Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 38 (1997), No. 2, 437-458.

A New Intrinsic Curvature Invariant for Centroaffine Hypersurfaces

Christine Scharlach, Udo Simon
Leopold Verstraelen, Luc Vrancken
Fachbereich Mathematik, MA 8-3, Technische Universität Berlin
Straße des 17. Juni 136, D - 10623 Berlin, Germany
Departement Wiskunde, Katholieke Universiteit Leuven
Celestijnenlaan 200 B, B - 3001 Leuven, Belgium

Abstract. We consider centroaffine hypersurfaces M^n and introduce a new intrinsic curvature invariant of the centroaffine metric. For $n > 2$ we derive an inequality between this invariant and the square norm of the Tchebychev field. We study the class of hypersurfaces which realize the equality and obtain a wide class of examples and a partial classification. Furthermore we discuss our results under polarization of the hypersurface and finally relate them to recent asymptotic spectral results on second order Laplace type operators in affine differential geometry.

MSC 1991: 53A15, 58G25

Key words and phrases. centroaffine hypersurfaces, curvature invariant, proper affine hyperspheres, polar hypersurfaces, asymptotic spectral geometry of Laplace type operators

1. Introduction

In this paper we study definite centroaffine hypersurfaces M^n in $\mathbf{R}^{n+1}$, $n \geq 2$. For such hypersurfaces, the centroaffine normal induces a definite symmetric bilinear form h on M^n. Therefore we can consider (M^n, h) as a Riemannian manifold and we can define the invariant ϵ_M by

$$\epsilon_M(p) = \tfrac{n(n-1)}{2}\hat{\kappa}(p) - (\sup \hat{K})(p),$$

where $\hat{\kappa}(p)$ is the normalized scalar curvature, $\hat{K}(\pi)$ is the sectional curvature of a plane section π in T_pM and

$$(\sup \hat{K})(p) := \sup\{\hat{K}(\pi) \mid \pi \text{ a plane section in } T_pM\}.$$

0138-4821/93 \$ 2.50

Since $\epsilon_M = 0$ for $n = 2$, a study of ϵ_M is only of interest for $n \geq 3$. A similar invariant, replacing supremum by infimum, was first introduced by B. Y. Chen (see [2]) in order to study submanifolds of real space forms. For further results see also [3], [4], [8] and [5].

The paper is organized as follows. In Section 2, we recall the basic facts about centroaffine hypersurfaces and introduce the necessary notations. We also introduce some notions about distributions on Riemannian manifolds which we will need in Section 5. In Section 3, we derive an inequality relating the invariant ϵ_M with the length of the Tchebychev vector field T. For a positive definite centroaffine hypersurface M^n in $\mathbf{R}^{n+1}$ we show that

$$\epsilon_M \geq -\tfrac{n^2(n-2)}{2(n-1)} h(T^\sharp, T^\sharp) + \tfrac{1}{2}(n+1)(n-2). \tag{1.1}$$

Then we continue with investigating those centroaffine hypersurfaces which realize the equality in (1.1). Clearly, if M is two-dimensional, the equality is trivially satisfied. If the dimension of M^n is greater than two, then hypersurfaces satisfying the equality can be characterized by a special property of the difference tensor K from (2.5) as shown in Section 3. In particular, for these hypersurfaces the Tchebychev vector field vanishes identically, which means they are proper affine hyperspheres in the sense of Blaschke.

In Section 4, we define on such hypersurfaces a distribution D by

$$D_p = \{v \in T_pM | K(v, w) = 0 \quad \forall w \in T_pM\}.$$

We show that the distribution D is differentiable (on an open and dense subset) and that (on that subset) D is integrable and autoparallel. We use these results in Section 5 to obtain classification theorems under the additional assumption that either M^n has constant scalar curvature (Thm. 5.3) or the distribution $D^\perp$ is integrable (Thm. 5.8). In particular, we obtain a wide class of examples related to equiaffine elliptic spheres in $\mathbf{R}^3$, realizing the equality in (1.1). Similar results can be obtained for negative definite centroaffine hypersurfaces and for positive definite hypersurfaces in relative geometry. These will be mentioned throughout the paper. Some of the extensions need subtle proofs which will appear in [15]. Finally some remarks about polarization and asymptotic spectral invariants will be made in Section 6 and 7.

2. Preliminaries

First, we recall the basic facts about centroaffine hypersurfaces. For more details and proofs see [14, Chapter 6]. Let M^n be a n-dimensional C^∞-manifold and let

$$f : M^n \to \mathbf{R}^{n+1}$$

be a non-degenerate hypersurface immersion whose position vector is nowhere tangent. Then f can be regarded as a transversal field along itself and we call $\xi = -f$ the centroaffine normal. Following Nomizu, we call f together with this normalisation a centroaffine hypersurface. We get the centroaffine structure equations:

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)(-f), \tag{2.1}$$
$$D_X \xi = -f_*(X). \tag{2.2}$$

Here D denotes the canonical flat connection of $\mathbf{R}^{n+1}$, ∇ is a torsionfree connection on M^n, called the induced centroaffine connection, and h is a non-degenerate symmetric (0,2)-tensor field, called the centroaffine metric. The corresponding integrability conditions (equations of Gauss (2.3) and Codazzi (2.4)) read

$$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y, \tag{2.3}$$
$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{2.4}$$

If we define $C(X,Y,Z) = (\nabla_X h)(Y,Z)$, we get a totally symmetric (0,3)-tensor field, called the cubic form.

From now on we restrict our attention to so called definite centroaffine hypersurfaces; that means h is definite. In order to obtain a more unified approach, we take for negative definite hypersurfaces $\xi = f$ as affine normal, so that the corresponding second fundamental form h again becomes positive definite; (2.1), (2.2) and (2.3) then read

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)f, \tag{2.1$'$}$$
$$D_X \xi = f_*(X), \tag{2.2$'$}$$
$$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y. \tag{2.3$'$}$$

Denote by $\hat\nabla$ the Levi-Civita connection of h, by $\hat R$, $\widehat{Ric}$ and $\hat\kappa$ the curvature tensor, the Ricci tensor and the normalized scalar curvature of $\hat\nabla$, respectively. The difference tensor K, which is defined by

$$K_X Y = K(X,Y) = \nabla_X Y - \hat\nabla_X Y, \tag{2.5}$$

is a symmetric (1,2)-tensor field related to the cubic form by $C(X,Y,Z) = -2h(K_X Y, Z)$. Hence, for every X, K_X is selfadjoint with respect to h. Furthermore, we define the Tchebychev form T, the Tchebychev vector field $T^\sharp$ and the Pick invariant J by

$$nT(X) = \operatorname{trace} K_X, \tag{2.6}$$
$$h(T^\sharp, X) = T(X), \tag{2.7}$$
$$4n(n-1)J = 4h(K,K) = h(C,C). \tag{2.8}$$

It is also well known that in centroaffine geometry

$$h(K_X Y, Z) = h(Y, K_X Z), \tag{2.9}$$
$$\hat R(X,Y)Z = K_Y K_X Z - K_X K_Y Z + \varepsilon(h(Y,Z)X - h(X,Z)Y), \tag{2.10}$$
$$(\hat\nabla K)(X,Y,Z) = (\hat\nabla K)(Y,X,Z), \tag{2.11}$$
$$\hat\kappa = J + \varepsilon - \tfrac{n}{n-1} h(T^\sharp, T^\sharp), \tag{2.12}$$

where $\varepsilon = 1$ (resp. $\varepsilon = -1$) if M is positive definite (resp. negative definite).

Next, we recall some definitions and properties of distributions on Riemannian manifolds. Details can be found e.g. in [11]. Let (M^n, h) be a Riemannian manifold with Levi Civita connection $\hat\nabla$. Let E define a distribution on M^n. Denote by $E^\perp$ its orthogonal distribution. Then E is called *parallel* if $\hat\nabla_X Y \in E$ for all vectorfields X tangent to M^n

and $Y \in E$; it is called *autoparallel* if $\hat{\nabla}_X Y \in E$ for all $X, Y \in E$; it is called *totally umbilical* if there exists a vector $H \in E^\perp$ such that $h(\hat{\nabla}_X Y, Z) = h(X, Y)h(H, Z)$ for all $X, Y \in E$ and for all $Z \in E^\perp$, in this case H is called the *mean curvature normal* of the distribution E. We call E *spherical* if it is totally umbilical and its mean curvature normal H satisfies $h(\hat{\nabla}_X H, Z) = 0$ for all $X \in E$ and $Z \in E^\perp$. If E is autoparallel, totally umbilical or spherical, then E is involutive and all the leaves of the foliation of M^n induced by E are totally geodesic, totally umbilical or spherical respectively.

3. A Centroaffine Inequality and its Applications

Unless otherwise mentioned, M^n will denote a positive definite centroaffine hypersurface in $\mathbf{R}^{n+1}$. Following the approach of Chen (see [2]), we will derive an inequality involving ϵ_M. We will need the following algebraic inequality from [2]:

Lemma 3.1. *Let $a_1, \ldots, a_n, c$ be $n+1$ real numbers, $n \geq 2$, such that*

$$(\sum_{i=1}^n a_i)^2 = (n-1)(\sum_{i=1}^n a_i^2 + c). \tag{3.1}$$

Then $2a_1a_2 \geq c$, with equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_n$.

Theorem 3.2. *Let M^n be an n-dimensional positive definite centroaffine hypersurface in $\mathbf{R}^{n+1}$. Then*

$$\epsilon_M \geq -\tfrac{n^2(n-2)}{2(n-1)} h(T^\sharp, T^\sharp) + \tfrac{1}{2}(n+1)(n-2).$$

Proof. Define the centroaffine invariant

$$\delta := n(n-1)\hat{\kappa} + \tfrac{n^2(n-2)}{n-1} h(T^\sharp, T^\sharp) - (n+1)(n-2). \tag{3.2}$$

Then (2.12) and (3.2) imply that

$$n^2 h(T^\sharp, T^\sharp) = (n-1)\left(h(K, K) - (\delta - 2)\right). \tag{3.3}$$

Let $\pi \subset T_pM$ be a plane section. We choose an h-orthonormal frame $\{e_1, \ldots, e_n\}$ such that $\pi = \mathrm{span}\{e_1, e_2\}$ and an h-orthonormal frame $\{u_1, \ldots, u_n\}$ such that u_1 is in the direction of the Tchebychev vector field $T^\sharp$. In the special case that $T^\sharp = 0$, we choose an arbitrary h-orthonormal frame $\{u_1, \ldots, u_n\}$. We introduce the notation $K^k_{ij} := h(K_{u_k} e_i, e_j)$; then (3.3) yields

$$(\sum_{i=1}^n K^1_{ii})^2 = (n-1)\left(\sum_{i=1}^n (K^1_{ii})^2 + \sum_{i \neq j}^n (K^1_{ij})^2 + \sum_{r=2}^n \sum_{i,j=1}^n (K^r_{ij})^2 - \delta + 2\right). \tag{3.4}$$

Applying now Lemma 3.1 gives

$$2K^1_{11}K^1_{22} \geq \sum_{i \neq j}^n (K^1_{ij})^2 + \sum_{r=2}^n \sum_{i,j=1}^n (K^r_{ij})^2 - \delta + 2. \tag{3.5}$$

From (2.10), we have

$$\hat{K}(\pi) = h(\hat{R}(e_1, e_2)e_2, e_1) = 1 - h(K(e_1, e_1), K(e_2, e_2)) + h(K(e_1, e_2), K(e_1, e_2)).$$

Therefore,

$$\hat{K}(\pi) = 1 - K_{11}^1 K_{22}^1 - \sum_{r=2}^{n} K_{11}^r K_{22}^r + \sum_{r=1}^{n} (K_{12}^r)^2,$$

and we deduce from (3.5) that

$$\begin{aligned} \hat{K}(\pi) &\le 1 - \tfrac{1}{2}\Big(\textstyle\sum_{i\neq j}(K_{ij}^1)^2 + \sum_{r=2}^n \sum_{i,j=1}^n (K_{ij}^r)^2 - \delta + 2\Big) - \sum_{r=2}^n K_{11}^r K_{22}^r + \sum_{r=1}^n (K_{12}^r)^2 \\ &= -\tfrac{1}{2}\textstyle\sum_{r=2}^n (K_{11}^r + K_{22}^r)^2 - \sum_{r=1}^n \sum_{j=3}^n \big((K_{1j}^r)^2 + (K_{2j}^r)^2\big) \\ &\qquad - \tfrac{1}{2}\textstyle\sum_{i\neq j>2}^n (K_{ij}^1)^2 - \tfrac{1}{2}\sum_{r=2}^n \sum_{i,j>2}^n (K_{ij}^r)^2 + \tfrac{1}{2}\delta \\ &\le \tfrac{1}{2}\delta. \end{aligned} \tag{3.6}$$

Combining (3.6) with (3.2) gives

$$\hat{K}(\pi) \le \tfrac{n(n-1)}{2}\hat{\kappa} + \tfrac{n^2(n-2)}{2(n-1)} h(T^\sharp, T^\sharp) - \tfrac{1}{2}(n+1)(n-2). \tag{3.7}$$

Hence the assertion follows from the definition of ϵ_M. □

Lemma 3.3. *Let M^n be an n-dimensional positive definite centroaffine hypersurface in $\mathbf{R}^{n+1}$. Then, M^n realizes the equality in (1.1) at a point p of M^n if and only if there exist tangent bases $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_n\}$ of T_pM such that the linear maps K_{u_i} take, with respect to the basis $\{e_1, \dots, e_n\}$, the following form:*

$$K_{u_1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}, \qquad K_{u_r} = \begin{pmatrix} K_{11}^r & K_{12}^r & 0 & \dots & 0 \\ K_{12}^r & -K_{11}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (r > 1),$$

where $a + b = \mu$.

Proof. If M^n satisfies the equality in (1.1), then the inequalities in (3.5) and (3.6) become equalities. Therefore, we get that, for $i, j > 2$, $r \ge 2$:

$$\begin{aligned} &K_{1j}^1 = K_{2j}^1 = 0, \\ &K_{ij}^1 = 0, \\ &K_{1j}^r = K_{2j}^r = K_{ij}^r = 0, \\ &K_{11}^2 + K_{22}^2 = \ldots = K_{11}^n + K_{22}^n = 0. \end{aligned}$$

Since $K_{u_1}: \pi \to \pi$ is a symmetric linear operator, we can still choose e_1 and e_2 which diagonalize K_{u_1}, i.e. $K^1_{12} = 0$. Moreover, by applying Lemma 3.1, we have that $K^1_{11} + K^1_{22} = K^1_{33} = \ldots = K^1_{nn}$. □

From now on, we assume that M^n is a centroaffine hypersurface which satisfies the equality in (1.1). The following lemmas extract the information of Lemma 3.3 in a form independent of the choice of the basis. They will be used to show that the Tchebychev vector field $T^\sharp$ vanishes identically; that means that the hypersurface is a proper affine hypersphere.

Lemma 3.4. *Let M^n be an n-dimensional positive definite centroaffine hypersurface in $\mathbf{R}^{n+1}$ satisfying the equality in (1.1). If y is a tangent vector at a point p of M^n, then the symmetric linear operator K_y has at most three eigenvalues and the dimension of one of the eigenspaces is at least $(n-2)$. Moreover, the following cases can occur:*
(1) *$K_y = 0$;*
(2) *K_y has a non-zero real eigenvalue α with a 2-dimensional eigenspace and an eigenvalue -2α with an $(n-2)$-dimensional eigenspace;*
(3) *K_y has zero as an eigenvalue with a 1-dimensional eigenspace and a non-zero real eigenvalue α with an $(n-1)$-dimensional eigenspace;*
(4) *K_y has a non-zero eigenvalue α with a 1-dimensional eigenspace, a non-zero eigenvalue β with a 1-dimensional eigenspace and an eigenvalue $\alpha + \beta$ with an $(n-2)$-dimensional eigenspace.*

Lemma 3.5. *Let M^n be as in Lemma 3.4, let $p \in M$ and let y be a tangent vector at p. Suppose that K_y satisfies case 4 of Lemma 3.4. Let v be an eigenvector belonging to one of the 1-dimensional eigenspaces and denote the $(n-2)$-dimensional eigenspace by W. Then*
(1) *$V = \{K_u v | u \in T_pM\}$ is a vector space containing v. The dimension of V is at most two and for all $u' \in T_pM$, $K_{u'}V \subset V$,*
(2) *for all $u' \in T_pM$, there exists a real number $\kappa(u')$ such that $K_{u'}w = \kappa(u')w$ for all $w \in W$.*

Recall that the equiaffine support function ρ satisfies $T = \frac{n+2}{2n} d \log \rho$ (see (5.1), (5.1.3) and (6.3.2) in [14]). Now, we can prove the following theorem.

Theorem 3.6. *Let M^n be a positive definite centroaffine hypersurface in $\mathbf{R}^{n+1}$, $n \geq 3$. If M^n realizes the equality in (1.1) at a point p of M^n, then the Tchebychev vector field $T^\sharp$ vanishes at p and the equiaffine support function is stationary at p. Moreover, if M^n realizes the equality for every $p \in M^n$, then M^n is a proper affine hypersphere centered at the origin.*

Proof. Our proof is based on a method from [9] which was extended in [16]. Let us assume that the equality holds at $p \in M$ and that the dimension $n > 2$. If for all tangent vectors u, v and w, $h(K(u,v),w) = 0$, there is nothing to prove. So we assume that this is not the case. We define a function $f_p : U_pM \to \mathbf{R}$ by $f_p(v) = h(K(v,v),v)$, where $U_pM = \{v \in T_pM | h(v,v) = 1\}$. Since U_pM is a compact set, there exists a vector u in U_pM such that f_p attains an absolute maximum at u. Then $f_p(u) > 0$ and $h(K(u,u),w) = 0$, for all w orthogonal to u. So u is an eigenvector of the symmetric operator K_u (cp. (2.5)). We put $e_1 = u$ and choose $e_2, \ldots, e_n$ in such a way that $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM and e_i is an eigenvector of K_{e_1} with eigenvalue λ_i. Since f_p attains an absolute maximum in e_1, the function $f_i : \mathbf{R} \to \mathbf{R}$, $i \geq 2$, defined by $f_i(t) = f(\cos te_1 + \sin te_i)$ has a relative maximum at $t = 0$. So, by computing $f_i''(0)$, we get $\lambda_1 - 2\lambda_i \geq 0$. Since $\lambda_1 > 0$, this implies $\lambda_1 \neq \lambda_i$ for all $i \geq 2$. Now we have $K_{e_i}e_1 = K_{e_1}e_i = \lambda_i e_i$, so by Lemma 3.5 with $y = v = e_1$, we may assume $\lambda_3 = \ldots = \lambda_n = 0$. From Lemma 3.4, we get $\lambda_1 + \lambda_2 = 0$. So, we have for $i \geq 3$:

$$\begin{aligned} K_{e_1}e_1 &= \lambda_1 e_1, \\ K_{e_1}e_2 &= -\lambda_1 e_2, \\ K_{e_1}e_i &= 0. \end{aligned}$$

Since $K_{e_2}e_1 = -\lambda_1 e_2$, we also get from Lemma 3.5 (with $y = v = e_1$) that the space spanned by e_1 and e_2 is invariant under all $K_{u'}$, where u' is an arbitrary tangent vector. Using Lemma 3.4 and Lemma 3.5, we can write for $i \geq 3$:

$$\begin{aligned} K_{e_2}e_1 &= -\lambda_1 e_2, \\ K_{e_2}e_2 &= -\lambda_1 e_1 + \alpha e_2, \\ K_{e_2}e_i &= \alpha e_i. \end{aligned}$$

However, since $K_{e_3}e_2 = K_{e_2}e_3 = \alpha e_3$, we get that $\alpha = 0$.

Consider now K_{e_k}, with $k \geq 3$. Since $0 = K_{e_k}e_1 = K_{e_k}e_2$, the space spanned by $\{e_1, e_2\}$ is an eigenspace with eigenvalue 0 and span$\{e_3, \ldots, e_n\}$ also is an eigenspace. Lemma 3.4 now implies that $K_{e_k} = 0$. Finally, we obtain $nT(e_i) = \operatorname{trace} K_{e_i} = 0$ for all i. □

The following lemma is an immediate consequence of the proof of Theorem 3.6 and (3.2).

Lemma 3.7. *Let M^n, $n \geq 3$, be a positive definite centroaffine hypersurface satisfying the equality in (1.1) and let $p \in M$. Then, there exist an h-orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and a number λ such that for $i, j \geq 3$:*

$$\begin{aligned} &K(e_1, e_1) = \lambda e_1, && K(e_1, e_2) = -\lambda e_2, && K(e_1, e_i) = 0, \\ &K(e_2, e_2) = -\lambda e_1, && K(e_2, e_i) = 0, && K(e_i, e_j) = 0. \end{aligned}$$

The function $\lambda (= \lambda_1)$ is a continuous positive function determined by $-4\lambda^2 = n(n-1) - n(n-1)\hat{\kappa}$.

In the case that M^n is a negative definite centroaffine hypersurface, taking f as affine normal and following exactly the same approach, we can prove:

Theorem 3.8. *Let M^n be a negative definite centroaffine hypersurface in $\mathbf{R}^{n+1}$, $n \geq 3$. Then*

$$\epsilon_M \geq -\tfrac{n^2(n-2)}{2(n-1)} h(T^\sharp, T^\sharp) - \tfrac{1}{2}(n+1)(n-2). \tag{3.8}$$

If M^n realizes the equality at a point p of M^n, then the Tchebychev vector field $T^\sharp$ vanishes at p and the equiaffine support function is stationary at p. Moreover, if M^n realizes the equality for every $p \in M^n$, then M^n is a proper affine hypersphere centered at the origin.

Also, in the case that $f\colon M \to \mathbf{R}^{n+1}$ is a (positive definite) hypersurface with relative affine normal ξ, in the sense of [14], a similar inequality can be derived. We denote by H the mean curvature and introduce a curvature like tensor by

$$\tilde{R}(X,Y)Z = \tfrac{1}{2}(R(X,Y)Z + \bar{R}(X,Y)Z),$$

where $\bar{R}$ is the curvature tensor associated with the conjugate connection $\bar{\nabla}$. We denote by $\tilde{K}(\pi)$ the sectional curvature of a plane π with respect to this curvature tensor. Then,

$$\sup(\hat{K}(\pi) - \tilde{K}(\pi)) \leq \tfrac{1}{2}n(n-1)(\hat{\kappa} - H) + \tfrac{n^2(n-2)}{2(n-1)}h(T^\sharp, T^\sharp). \tag{3.9}$$

Similar computations as in Theorem 3.6 and Lemma 3.7 show that if M^n satisfies the equality in (3.9), then the Tchebychev vector field has to vanish identically (hence the relative normal is a constant multiple of the equiaffine normal) and at each point there exists an h-orthonormal basis $\{e_1, \ldots, e_n\}$ such that for $i, j \geq 3$:

$$\begin{array}{lll} K(e_1,e_1) = \lambda e_1, & K(e_1,e_2) = -\lambda e_2, & K(e_1,e_i) = 0, \\ K(e_2,e_2) = -\lambda e_1, & K(e_2,e_i) = 0, & K(e_i,e_j) = 0. \end{array}$$

4. A Differentiable Choice of a Basis

Throughout this section, we will assume that M^n is a centroaffine positive definite hypersurface satisfying the equality in (1.1). We define on such a hypersurface a distribution D by

$$D_p = \{v \in T_pM | K(v,w) = 0 \quad \forall w \in T_pM\}.$$

Denote by $V = \{p \in M | K_p \neq 0\}$. It is clear that V is an open subset. Denote by W the interior of $M \backslash V$. It is well known that either $W = \emptyset$ or W is an open part of a quadric. Then, on V, D defines at each point p an $(n-2)$-dimensional linear subset of T_pM and λ varies continuously. Let $p \in V$ and let $\{e_1, \ldots, e_n\}$ be an h-orthonormal basis of T_pM satisfying Lemma 3.7. In the following we show that we can extend this basis to local vector fields $\{E_1, \ldots, E_n\}$ which satisfy Lemma 3.7 in a neighborhood of the point p of V. In order to do so, we again consider the function $f_q(v) = h(K(v,v),v)$, $q \in M$ (see the proof of Theorem 3.6).

Lemma 4.1. *Let $q \in M$ and let s_q be a critical value of f_q. Then $s_q \in \{-\lambda(q), 0, \lambda(q)\}$.*

Proof. Let s_q be a critical value. Then there exists a $v \in U_pM$ such that $f_q(v) = s_q$ and $h(K(v,v),w) = 0$ for all w orthogonal to v. We take an orthonormal basis $\{e_1, \ldots, e_n\}$ satisfying Lemma 3.7. If necessary by changing $\{e_3, \ldots, e_n\}$, we may assume that $v =$

$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. So $K(v,v) = \lambda(q)(\alpha_1^2 - \alpha_2^2)e_1 - 2\lambda(q)\alpha_1\alpha_2 e_2$. Then $s_q = h(K(v,v),v)$ is a critical value of f_q if and only if the following two equations are satisfied:

$$0 = h(K(v,v), \alpha_2 e_1 - \alpha_1 e_2) = \lambda(q)(\alpha_1^2 - \alpha_2^2)\alpha_2 + 2\lambda(q)\alpha_1^2\alpha_2 = \lambda(q)(3\alpha_1^2 - \alpha_2^2)\alpha_2,$$
$$0 = h(K(v,v), \alpha_3 e_1 - \alpha_1 e_3) = \lambda(q)(\alpha_1^2 - \alpha_2^2)\alpha_3.$$

First we assume $\alpha_2 = 0$. The second equation becomes $\alpha_1^2\alpha_3 = 0$. Since $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, we get that either $v = \pm e_1$ or $v = \pm e_3$. In the first case $h(K(v,v),v) = \pm\lambda(q)$, while in the second case $h(K(v,v),v) = 0$. If $\alpha_2 \neq 0$, the first equation implies that $\alpha_2 = \epsilon\sqrt{3}\alpha_1$, $\epsilon = \pm 1$. Then the second equation gives that $\alpha_3 = 0$ and therefore $h(K(v,v),v) = \pm\lambda(q)$. This completes the proof of the lemma. □

Proposition 4.2. *Let M^n be a positive definite centroaffine hypersurface in $\mathbf{R}^{n+1}$ which realizes the equality in (1.1). Let V be the set of points at which the difference tensor does not vanish identically. Then, there exist local h-orthonormal vector fields $E_1, \ldots, E_n$ in a neighborhood of $p \in V$ such that for $i, j \geq 3$:*

$$\begin{aligned} K(E_1,E_1) &= \lambda E_1, & K(E_1,E_i) &= 0,\\ K(E_1,E_2) &= -\lambda E_2, & K(E_2,E_i) &= 0,\\ K(E_2,E_2) &= -\lambda E_1, & K(E_i,E_j) &= 0. \end{aligned}$$

Proof. We use a similar construction technique as in [16]. Let $p \in V$. First, we show that if f_p attains an absolute maximum in v, we can extend v differentiably to a vector field V on a neighborhood U of p such that $h(V,V) = 1$ and, at every point q of U, f_q attains an absolute maximum in $V(q)$. In order to do this, let $U_1, \ldots, U_n$ be an arbitrary local differentiable orthonormal basis on a neighborhood U of p such that $U_i(p) = e_i$, with $e_1 = v$, satisfying Lemma 3.7. Then we define a function $\gamma\colon \mathbf{R}^n \times U \longrightarrow \mathbf{R}^n$ by $\gamma(a_1, \ldots, a_n, q) = (b_1, \ldots, b_n)$, where for $k = 1, 2, \ldots, n$:

$$b_k = \sum_{i,j=1}^{n} a_i a_j h(K(U_i,U_j),U_k) - \lambda(p)a_k.$$

Using the fact that $\{U_1(p), \ldots, U_n(p)\}$ satisfies Lemma 3.7, we get

$$\begin{aligned} \frac{\partial b_k}{\partial a_m}(1,0,\ldots,0,p) &= 2h(K(U_1(p),U_k(p)),U_m(p)) - \lambda(p)\delta_{km}\\ &= 0, \qquad \text{if} \quad k \neq m,\\ &= \lambda(p), \qquad \text{if} \quad k = m = 1,\\ &= -3\lambda(p), \qquad \text{if} \quad k = m = 2,\\ &= -\lambda(p), \qquad \text{if} \quad k = m > 2. \end{aligned}$$

The implicit function theorem shows that there exist differentiable functions $a_1, \ldots, a_n$ defined on a neighborhood of p such that the local vector field V defined by

$$V = a_1 U_1 + a_2 U_2 + \ldots + a_n U_n$$

satisfies $V(p) = v$ and $K(V, V) = \lambda(p)V$. Therefore the vector field $W = V/\sqrt{h(V,V)}$ satisfies $K(W, W) = (\lambda(p)/\sqrt{h(V,V)})W$. Hence, on a neighborhood of p, f_q has a critical point $W(q)$. Therefore $\lambda(p)/\sqrt{h(V,V)}$ is a critical value, and hence equal to $\lambda(q)$, 0 or $-\lambda(q)$. The latter cases are impossible, so $\lambda(q) = \lambda(p)/\sqrt{h(V,V)}$ and W is the desired vector field.

We now take $V = E_1$ and look at K_{E_1}. We know that E_1 is an eigenvector of K_{E_1} with eigenvalue λ. Since $\lambda \neq 0$, Lemma 3.7 implies that K_{E_1} has two more eigenspaces. The first one is one-dimensional and corresponds to the eigenvalue $-\lambda$ and the second one is $(n-2)$-dimensional and corresponds to the eigenvalue 0. Since these three eigenvalues are different and have constant multiplicities on a neighborhood of p, a result of [11] allows us to find local orthonormal vector fields $\{E_2, \ldots, E_n\}$ such that for $i > 2$:

$$\begin{aligned} K_{E_1}E_2 &= -\lambda E_2, \\ K_{E_1}E_i &= 0. \end{aligned}$$

Combining this with Lemma 3.7 completes the proof of the lemma. □

Since on V, $D = \operatorname{span}\{E_3, \ldots, E_n\}$, it immediately follows that D is differentiable on an open and dense subset of M^n, and thus determines a distribution on that set. Let $p \in V$. We take the local orthonormal basis $\{E_1, \ldots, E_n\}$ constructed in Lemma 4.2. We introduce local functions $\hat{\gamma}_{ij}^k$ by $\hat{\gamma}_{ij}^k := h(\hat{\nabla}_{E_i}E_j, E_k)$. Obviously, $\hat{\gamma}_{ij}^k + \hat{\gamma}_{ik}^j = 0$. Then, we have the following lemma.

Lemma 4.3. *For $i, j > 2$ we have*

$$\hat{\gamma}_{11}^i - \hat{\gamma}_{22}^i = 0, \tag{4.1}$$
$$\hat{\gamma}_{12}^i + \hat{\gamma}_{21}^i = 0, \tag{4.2}$$
$$\hat{\gamma}_{ij}^1 = \hat{\gamma}_{ij}^2 = 0, \tag{4.3}$$
$$\hat{\gamma}_{i1}^2 = -\tfrac{1}{3}(\hat{\gamma}_{12}^i). \tag{4.4}$$

Moreover, the function λ satisfies the following system of differential equations:

$$E_1(\lambda) = -3\lambda\hat{\gamma}_{21}^2, \tag{4.5}$$
$$E_2(\lambda) = 3\lambda\hat{\gamma}_{11}^2, \tag{4.6}$$
$$E_i(\lambda) = -\lambda\hat{\gamma}_{1i}^1. \tag{4.7}$$

Proof. We write $(\hat{\nabla}K)(E_i, E_j, E_k) = (\hat{\nabla}_{E_i}K)(E_j, E_k)$. Since, for $i, j > 2$,

$$\begin{aligned} (\hat{\nabla}K)(E_1, E_i, E_j) &= \hat{\nabla}_{E_1}K(E_i, E_j) - K(\hat{\nabla}_{E_1}E_i, E_j) - K(E_i, \hat{\nabla}_{E_1}E_j) = 0, \\ (\hat{\nabla}K)(E_i, E_1, E_j) &= \hat{\nabla}_{E_i}K(E_1, E_j) - K(\hat{\nabla}_{E_i}E_1, E_j) - K(E_1, \hat{\nabla}_{E_i}E_j) \\ &= -K(E_1, \hat{\nabla}_{E_i}E_j), \end{aligned}$$

Codazzi's equation (2.4) yields $\hat\gamma^1_{ij} = \hat\gamma^2_{ij} = 0$. This proves (4.3). Next we compute

$$\begin{aligned}(\hat\nabla K)(E_i, E_1, E_1) &= \hat\nabla_{E_i} K(E_1, E_1) - 2K(\hat\nabla_{E_i} E_1, E_1)\\ &= E_i(\lambda)E_1 + \lambda\hat\nabla_{E_i} E_1 + 2h(\hat\nabla_{E_i} E_1, E_2)\lambda E_2,\\ (\hat\nabla K)(E_1, E_i, E_1) &= \hat\nabla_{E_1} K(E_i, E_1) - K(\hat\nabla_{E_1} E_i, E_1) - K(E_i, \hat\nabla_{E_1} E_1)\\ &= -h(\hat\nabla_{E_1} E_i, E_1)\lambda E_1 + h(\hat\nabla_{E_1} E_i, E_2)\lambda E_2.\end{aligned}$$

Since $h(\hat\nabla_{E_i} E_j, E_1) = 0$, for $i, j > 2$, by comparing components, we get

$$\begin{aligned}E_i(\lambda) &= -\lambda h(\hat\nabla_{E_1} E_i, E_1) = -\lambda\hat\gamma^1_{1i},\\ 3h(\hat\nabla_{E_i} E_1, E_2) &= -h(\hat\nabla_{E_1} E_2, E_i).\end{aligned}$$

This proves (4.4) and (4.7). Similarly, from $(\hat\nabla K)(E_i, E_2, E_2) = (\hat\nabla K)(E_2, E_i, E_2)$, we obtain (4.1): $\hat\gamma^i_{22} = \hat\gamma^i_{11}$. Finally, we have

$$\begin{aligned}(\hat\nabla K)(E_2, E_1, E_1) &= \hat\nabla_{E_2} K(E_1, E_1) - 2K(\hat\nabla_{E_2} E_1, E_1)\\ &= E_2(\lambda)E_1 + \lambda\hat\nabla_{E_2} E_1 + 2h(\hat\nabla_{E_2} E_1, E_2)\lambda E_2,\\ (\hat\nabla K)(E_1, E_2, E_1) &= \hat\nabla_{E_1} K(E_2, E_1) - K(\hat\nabla_{E_1} E_2, E_1) - K(E_2, \hat\nabla_{E_1} E_1)\\ &= -E_1(\lambda)E_2 - \lambda\hat\nabla_{E_1} E_2 - h(\hat\nabla_{E_1} E_2, E_1)\lambda E_1 + h(\nabla_{E_1} E_1, E_2)\lambda E_1.\end{aligned}$$

So, by comparing components, we get

$$\begin{aligned}&h(\hat\nabla_{E_2} E_1 + \hat\nabla_{E_1} E_2, E_i) = 0,\\ &E_2(\lambda) = 3\lambda\hat\gamma^2_{11},\\ &E_1(\lambda) = -3\lambda\hat\gamma^2_{21}.\end{aligned}$$

This completes the proof of the lemma. □

Clearly, Lemma 4.3 implies that the distribution D is integrable and autoparallel.

5. Some Classification Theorems

The inequality (1.1) gives rise to a global centroaffine curvature invariant for hyperovaloids and a characterization of hyperellipsoids in terms of this global invariant.

Proposition 5.1. *Consider a centroaffine hyperovaloid $f\colon M^n \to \mathbf{R}^{n+1}$ of dimension $n \geq 3$ with normalization as in (2.1). Then,*

$$\int (\epsilon_M + \tfrac{n^2(n-2)}{2(n-1)} h(T^\sharp, T^\sharp))\omega(h) \geq \tfrac{1}{2}(n+1)(n-2)\,\mathrm{vol}(M, h),$$

where $\omega(h)$ denotes the Riemannian volume form of (M^n, h) and $\mathrm{vol}(M, h)$ *the total volume. Moreover, we have equality if and only if the hyperovaloid is an hyperellipsoid.*

Proof. We only need to discuss the equality. Theorem 3.6 gives an affine hypersphere and finally an hyperellipsoid from the theorem of Blaschke and Deicke ([10], Theorem 2.4.7).

Remark 5.2. According to Sections 3 and 4 in [13], h and $T^\sharp$ form a complete system of centroaffine invariants on hyperovaloids; that means a centroaffine hyperovaloid can be uniquely determined from $\{h, T^\sharp\}$. The global centroaffine invariant on the left hand side depends only on this system.

Theorem 5.3. *Let M^n, $n \geq 3$, be a positive definite centroaffine hypersurface realizing the equality in (1.1). If M^n has constant scalar curvature, then M^n is an open part of an hyperellipsoid centered at the origin.*

Proof. Let M^n be a centroaffine hypersurface with constant scalar curvature satisfying (1.1). Then, it follows from Lemma 3.7 that either K vanishes identically on M^n or λ is a strictly positive constant. In the first case, it is well known (see e.g. [14], Section 7.1.1) that M^n is affinely equivalent with an open part of an hyperellipsoid centered at the origin.

Therefore, we may assume that λ is a positive constant. Lemma 4.3 then implies that, for $i, j > 2$:

$$\hat\gamma_{11}^i - \hat\gamma_{22}^i = 0, \tag{5.1}$$
$$\hat\gamma_{12}^i + \hat\gamma_{21}^i = 0, \tag{5.2}$$
$$\hat\gamma_{ij}^1 = \hat\gamma_{ij}^2 = 0, \tag{5.3}$$
$$\hat\gamma_{i1}^2 = -\tfrac{1}{3}(\hat\gamma_{12}^i), \tag{5.4}$$
$$0 = -3\lambda\hat\gamma_{21}^2, \tag{5.5}$$
$$0 = 3\lambda\hat\gamma_{11}^2, \tag{5.6}$$
$$0 = -\lambda\hat\gamma_{1i}^1. \tag{5.7}$$

Combining (5.1), (5.5), (5.6) and (5.7) we immediately get that

$$\hat\nabla_{E_1}E_1 = \hat\nabla_{E_2}E_2 = 0.$$

Clearly, we can still assume that $\{E_3, \dots, E_n\}$ are chosen in such a way that

$$\hat\nabla_{E_1}E_2 = bE_3, \tag{5.8}$$

where b is a locally defined function. Then, (5.2) implies

$$\hat\nabla_{E_2}E_1 = -bE_3. \tag{5.9}$$

If $n > 3$, then by (2.10) we have $1 = h(\hat R(E_1, E_4)E_4, E_1)$. Since by (5.3) resp. (5.7) and (5.8), $\hat\nabla_{E_4}E_4$ and $\hat\nabla_{E_1}E_4$ are orthogonal to E_1 and E_2, this becomes

$$1 = h(\hat\nabla_{\hat\nabla_{E_4}E_1}E_4, E_1) = 0,$$

which is a contradiction. Thus $K \equiv 0$. So, $n = 3$, and $\hat{\nabla}$ is given by

$$\begin{array}{lll}
\hat{\nabla}_{E_1} E_1 = 0, & \hat{\nabla}_{E_2} E_2 = 0, & \hat{\nabla}_{E_3} E_3 = 0, \\
\hat{\nabla}_{E_1} E_2 = bE_3, & \hat{\nabla}_{E_1} E_3 = -bE_2, & \hat{\nabla}_{E_2} E_3 = bE_1, \\
\hat{\nabla}_{E_2} E_1 = -bE_3, & \hat{\nabla}_{E_3} E_1 = -\frac{1}{3} bE_2 & \hat{\nabla}_{E_3} E_2 = \frac{1}{3} bE_1.
\end{array}$$

Using again (2.10) we get

$$1 = h(\hat{R}(E_1, E_3)E_3, E_1) = b^2. \tag{5.10}$$

Hence $b = \pm 1$. To obtain now the desired contradiction, we use again (2.10):

$$1 + 2\lambda^2 = h(\hat{R}(E_1, E_2)E_2, E_1) = -\tfrac{5}{3}.$$

This completes the proof of the theorem. □

From now on, we will concentrate on classifying those hypersurfaces for which the distribution $D^\perp$ is integrable. Clearly, we may restrict ourselves to the open set V consisting of the points p with $K_p \neq 0$. Since this distribution is locally spanned by the vector fields E_1 and E_2, its integrability is equivalent to $h(\hat{\nabla}_{E_1} E_2 - \hat{\nabla}_{E_2} E_1, E_i) = 0$, for $i \geq 3$. Combining this with Lemma 4.3 gives us, for $i > 2$:

$$\hat{\gamma}_{12}^i = \hat{\gamma}_{21}^i = 0. \tag{5.11}$$

If we suppose that $D^\perp$ is also autoparallel on some open set, then both D and $D^\perp$ are parallel distributions and therefore $h(\hat{R}(E_3, E_1)E_1, E_3) = 0$. On the other hand, (2.10) implies that $h(\hat{R}(E_3, E_1)E_1, E_3) = 1$, which gives us a contradiction.

Therefore we can define an open dense set U of V as the set of points where $D^\perp$ is not autoparallel. On U, we can define a unique (up to sign) vector field E_3 (which belongs to D) such that by (4.1):

$$\hat{\nabla}_{E_1} E_1 = a_1 E_2 + a_2 E_3,$$
$$\hat{\nabla}_{E_2} E_2 = -a_3 E_1 + a_2 E_3,$$

where a_1, a_2, a_3 are locally defined functions with $a_2 \neq 0$. Our assumption (5.11) together with Lemma 4.3 implies that for $i \geq 3$:

$$\hat{\nabla}_{E_1} E_2 = -a_1 E_1, \tag{5.12}$$
$$\hat{\nabla}_{E_2} E_1 = a_3 E_2, \tag{5.13}$$
$$\hat{\nabla}_{E_i} E_1 = \hat{\nabla}_{E_i} E_2 = 0. \tag{5.14}$$

We now define 3 local distributions H_0, H_1 and H_2 by

$$H_0 = \operatorname{span}\{E_3\}, \quad H_1 = \operatorname{span}\{E_1, E_2\}, \quad H_2 = \operatorname{span}\{E_4, \ldots, E_n\}.$$

In order to simplify our notation, we introduce different ranges of indices. From now on, $p,q \in \{1,2\}$ and $i,j,k \in \{4,\dots,n\}$.

Lemma 5.4. *The distribution H_0 is autoparallel and its integral curves are ellipses centered at the origin in $\mathbf{R}^{n+1}$.*

Proof. By Lemma 4.3, we already know that $\hat\nabla_{E_3}E_3$ has no H_1-component. In order to prove that $\hat\nabla_{E_3}E_3$ has no H_2-component, we use (2.10). For $i \geq 4$, we have

$$\begin{aligned}0 &= h(\hat R(E_1,E_3)E_1,E_i)\\ &= h(\hat\nabla_{E_1}\hat\nabla_{E_3}E_1 - \hat\nabla_{E_3}\hat\nabla_{E_1}E_1 - \hat\nabla_{\hat\nabla_{E_1}E_3-\hat\nabla_{E_3}E_1}E_1,E_i)\\ &= -h(\hat\nabla_{E_3}(a_1E_2+a_2E_3),E_i) = -a_2h(\hat\nabla_{E_3}E_3,E_i).\end{aligned}$$

Since $a_2 \neq 0$, this shows that $\hat\nabla_{E_3}E_3$ has no H_2-component and thus $\hat\nabla_{E_3}E_3 = 0$.

Let γ be an integral curve of the vector field E_3. Since, we have that $D_{E_3}E_3 = -f$ (cf. (2.1) and (2.5)), the curve γ satisfies the differential equation $\gamma'' = -\gamma$. Hence γ is an ellipse centered at the origin. □

Lemma 5.5. *The distribution H_1 is spherical and the distribution $H_0 \oplus H_1$ is autoparallel.*

Proof. For $p,q \in \{1,2\}$, we know that

$$\hat\nabla_{E_p}E_q - a_2\delta_{pq}E_3 \in H_1, \tag{5.15}$$

showing that H_1 is totally umbilical with mean curvature normal $\eta = a_2E_3$. So, in order to show that H_1 is spherical, it is sufficient to prove that the following two conditions are satisfied:

$$E_1(a_2) = E_2(a_2) = 0, \tag{5.16}$$

$$h(\hat\nabla_{E_1}E_3,E_i) = h(\hat\nabla_{E_2}E_3,E_i) = 0, \quad i \geq 4. \tag{5.17}$$

In order to prove (5.16) and (5.17), we use again the equation (2.10). Modulo the distribution H_1, we have

$$\begin{aligned}0 &\equiv \hat R(E_1,E_2)E_1\\ &\equiv \hat\nabla_{E_1}\hat\nabla_{E_2}E_1 - \hat\nabla_{E_2}\hat\nabla_{E_1}E_1 - \hat\nabla_{\hat\nabla_{E_1}E_2-\hat\nabla_{E_2}E_1}E_1\\ &\equiv \hat\nabla_{E_1}(a_3E_2) - \hat\nabla_{E_2}(a_1E_2+a_2E_3) - \hat\nabla_{-a_1E_1-a_3E_2}E_1\\ &\equiv -a_1a_2E_3 - E_2(a_2)E_3 - a_2\hat\nabla_{E_2}E_3 + a_1a_2E_3,\\ &\equiv -E_2(a_2)E_3 - a_2\hat\nabla_{E_2}E_3 \qquad (\mathrm{mod}\, H_1),\end{aligned}$$

which shows that $E_2(a_2) = 0$ and $h(\hat\nabla_{E_2}E_3,E_i) = 0$ for $i \geq 4$. The other equations follow in a similar manner by computing $\hat R(E_1,E_2)E_2$. This shows that H_1 is spherical.

The fact that $H_0 \oplus H_1$ is autoparallel now follows immediately from (5.14), (5.15) and (5.17). $\square$

Lemma 5.6. *The distribution H_2 is spherical and the distribution $H_0 \oplus H_2$ is autoparallel.*

Proof. First, we notice that the distribution $H_0 \oplus H_2 = D$ is autoparallel. To show that H_2 is spherical, we first compute $\hat{\nabla}_{E_i} E_j$ modulo H_2. It follows from Lemma 4.3 that there exist local functions α_{ij} such that

$$\hat{\nabla}_{E_i} E_j \equiv \alpha_{ij} E_3 \qquad (\operatorname{mod} H_2),$$

for $i, j \geq 4$. From (2.10), we then get

$$\begin{aligned}\delta_{ij} &= h(\hat{R}(E_i, E_1)E_1, E_j) \\ &= h(\hat{\nabla}_{E_i} \hat{\nabla}_{E_1} E_1 - \hat{\nabla}_{E_1} \hat{\nabla}_{E_i} E_1 - \hat{\nabla}_{\hat{\nabla}_{E_i} E_1 - \hat{\nabla}_{E_1} E_i} E_1, E_j) \\ &= h(\hat{\nabla}_{E_i} (a_1 E_2 + a_2 E_3) + \hat{\nabla}_{\hat{\nabla}_{E_1} E_i} E_1, E_j) \\ &= a_2 h(\hat{\nabla}_{E_i} E_3, E_j),\end{aligned}$$

from which we deduce $\alpha_{ij} = -\delta_{ij} 1/a_2$.

Since $\hat{\nabla}_{E_i} E_j \in H_0 \oplus H_2$, we see that H_2 is spherical if and only if a_2 is constant along H_2. We have, for $i \geq 4$,

$$\begin{aligned}0 &= h(\hat{R}(E_i, E_1)E_1, E_3) \\ &= h(\hat{\nabla}_{E_i} \hat{\nabla}_{E_1} E_1 - \hat{\nabla}_{E_1} \hat{\nabla}_{E_i} E_1 - \hat{\nabla}_{\hat{\nabla}_{E_i} E_1 - \hat{\nabla}_{E_1} E_i} E_1, E_3) \\ &= h(\hat{\nabla}_{E_i} (a_1 E_2 + a_2 E_3) + \hat{\nabla}_{\hat{\nabla}_{E_1} E_i} E_1, E_3) = E_i(a_2).\end{aligned}$$

Thus a_2 is constant along H_2 and H_2 is spherical. $\square$

Lemma 5.7. *The function a_2 satisfies the following system of differential equations:*

$$\begin{aligned}&E_1(a_2) = E_2(a_2) = E_i(a_2) = 0, \qquad i \geq 4, \\ &E_3(a_2) = 1 + a_2^2.\end{aligned}$$

Proof. The first statement is clear from Lemma 5.5 and Lemma 5.6. In order to show the second statement, we use again (2.10). We have

$$\begin{aligned}1 &= h(\hat{R}(E_3, E_1)E_1, E_3) \\ &= h(\hat{\nabla}_{E_3} \hat{\nabla}_{E_1} E_1 - \hat{\nabla}_{E_1} \hat{\nabla}_{E_3} E_1 - \hat{\nabla}_{\hat{\nabla}_{E_3} E_1 - \hat{\nabla}_{E_1} E_3} E_1, E_3) \\ &= h(\hat{\nabla}_{E_3} (a_1 E_2 + a_2 E_3) + \hat{\nabla}_{\hat{\nabla}_{E_1} E_3} E_1, E_3) = E_3(a_2) - a_2^2,\end{aligned}$$

which completes the proof. $\square$

Theorem 5.8. *Let M^n, $n \geq 3$, be a simply connected positive definite centroaffine hypersurface satisfying the equality in (1.1). Suppose moreover that the distribution $D^\perp$ is integrable. Then M^n is locally centroaffinely equivalent to the hypersurface*

$$f(t, y, x) = \cos t\,(\psi(y), 0, \ldots, 0) + \sin t\,(0, 0, 0, \phi(x)),$$

where the map $x \mapsto \phi(x)$ describes an hyperellipsoid in $\mathbf{R}^{n-2}$ centered at the origin and the map $y \mapsto \psi(y)$ describes a proper affine sphere in $\mathbf{R}^3$.

Proof. Our considerations are of local character. Here, we use the notation M^n for an appropriate local neighborhood. Using Nölker's result [11] and using the formalism of [6], the above lemmas imply that M^n is locally isometric to a warped product manifold, say $M^n = (-\epsilon, \epsilon) \times_{\rho_1} M_1 \times_{\rho_2} M_2$, where the warping functions ρ_1 and ρ_2 are respectively determined by

$$a_2 = -E_3(\log \rho_1),$$
$$(-\tfrac{1}{a_2}) = -E_3(\log \rho_2),$$

with the initial conditions that $\rho_1(0) = \rho_2(0) = 1$. So, if we denote the variable on $(-\epsilon, \epsilon)$ by t, with $\frac{\partial}{\partial t} = E_3$, we get that

$$a_2 = -\tfrac{\partial}{\partial t}(\log \rho_1),$$
$$\tfrac{1}{a_2} = \tfrac{\partial}{\partial t}(\log \rho_2).$$

Since, by the previous lemma,

$$\tfrac{\partial a_2}{\partial t} = 1 + a_2^2,$$

we get that the function a_2, which only depends on t, is given by

$$a(t) = \tan(t - s),$$

where s is an arbitrary constant. By translating the variable t, we may assume that $s = 0$. It now follows that

$$\rho_1(t) = D_1 \cos t,$$
$$\rho_2(t) = D_2 \sin t,$$

where D_1 and D_2 are determined by the initial conditions.

We now denote by u and v local coordinates on M_1 and by x a point on M_2. We identify M^n with its image under f and define a map ψ from M^n into $\mathbf{R}^{n+1}$ by

$$\psi(t, u, v, x) = \cos t\, f(t, u, v, x) - \sin t\, E_3(t, u, v, x). \tag{5.18}$$

Then, for $i \geq 4$,

$$\begin{aligned} \psi_*(E_3) &= -\sin t\, f(t, u, v, x) + \cos t\, E_3(t, u, v, x) \\ &\qquad - \cos t\, E_3(t, u, v, x) - \sin t\,(-f(t, u, v, x)) = 0. \\ \psi_*(E_i) &= \cos t\, E_i - \sin t\, \tfrac{1}{a_2(t)} E_i = 0. \end{aligned}$$

Hence ψ actually reduces to a well-defined map from M_1 into $\mathbf{R}^{n+1}$. We now put $\psi(u,v) = \psi(t,u,v,x)$. We compute

$$\begin{aligned}\psi_*(E_1) &= (\cos t + \sin t\, a_2(t))E_1 = \tfrac{1}{\cos t}E_1,\\ \psi_*(E_2) &= (\cos t + \sin t\, a_2(t))E_2 = \tfrac{1}{\cos t}E_2.\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}D_{E_1}\psi_*(E_1) &= \tfrac{1}{\cos t}(\lambda E_1 + a_1E_2 + a_2E_3 - f)\\ &= \tfrac{1}{\cos t}(\lambda E_1 + a_1E_2) - \tfrac{1}{\cos^2 t}(\cos t\, f - \sin t\, E_3)\\ &= \tfrac{1}{\cos t}(\lambda E_1 + a_1E_2) - \tfrac{1}{\cos^2 t}\psi.\end{aligned}$$

Similarly we get that

$$\begin{aligned}D_{E_1}\psi_*(E_2) &= \tfrac{1}{\cos t}(-a_1E_1 - \lambda E_2),\\ D_{E_2}\psi_*(E_1) &= \tfrac{1}{\cos t}(a_3 - \lambda)E_2,\\ D_{E_2}\psi_*(E_2) &= \tfrac{1}{\cos t}(-(a_3+\lambda)E_1) - \tfrac{1}{\cos^2 t}\psi.\end{aligned}$$

Using the previous equations, it follows immediately that the image of ψ is contained in a three-dimensional linear subspace A_1 (through the origin) and that $\psi: M_1 \to A_1$ is a positive definite centroaffine immersion with vanishing Tchebychev vector field.

Finally, we define still another map from $\phi: M \to \mathbf{R}^{n+1}$ by

$$\phi(t,u,v,x) = \tfrac{1}{\sin t}\left(f(t,u,v,x) - \cos t\, \psi(u,v)\right). \tag{5.19}$$

Then, using (5.18), we have

$$\begin{aligned}\phi_*(E_1) &= \tfrac{1}{\sin t}\left(E_1 - \cos t\, \psi_*(E_1)\right) = 0,\\ \phi_*(E_2) &= \tfrac{1}{\sin t}\left(E_2 - \cos t\, \psi_*(E_2)\right) = 0,\\ \phi_*(E_3) &= -\tfrac{1}{\sin^2 t}\cos t\,(f - \cos t\, \psi)\\ &\quad + \tfrac{1}{\sin t}(E_3 + \sin t\, \psi)\\ &= \tfrac{1}{\sin^2 t}(-\cos t\, f + \sin t\, E_3 + \psi) = 0.\end{aligned}$$

Therefore ϕ actually defines a mapping from M_2 in $\mathbf{R}^{n+1}$. If we denote by $\nabla^*_X Y$ the H_2-component of $\hat{\nabla}_X Y$, we find that for $i, j \geq 4$,

$$\phi_*(E_i) = \tfrac{1}{\sin t}E_i,$$

and

$$\begin{aligned}D_{E_i}\phi_*(E_j) &= \tfrac{1}{\sin t}D_{E_i}E_j\\ &= \tfrac{1}{\sin t}(K(E_i,E_J) + \hat{\nabla}_{E_i}E_j - \delta_{ij}f)\\ &= \tfrac{1}{\sin t}(\nabla^*_{E_i}E_j) + \delta_{ij}\tfrac{1}{\sin t}(-\tfrac{\cos t}{\sin t}E_3 - f)\\ &= \tfrac{1}{\sin t}(\nabla^*_{E_i}E_j) + \delta_{ij}\tfrac{1}{\sin t}(-\tfrac{\cos t}{\sin^2 t}(cost\, f - \psi) - f)\\ &= \tfrac{1}{\sin t}\nabla^*_{E_i}E_j - \tfrac{1}{\sin^2 t}\delta_{ij}\phi.\end{aligned}$$

So we notice that $\phi(M_2)$ is contained in an $(n-2)$-dimensional affine space containing the origin and that ϕ is a tranversal vector field to the immersion. So, if we take $-\phi$ as centroaffine normal and denote the induced connection by ∇^ϕ and the induced centroaffine metric by h^ϕ, we notice that ϕ is positive definite and

$$\begin{aligned} h^\phi(E_i, E_j) &= \tfrac{1}{\sin^2 t}\delta_{ij} = \tfrac{1}{\sin^2 t} h(E_i, E_j), \\ \nabla^\phi_{E_i} E_j &= \nabla^*_{E_i} E_j. \end{aligned}$$

Since $\hat{\nabla}$ is the Levi Civita connection for h, a straightforward computation shows that ∇^ϕ is the Levi Civita connection for h^ϕ. Hence the cubic form vanishes and M_2 is centroaffine equivalent with an hyperellipsoid centered at the origin in an $(n-2)$-dimensional linear subspace A_2 (cp. [14], Section 7.1.1). Since M^n is linearly full we have that A_1 and A_2 are complementary linear subspaces. Hence it follows from (5.19) that there exists a centroaffine transformation such that

$$f(t, y, x) = \cos t\,(\psi(y), 0, \ldots, 0) + \sin t\,(0, 0, 0, \phi(x)). \qquad \square$$

Also, in the negative definite case, similar results can be obtained. By slightly adapting the proof of Theorem 5.3 it is possible to show that

Theorem 5.9. *Let M^n, $n \geq 3$, be a negative definite centroaffine hypersurface realizing the equality in (3.8). If M^n has constant scalar curvature, then M^n is an open part of a hyperboloid centered at the origin.*

Adapting the proof of Theorem 5.8 to the negative definite case is subtle (cp. [15]). However, it is still possible to show that M^n is centroaffinely equivalent with

(1) a quadric or

(2) with a hypersurface

$$f_1(t, y, x) = \cosh t\,(\psi(y), 0, \ldots, 0) + \sinh t\,(0, 0, 0, \phi(x)),$$

where the map $x \mapsto \phi(x)$ describes an ellipsoid in $\mathbf{R}^{n-2}$ centered at the origin and the map $y \mapsto \psi(y)$ describes a negative definite proper affine sphere in $\mathbf{R}^3$ or

(3) with a hypersurface

$$f_2(t, y, x) = -\sinh t\,(\psi(y), 0, \ldots, 0) + \cosh t\,(0, 0, 0, \phi(x)),$$

where the map $x \mapsto \phi(x)$ describes a two-sheeted hyperboloid in $\mathbf{R}^{n-2}$ centered at the origin and the map $y \mapsto \psi(y)$ describes a positive definite proper affine sphere in $\mathbf{R}^3$ or

(4) with a hypersurface

$$f_3(t, y, x) = e^{-t}(1, e^{2t} + \phi(y) + \tfrac{1}{2}\textstyle\sum_{i=1}^{n-3} x_i^2, y, x_1, \ldots, x_{n-3}),$$

where $(y, \phi(y))$, $\phi: N^2 \to \mathbf{R}$, defines an equiaffine improper affine sphere with constant normal vector field $(0, 0, 1)$. This last example can also be seen as a semi-projective composition of improper affine spheres (in the sense of [7]).

6. Remarks on Polarization

Polarity of hypersurfaces is one of the classical concepts of Euclidean hypersurface theory. In [12] it was studied under affine view points: from this, polarity of a non-degenerate pair of hypersurfaces exactly describes the relation between a centroaffine hypersurface and its centroaffine Gauss conormal immersion. We briefly recall this situation.

Let $V := \mathbf{R}^{n+1}$, V^* be its dual space and $\langle\ ,\ \rangle : V^* \times V \to \mathbf{R}$ be the canonical scalar product. Then, the centroaffine hypersurfaces $f: M \to V$ and $F: M \to V^*$ define a polar pair if

$$\langle F, f\rangle = -1, \qquad \langle F, f_*\rangle = 0, \qquad \langle F_*, f\rangle = 0. \tag{6.1}$$

The duality of the situation is obvious: F is the Gauss conormal immersion of f and vice versa. Polarization in the classical terminology means the interchange of f and F, when V is equipped with a Euclidean structure and V and V^* can be identified; in this situation (6.1) can be described in Euclidean terms. In our situation polarization means the interchange of f and F. The Gauss structure equation for F, corresponding to (2.1), read

$$D_X F_*(Y) = F_*(\bar{\nabla}_X Y) + \bar{h}(X, Y)(-F), \tag{6.2}$$

while the normalisation in (2.1') implies

$$D_X F_*(Y) = F_*(\bar{\nabla}_X Y) + \bar{h}(X, Y)F. \tag{6.2'}$$

Proposition 7.2.1 in [12] contains a list of affine invariants of f and F which are also invariant under polarization. From this list, we only need the following:

Lemma 6.1. *For a polar pair with $n \geq 2$, we have the following centroaffine relations:*
(1) $\bar{h} = h$,
(2) $h(\bar{T}^\sharp, \bar{T}^\sharp) = h(T^\sharp, T^\sharp)$,
(3) *the connections ∇ and $\bar{\nabla}$ interchange under polarization.*

Recall from Remark 5.2 that $\{h, T^\sharp\}$ define a complete system of invariants which globally determine a hyperovaloid. Theorem 3.6 (resp. Theorem 3.8) and Lemma 6.1 imply for the centroaffine invariants ϵ_M and δ (see (3.2)):

Corollary 6.2. *For $n \geq 3$, we have:*
(1) *the local centroaffine invariants ϵ_M, δ and $\epsilon_M + \frac{n^2(n-2)}{2(n-1)} h(T^\sharp, T^\sharp)$ are also invariant under polarization;*
(2) *the inequality*

$$\tfrac{1}{\mathrm{vol}(M,h)} \int (\epsilon_M + \tfrac{n^2(n-2)}{2(n-1)} h(T^\sharp, T^\sharp))\omega(h) \geq \varepsilon \tfrac{1}{2}(n+1)(n-2)$$

gives a lower bound, depending only on the dimension n, for the global polarization invariant on the left-hand side; here $\varepsilon = 1$ ($\varepsilon = -1$) corresponding to the normalization in (2.1) (and (2.1'), respectively);

(3) *if we have equality in (2) then f and F both are proper affine hyperspheres (of elliptic type if $\varepsilon = 1$, of hyperbolic type if $\varepsilon = -1$).*

7. Spectral relation

Recently [1] studied asymptotic spectral invariants of some second order differential operators of Laplace type which are of importance in affine hypersurface theory. We recall the definitions and main results and refer to [1] for details. We restrict to compact M^n with $\partial M^n = \emptyset$ and thus to the case that h is positive definite and $f(M^n)$ is a hyperovaloid which is normalized by $\xi = -f$ (see (2.1)).

Consider the induced connection ∇ from (2.1), its Ricci tensor $\mathrm{Ric}(\nabla)$ and its Hessian

$$(\mathrm{Hess}_\nabla f)(V, W) := V(Wf) - df(\nabla_V W). \tag{7.1}$$

We define the second order operator D by

$$Df := D(h, \nabla)f := \mathrm{trace}_h(\mathrm{Hess}_\nabla f + \tfrac{1}{n-1}\,\mathrm{Ric}(\nabla)f) = \mathrm{trace}_h \mathrm{Hess}_\nabla f + nf. \tag{7.2}$$

Here, we used that $\mathrm{Ric}(\nabla) = (n-1)h$ (see [14], Section 6.3.3.2). Let $\{\lambda_v\}$ be the eigenvalues of the elliptic operator D where each eigenvalue is repeated according to its multiplicity and where $\lambda_1 \leq \lambda_2 \leq \ldots$. If $t > 0$, then $\exp(-tD)$ is trace class on $L^2(M)$ and the series

$$\operatorname*{Tr}_{L^2}(\exp(-tD)) = \sum_v \exp(-t\lambda_v)$$

converges uniformly on compact subsets of $(0, \infty)$. As $t \searrow 0$, there is an asymptotic series of the form

$$\mathrm{Tr}_{L^2}(\exp(-tD)) \sim \sum_{k \geq 0} a_k(D) t^{\frac{1}{2}(k-n)}.$$

The coefficients $a_k(D)$ are spectral invariants which are locally computable in terms of (M, h, ∇) as follows: there exist local invariants $a_k(p, D)$, $p \in M$, depending both on h and ∇ such that

$$a_k(D) = \int a_k(p, D)\omega(h). \tag{7.3}$$

Analogously, we can consider the conormal connection $\bar{\nabla}$ from (6.2) and the differential operator $\bar{D} := D(h, \bar{\nabla})$. From (1.22) and Theorem 6.1.3 in [1] we have (observe that the centroaffine Weingarten operator is the identity):

Lemma 7.1. *Let $f\colon M \to \mathbf{R}^{n+1}$ be a centroaffine hyperovaloid, $n \geq 2$. Then, we have the following relations for asymptotic spectral invariants of D and $\bar{D}$:*

(1) $a_{2k+1}(D) = 0 = a_{2k+1}(\bar{D})$, *for $k \in \mathbf{N}$,*

(2) $a_0(D) = a_0(\bar{D}) = (4\pi)^{-\frac{n}{2}} \operatorname{vol}(M, h)$,
(3) $a_2(D) = a_2(\bar{D}) = (4\pi)^{-\frac{n}{2}} \int(\frac{1}{6}n(n-1)\hat{\kappa} + n - \frac{1}{4}n^2 h(T^\sharp, T^\sharp))\omega(h)$.

As a consequence we get relations between intrinsic invariants of (M, h) and the asymptotic spectrum.

Corollary 7.2. *Let f be a centroaffine hyperovaloid as above, and $n \geq 3$. Then we have the following inequality between global intrinsic invariants of (M, h) and invariants of the asymptotic spectrum:*
(1) *for $\tilde{D} = D$ or $\tilde{D} = \bar{D}$, we have*

$$(n-1)\int(\epsilon_M + \tfrac{1}{3}n(n-2)\hat{\kappa})\omega(h) \geq (n-2)(4\pi)^{\frac{n}{2}}(\tfrac{1}{2}(n^2-4n-1)a_0(\tilde{D}) + 2a_2(\tilde{D})),$$

(2) *equality in (1) holds if and only if f is an ellipsoid centered at the origin;*
(3) *both sides of the inequality in (1) are invariant under polarization, and in case of equality the polar hyperovaloids are centered ellipsoids.*

Proof. The inequality in (1) can be easily verified from Proposition 5.1 and Theorem 7.1. Equality yields if and only if equality yields in Proposition 5.1. Finally, the polarization statement follows from Lemma 6.1, as $\bar{h} = h$ and D and $\bar{D}$ are interchanged. □

8. Acknowledgements

The geometry groups at K.U. Leuven and T.U. Berlin participate in the GADGET program of the EU. The first author was supported by the DFG-Forschungsstipendium Scha 698/1-1. The first, the second and the fourth author were partially supported by the DFG-project Si 163/4-1,2. The second, the third and the fourth author were partially supported by the grant OT/TBA/95/9 of the Research Council of the K.U. Leuven. The forth author is a Senior Research Assistant of the National Fund for Scientific Research (Belgium).

References

[1] N. Bokan, P. B. Gilkey and U. Simon: *Applications of spectral geometry to affine and projective geometry.* Contributions to Algebra and Geometry **35** (1994), 283–314.

[2] B. Y. Chen: *Some pinching and classification theorems for minimal submanifolds.* Arch. Math. **60** (1993), 568–578.

[3] B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken: *Two equivariant totally real immersions into the nearly Kähler 6-sphere and their characterization.* Japan. J. Math. **21** (1995), 207–222.

[4] B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken: *Characterizing a class of totally real submanifolds of $S^6(1)$ by their sectional curvatures.* Tôhoku Math. J. **47** (1995), 185–198.

[5] B. Y Chen and L. Vrancken: *Minimal hypersurfaces in $S^{n+1}(1)$ and $H^{n+1}(-1)$.* Proc. Amer. Math.Soc. **123** (1995), 3851-3854.

[6] F. Dillen and S. Nölker: *Semi-parallelity, multi-rotation surfaces and the helix-property.* J. Reine Angew. Math. **435** (1993), 33–63.

[7] F. Dillen and L. Vrancken: *Calabi Type composition of affine spheres.* Diff. Geom. and its Appl. **4** (1994), 303–328.

[8] F. Dillen and L. Vrancken: *Totally real submanifolds in S^6 satisfying Chen's equality.* Trans. Amer. Math. Soc. **348** (1996), 1633–1646.

[9] N. Ejiri: *Totally real minimal immersions of n-dimensional real space forms into n-dimensional complex space forms.* Proc. Amer. Math. Soc. **84** (1982), 243–246.

[10] A. M. Li, U. Simon and G. Zhao: *Global affine differential geometry of hypersurfaces.* W. De Gruyter, Berlin-New York 1993.

[11] S. Nölker: *Isometric immersions of warped products.* Diff. Geom. and its Appl. **6** (1996), 1–30.

[12] V. Oliker and U. Simon: *Affine geometry and polar hypersurfaces.* In: Analysis and Geometry: Trends in Research and Teaching (B. Fuchssteiner and W. A. J. Luxemburg, eds.), BI-Mannheim-Zürich, 1992, 87–112.

[13] R. Schneider: *Zur affinen Differentialgeometrie im Großen I.* Math. Z. **101** (1967), 375–406.

[14] U. Simon, A Schwenk-Schellschmidt and H. Viesel: *Introduction to the affine differential geometry of hypersurfaces.* Science University of Tokyo, 1991.

[15] Ch. Scharlach and L. Vrancken: *A curvature invariant for centroaffine hypersurfaces, Part II.* Geometry and Topology of Submanifolds, VIII, ed. by F. Dillen *et al.*, World Scientific, Singapore, 1996, 341–350.

[16] L. Vrancken, A. M. Li and U. Simon: *Affine spheres with constant affine sectional curvature.* Math. Z. **206** (1991), 651–658.

Received May 13, 1996