 2 is either a generalized digon, a generalized m-gon, $m \geq 3$, coming from some group of Lie type in char 2 or the 3-fold cover of the generalized $Sp_4(2)$-quadrangle. Since some of the sporadic simple groups act on a semiclassical geometry (see [8]), these geometries might be interesting in the context of trying to find a uniform approach to the classification of the finite simple groups via their action on certain "nice" geometries.

For example, apart from the above mentioned group $3^7Sp_6(2)$, also the sporadic simple groups M_{24} and He act flag-transitively on a geometry with diagram  . But it was shown in [3] that these three are the only possibilities. The group $3^7Sp_6(2)$ belongs to an infinite series of non-split extensions $3^kSp_{2n}(2)$, $k = (2^n-1)(2^{n-1}-1)/3$, acting on a tilde geometry, i.e. a geometry with diagram



The existence and uniqueness of these groups/geometries was proved in [4], [10]. The M_{24}-geometry extends to tilde geometries of rank 4 resp. 5 for the groups Co_1 and the Monster

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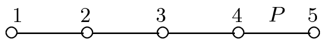
(see [6]), but it is shown in [2] that the geometries for M_{24} and He cannot arise as residues in a geometry with diagram $(\tilde{F}_4)$.

In the case that G is a group acting flag-transitively on a geometry with diagram $(\tilde{F}_4)$ where the rank-3-residues of elements of types 1 and 4 belong to $3^7 Sp_6(2)$, it is shown in [11] that there exists a homomorphism of G onto the finite simple group of Lie type $F_4(2)$. It is also shown that, under the assumption that the kernel of this homomorphism is abelian, G is a non-split extension of $F_4(2)$ by an elementary abelian subgroup of order 3^{833}, which can be viewed as an irreducible module over $GF(3)$. In this paper we prove the existence of such a group $G \cong 3^{833} F_4(2)$ with diagram of type $(\tilde{F}_4)$.

2. A group of type $(\tilde{F}_4)$

It is well-known that $(2^2 \times F_4(2))2$ is the centralizer of an involution of type $2C$ in the Babymonster sporadic simple group (see [1]), and the idea is to look at the corresponding subgroup of the non-split extension of the Babymonster by an elementary abelian group of order 3^{4371} (which exists by [5]). First we describe how one can get the maximal parabolics of $F_4(2)$ from certain subgroups of the Babymonster.

Let's denote the Babymonster by H. The group H acts on a rank-5-geometry $\mathcal{G}(H)$ of Petersen-type (for short P-geometry) with diagram



(see [7]). If $\{a_1, a_2, a_3\}$ is a flag in $\mathcal{G}(H)$ with a_i of type i and $H_i = H_{a_i}$ the stabilizer in H of a_i, then

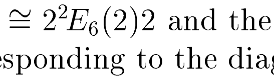
$$\begin{aligned} H_1 &\cong 2^{1+22}.Co_2, \\ H_2 &\cong [2^{32}].(\Sigma_3 \times Aut(M_{22})), \\ H_3 &\cong [2^{35}].(L_3(2) \times \Sigma_5). \end{aligned}$$

Further, the geometry $\mathcal{G}(H)$ contains a family of rank-4-subgeometries (including types 1 to 4 of the above diagram), which are classical symplectic spaces over $GF(2)$. If A denotes the stabilizer in H of such a geometry, then $A \cong 2^{9+16} Sp_8(2)$ and $A/O_2(A)$ induces $Sp_8(2)$ on the subgeometry $\mathcal{G}(A)$. Moreover, we may choose A such that $\mathcal{G}(A)$ contains the flag $\{a_1, a_2, a_3\}$.

Now take an involution $\sigma \in O_2(H_1)$ of type $2A$ and let $E_i = C_{H_i}(\sigma)$, $i = 1, 2, 3$, and $E_4 = C_A(\sigma)$. Then

$$\begin{aligned} &E_1 \cong 2^{2+20}.U_6(2).2, && E_2 \cong [2^{30}].(\Sigma_3 \times L_3(4).2), \\ &E_3 \cong [2^{32}].(L_3(2) \times \Sigma_5), && E_4 \cong [2^{25}].O_8^-(2) \end{aligned}$$

and if $E = \langle E_1, E_2, E_3, E_4 \rangle$, then $E \cong 2^2 E_6(2)2$ and the groups $E_i/\langle \sigma \rangle$ are just the maximal parabolics of $E/\langle \sigma \rangle \cong {}^2E_6(2)2$ corresponding to the diagram



Let τ be the outer automorphism in $2^2 E_6(2)2$, $F_i = C_{E_i}(\tau)$ and $F = \langle F_1, F_2, F_3, F_4 \rangle = C_E(\tau)$. Then $F/\langle \sigma, \tau \rangle \cong F_4(2)$, and it is not difficult to see that the groups $F_i/\langle \sigma, \tau \rangle$ are the maximal parabolics of $F_4(2)$. (The construction of the group $2^2E_6(2)2$ was carried out in detail in [7].)

It was shown in [5] that the universal 2-cover $\mathcal{G}(\tilde{H})$ of the P-geometry $\mathcal{G}(H)$ admits a flag-transitive automorphism group $\tilde{H} \cong 3^{4371}H$, and we are now going to apply the above procedure to the group $\tilde{H}$. So let $\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\}$ be a flag in $\mathcal{G}(\tilde{H})$ mapping onto $\{a_1, a_2, a_3\}$ and $\tilde{H}_i = \tilde{H}_{\tilde{a}_i}$. By [5]

$$\begin{aligned}\tilde{H}_1 &\cong (2^{1+22} \times 3^{23}).Co_2,\\ \tilde{H}_2 &\cong [2^{32}].(\Sigma_3 \times 3 \cdot Aut(M_{22})),\\ \tilde{H}_3 &\cong H_3 \cong [2^{35}].(L_3(2) \times \Sigma_5).\end{aligned}$$

Also by [5] $\mathcal{G}(\tilde{H})$ contains a family of symplectic type tilde-subgeometries with stabilizer $\tilde{A} \cong (2^{25} \times 3^{35})Sp_8(2)$.

Lemma 1. *Let $\tilde{\sigma} \in O_2(\tilde{H}_1)$ be an involution mapping onto σ under the natural epimorphism from $\tilde{H}$ onto H, $\tilde{E}_i = C_{\tilde{H}_i}(\tilde{\sigma})$, $i = 1, 2, 3$, and $\tilde{E}_4 = C_{\tilde{A}}(\tilde{\sigma})$. Then*

(1) $\tilde{E}_1 \cong (2^{2+20} \times 3^{23}).U_6(2).2$,

(2) $\tilde{E}_2 \cong [2^{30}].(\Sigma_3 \times SL_3(4).2)$,

(3) $\tilde{E}_3 \cong [2^{32}].(L_3(2) \times \Sigma_5)$,

(4) $\tilde{E}_4 \cong (2^{25} \times 3^{35}).O_8^-(2)$,

and $\tilde{E}/O_3(\tilde{E}) \cong 2^2E_6(2)2$, where $\tilde{E} = \langle \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4 \rangle$.

Proof. As $\tilde{\sigma} \in O_2(\tilde{H}_1)$ we have $[\tilde{\sigma}, O_3(\tilde{H}_1)] = 1$. Let $\hat{E}_1$ be the preimage of E_1 in $\tilde{H}_1$. Then clearly $[\tilde{\sigma}, \hat{E}_1] \leq O_3(\tilde{H}) \cap \tilde{H}_1 = O_3(\tilde{H}_1)$. So $[\tilde{\sigma}, \hat{E}_1] \leq O_3(\tilde{H}_1) \cap O_2(\tilde{H}_1) = 1$, i.e. $\hat{E}_1 = \tilde{E}_1$ and (1) is proved.

Since $O_3(\tilde{H}_2) \leq O_3(\tilde{H}_1)$, in (2) we get $SL_3(4)$ instead of $L_3(4)$, and $\tilde{E}_3 \cong E_3$ because $\tilde{H}_3 \cap O_3(\tilde{H}) = 1$.

Further, $\tilde{H}_1 \cap \tilde{A}$ is the stabilizer of the element $\tilde{a}_1 \in \mathcal{G}(\tilde{A})$, which implies $(\tilde{H}_1 \cap \tilde{A})/O_2(\tilde{H}_1 \cap \tilde{A}) \cong 3^7Sp_6(2)$ and $|O_3(\tilde{H}_1) \cap O_3(\tilde{A})| = 3^7$. Hence $|C_{O_3(\tilde{A})}(\tilde{\sigma})| \geq 3^7$. But $O_8^-(2)$ acts on $C_{O_3(\tilde{A})}(\tilde{\sigma})$ and $O_3(\tilde{A})$ possesses the decomposition $1 + 34$ with respect to the action of $O_8^-(2)$. So $C_{O_3(\tilde{A})}(\tilde{\sigma}) = O_3(\tilde{A})$ and we can argue as above to get (4). □

Lemma 2. *Let $\tilde{\tau} \in \tilde{E}$ be an involution mapping onto τ under the natural epimorphism from $\tilde{H}$ onto H and $\tilde{F}_i = C_{\tilde{E}_i}(\tilde{\tau})$, $i = 1, \ldots, 4$. Then*

(1) $\tilde{F}_1 \cong 2^2 \times (2^{15} \times 3^7).Sp_6(2)$,

(2) $\tilde{F}_2 \cong 2^2 \times [2^{20}]..(\Sigma_3 \times L_3(2))$,

(3) $\tilde{F}_3 \cong 2^2 \times [2^{20}].(L_3(2) \times \Sigma_3)$,

(4) $\tilde{F}_4 \cong 2^2 \times ([2^{15}] \times 3^7).Sp_6(2)$,

$\tilde{F}/O_3(\tilde{F}) \cong 2^2 \times F_4(2)$, where $\tilde{F} = \langle \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4 \rangle$, and the cosets of the groups $\tilde{F}_i$ form a geometry with diagram of type $(\tilde{F}_4)$.

Proof. (3) is clear because $\tilde{E}_3 \cong E_3$, and as $\tau \in Aut(M_{22}) \backslash M_{22}$, $\tilde{\tau}$ inverts $O_3(\tilde{H}_2)$, which shows (2). Next we look at $\tilde{E}_4$. We have $\tau \in A \backslash O_2(A)$ and $C_A(\langle \sigma, \tau \rangle)/O_2(C_A(\langle \sigma, \tau \rangle)) \cong Sp_6(2)$, i.e. τ corresponds to the involution in the center of a maximal parabolic $2^{1+6}Sp_6(2)$ of $Sp_8(2)$. Now by [9, (3.5)] $|C_{O_3(\tilde{A})}(\tilde{\tau})| = 3^7$ and we get (4).

Also $\tilde{F}_1/O_2(\tilde{F}_1)$ is an extension of $Sp_6(2)$ by a 3-group. In order to determine $O_3(\tilde{F}_1)$ we look at the groups $\tilde{F}_{1i} = \tilde{F}_1 \cap \tilde{F}_i$ for $i \neq 1$. Obviously $\tilde{F}_{12}/O_2(\tilde{F}_{12}) \cong L_3(2)$ and $\tilde{F}_{13}/O_2(\tilde{F}_{13}) \cong \Sigma_3 \times \Sigma_3$. Further $\tilde{F}_{14}/O_{23}(\tilde{F}_4) \cong \Sigma_6$ and it follows from [9] that $|O_3(\tilde{F}_{14})| = 3$ or 3^7. Assume the latter possibility holds. Then $O_3(\tilde{F}_4) = O_3(\tilde{F}_{14}) \leq O_3(\tilde{H}_1 \cap \tilde{A})$. But as said in the proof of Lemma 1 $|O_3(\tilde{H}_1 \cap \tilde{A})| = 3^7$, so $O_3(\tilde{F}_4) = O_3(\tilde{H}_1 \cap \tilde{A})$. This yields $\tilde{F}_4 \leq N_{\tilde{A}}(O_3(\tilde{H}_1 \cap \tilde{A})) = \tilde{H}_1 \cap \tilde{A}$ which contradicts the fact that $F_4 \not\leq H_1$. So $\tilde{F}_{14}/O_2(\tilde{F}_{14}) \cong 3\Sigma_6$, and as $\tilde{F}_1 = \langle \tilde{F}_{12}, \tilde{F}_{13}, \tilde{F}_{14} \rangle$, (1) and the last assertion follow. □

For the determination of $|O_3(\tilde{F})|$ we need a little lemma about the action of dihedral groups on vector spaces over fields of odd characteristic.

Lemma 3. *Let $P \cong D_8^n$ act on a vectorspace V over a field F such that 2 does not divide* char$F$. *If $Z(P)$ acts fixpointfreely on V, then 2^n divides* $\dim V$.

Proof. We proceed by induction on n. If $n = 1$, then

$$P = \langle x, y | x^2 = y^2 = (xy)^4 = 1 \rangle.$$

Let $\langle z \rangle = Z(P)$ and consider the action of $\langle x, z \rangle \cong E_4$ on V. Since 2 does not divide char$F$ and $C_V(z) = 0$ we have $V = C_V(x) \oplus C_V(xz)$. But $xz = x^y$, so $\dim V = \dim C_V(xz) + \dim C_V(x) = 2 \cdot \dim C_V(x)$ and the induction is anchored.

If $n > 1$, take $x \in P$, $o(x) = 2$. Then $C_P(x) \cong D_8^{n-1} \times \langle x \rangle$ and the pair $P_1 = C_P(x)/\langle x \rangle$, $V_1 = C_V(x)$ satisfies the hypothesis of the lemma, so by induction 2^{n-1} divides $\dim V_1$. As before we can show $\dim V = 2 \cdot \dim V_1$, so the assertion follows. □

Lemma 4. $\tilde{F} \cong 2^2 \times 3^{833}.F_4(2)$.

Proof. We set $U = O_3(\tilde{F})$, $V = O_3(\tilde{H})$ and regard the images $\sigma, \tau \in H$ of $\tilde{\sigma}, \tilde{\tau}$ as involutions of $H \cong \tilde{H}/V$ acting on the $GF(3)H$-module V (which we write additively). Then $|U| = 3^k$ for some $k \leq \dim C_V(\langle \sigma, \tau \rangle)$. It follows from [5] that we can read $\dim C_V(\sigma)$ and $\dim C_V(\tau)$ from the usual character table of H in [1] and that χ_1 is the character corresponding to V. As $\sigma \in 2A$ and $\tau \in 2C$ we have $\dim C_V(\sigma) = 1939$, $\dim[V, \sigma] = 2432$, $\dim C_V(\tau) = 2159$ and $\dim[V, \tau] = 2212$. Now $\langle \sigma, \tau \rangle = \langle \sigma, \sigma\tau \rangle$ acts on $W = [V, \tau]$ and

$$W = C_W(\sigma) \oplus [W, \sigma] = C_W(\sigma\tau) \oplus [W, \sigma\tau].$$

From $C_W(\sigma) \cap C_W(\sigma\tau) \leq C_W(\tau) = 0$ follows $W = C_W(\sigma) \oplus C_W(\sigma\tau)$. But since the whole centralizer of an involution of type $2C$ in the Babymonster is isomorphic to $(2^2 \times F_4(2))2$, σ and $\sigma\tau$ are conjugate in $C_H(\tau)$. So $\dim C_W(\sigma) = \frac{1}{2} \dim W = 1106$. Further $C_W(\sigma) = C_{[V,\tau]}(\sigma) = [C_V(\sigma), \tau]$ and $\dim C_V(\langle \sigma, \tau \rangle) = \dim C_V(\sigma) - \dim[C_V(\sigma), \tau] = 1939 - 1106 = 833$.

So we have $k \leq 833$ and it remains to show that also $k \geq 833$. Let $C = C_{\tilde{F}}(U)$, $F = \tilde{F}/C \cong F_4(2)$, $F_i = \tilde{F}_i C/C$, and consider the action of F on U. Further, let $V_i = V \cap \tilde{F}_i$,

$i = 1, 4$, and $U_1 = \langle (V_1 V_4)^{\tilde{F}_1} \rangle \leq U$. We have $|V_1| = |V_4| = 3^7$, $|V_1 \cap V_4| = 3$ and $[V_i, O_2(F_i)] = 1$. Let $Z_1 = Z(O_2(F_1))$. Then

$$U_1 = C_{U_1}(Z_1) \oplus [U_1, Z_1] \text{ and } [U_1, Z_1] = \bigoplus_{Z \in \mathcal{H}} C_{U_1}(Z),$$

where $\mathcal{H}$ denotes the set of hyperplanes $Z \leq Z_1$ such that $C_{U_1}(Z) \neq 0$. It follows from the structure of $F_4(2)$ that $|Z_1 : Z_1 \cap O_2(F_4)| = 2$, and by [9, (3.5)] any element $t \in Z_1 \setminus O_2(F_4)$ acts fixpointfreely on $V_4/V_4 \cap V_1$. This implies that $\dim[V_4, Z_1] = 6$ and that $C_{U_1}(Z_1) = V_1$. Hence

$$\dim U_1 = 7 + |F_1 : F_1 \cap F_4| \cdot 6 = 7 + 63 \cdot 6 = 385.$$

Now let $\langle z_1 \rangle = Z(F_1)$. Then $z_1 \in Z(O_2(F_4))$ and $[V_4, z_1] = 0$. Hence also $[z_1, U_1] = [z_1, V_1 V_4]^{F_1} = 0$. Choose $g \in F_4$ such that $z_1 \in F_1^g$, but $z_1 \notin O_2(F_1^g)$ (such a g exists by the structure of $F_4(2)$). Then z_1 acts nontrivially on V_1^g. By similar arguments as before we have $\dim[V_1^g, z_1] = 6$ and $C_{V_1^g}(z_1) = V_1^g \cap V_4 \leq U_1$, i.e. z_1 acts fixpointfreely on $V_1^g / V_1^g \cap U_1$. Let $U_2 = \langle (U_1 V_1^g)^{F_1} \rangle$ and $\bar{U}_2 = U_2/U_1$. Then z_1 is also fixpointfree on $\bar{U}_2$. As above

$$\bar{U}_2 = \bigoplus_{Z \in \mathcal{H}} C_{\bar{U}_2}(Z), \text{ where } \mathcal{H} = \{Z \leq Z_1 | |Z_1 : Z| = 2, C_{\bar{U}_2}(Z) \neq 0\}.$$

Choose $Z \in \mathcal{H}$. Then $z_1 \notin Z$, so $N_{F_1}(Z)/Z \cong 2^{1+8} O_6^-(2)$ or $2^{1+8} O_6^+(2)$. Moreover, $O_2(N_{F_1}(Z)/Z) \cong D_8^4$ and $Z(O_2(N_{F_1}(Z)/Z)) = \langle z_1 \rangle Z/Z$ acts fixpointfreely on $\bar{U}_2$. Now Lemma 3 yields that 16 must divide $\dim C_{\bar{U}_2}(Z)$, and we get

$$\dim \bar{U}_2 \geq |F_1 : N_{F_1}(Z)| \cdot 16 = \begin{cases} 28 \cdot 16 = 448 \text{ resp.} \\ 36 \cdot 16 = 576. \end{cases}$$

Since $\dim \bar{U}_2 \leq \dim U - \dim U_1 \leq 833 - 385 = 448$, we must have $U = U_2$, $N_{F_1}(Z)/Z \cong 2^{1+8} O_6^-(2)$ and $\dim U = 833$. □

Theorem. *There exists a non-split extension $G \cong 3^{833} F_4(2)$ which acts on a geometry of type $(\tilde{F}_4)$.*

Proof. Let $G = \tilde{F}/\langle \tilde{\sigma}, \tilde{\tau} \rangle$ with $\tilde{F}, \tilde{\sigma}, \tilde{\tau}$ as above, and let $M = O_3(G)$. Then G acts on a geometry of type $(\tilde{F}_4)$ by Lemma 2 and $|M| = 3^{833}$ by Lemma 4. So it remains to show that the extension $M \trianglelefteq G$ is non-split.

Suppose $G = MF$ with $F \cong F_4(2)$. Let $\varphi : G \to F$ be the canonical epimorphism, $G_1 = \tilde{F}_1/\langle \tilde{\sigma}, \tilde{\tau} \rangle \cong (2^{15} \times 3^7) Sp_6(2)$, $F_1 = \varphi(G_1) \cong 2^{15} Sp_6(2)$ and $K_1 = O_2(F_1)$. Since F_1 and G_1 both contain a Sylow-2-subgroup of G, we may assume $K_1 = O_2(G_1)$. Now $N_M(K_1) = C_M(K_1)$ and in the proof of Lemma 4 we have seen that $C_M(K_1) = M_1$. This implies that $G_1 = N_G(K_1) = M_1 F_1$, a contradiction to the fact that $G_1/K_1 \cong 3^7 Sp_6(2)$ is a non-split extension. □

References

[1] Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; Wilson, R. A.: *Atlas of Finite Groups.* Oxford 1985.

[2] Fukshansky, A.; Stroth, G.: *Semiclassical parabolic systems related to M_{24}.* To appear.

[3] Heiss, S.: *On a minimal parabolic system of type M_{24}.* J. Algebra **142** (1991), 188–200.

[4] Ivanov, A. A.; Shpectorov, S. V.: *An infinite family of simply connected flag-transitive tilde geometries.* Geom. Dedicata **45** (1993), 1–23.

[5] Ivanov, A. A.; Shpectorov, S. V.: *The last flag-transitive P-geometry.* Israel J. Math. **82** (1993), 341–362.

[6] Ivanov, A. A.; Shpectorov S. V.: *The flag-transitive tilde and Petersen-type geometries are all known.* Bull. Amer. Math. Soc. **31** (1994), 173–184.

[7] Ivanov, A. A.: *A geometric characterization of Fischer's Baby Monster.* J. Alg. Comb. **1** (1992), 45–69.

[8] Ronan, M.; Stroth, G.: *Minimal parabolic geometries for the sporadic groups.* Europ. J. Comb. **5** (1984), 59–91.

[9] Shpectorov, S. V.: *Natural representations of some tilde and Petersen type geometries.* Geom. Dedicata **54** (1995), 87–102.

[10] Shpectorov, S. V.; Stroth, G.: *Classification of certain types of tilde geometries.* Geom. Dedicata **49** (1994), 155–172.

[11] Wiedorn, C.: Dissertation, to appear.

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