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Spacefilling Knots

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Abstract. This note describes how to construct toroidal polyhedra which are homotopic to a given type of knot and which admit an isohedral tiling of 3-space.

1. Introduction

A *knotted* (toroidal) polyhedron (of a certain type) is a polyhedron which is homeomorphic to a closed torus and homotopic to a knot of this type. This note describes in detail the construction of knotted polyhedra which are spacefillers, i.e., which admit a tiling of 3-space. The type of the knot may be chosen arbitrarily, and the resulting tiling is both periodic and isohedral, i.e., the group of symmetries acts transitively on the tiles. This construction is a generalization of that used in [4] and [5]. It has been briefly indicated in [6].

These investigations were initiated by a lecture of Schulte [7] during which he suggested to search for a spacefilling knotted polyhedron. (However, the published version [8] does not mention knotted tiles.) Other constructions of spacefilling knots (independently, but also initiated by Schulte) are given by Kuperberg [3] and Adams [1]. Both authors describe knotted polyhedra which admit a lattice tiling of 3-space, without interlocking. Kuperberg's approach is simplified by Zaks [9].

2. An outline of the construction

Consider the graph in 3-space formed by the edges of a cube lattice. Obviously, it is possible to embed a knot of any desired type into this graph (Fig. 1 shows one of the possible realizations of the simplest case: a trefoil knot embedded into a $3 \times 3 \times 3$ cubic grid.) Apply all the translational symmetries of the lattice to obtain a family of congruent copies of embedded knots. This family covers the graph. Duplicate the family by translating the vertices to the centres of the cubes. Intuitively, it is clear that in three dimensions there

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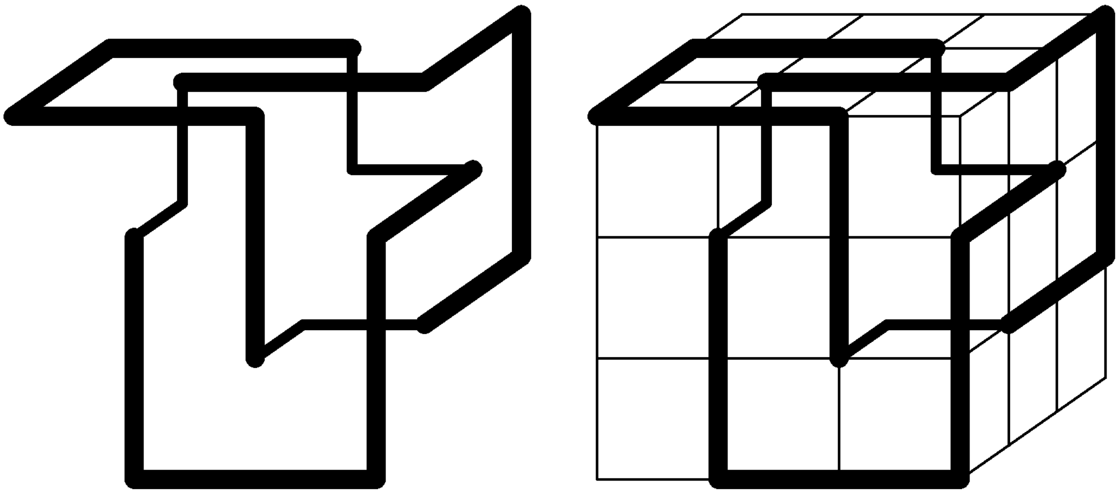

Fig. 1: A trefoil knot and its embedding into a cubic grid

is enough freedom to modify the embedded knot slightly to obtain a knot (homotopic and 'close' to the original one, but no longer contained in the edge graph) such that its copies by translation are pairwise disjoint. Finally, blow up the knot to a topological torus such that its congruent copies form a tiling of 3-space, while taking care that the homotopy class remains unchanged by this process.

The procedure just outlined can, of course, be realized in several ways. One specific construction is described in the next section. It is essentially straightforward, but the details are partially rather cumbersome to write down explicitly.

3. The construction in detail

Step 1: A partition of 3-space

Consider the tiling $\mathcal{T}$ of 3-space by unit cubes with centres at the elements of the integer lattice $\mathbb{L}$. For the purpose of the construction this tiling is split into two families, $\mathcal{T}_1$ and $\mathcal{T}_2$, of tiles according to the majority of even or odd integers among the coordinates of the centres. The unions S_1 and S_2 of these two families are congruent, have disjoint interiors, and (together) cover space. (More formally, $\mathcal{T} := \{C + p \mid p \in \mathbb{L}\}$, $\mathcal{T}_i := \{C + p \mid p \in \mathbb{L}_i\}$ and $S_i := \bigcup \mathcal{T}_i = C + \mathbb{L}_i$ where $C := [-1/2, 1/2]^3$, $\mathbb{L} = \mathbb{Z}^3$, and
$\mathbb{L}_1 := \{p \in \mathbb{L} \mid \text{at most one coordinate odd}\} \subset \mathbb{L}$ and
$\mathbb{L}_2 := \{p \in \mathbb{L} \mid \text{at most one coordinate even}\} = \mathbb{L}_1 + (1,1,1)$.
This partition of 3-space into two sets has also been used by Debrunner [2].)

Therefore, if S_1 can be tiled monohedrally by some prototile K, then $\mathbb{R}^3$ can be tiled by it, as well. Furthermore, note that S_1 and S_2 are homotopic to the 3-dimensional grid G (of lines parallel the axes)

$$\mathrm{G} := \{p \in \mathbb{R}^3 \mid \text{at least two coordinates are integers}\}$$

generated by the integer lattice or, equivalently, to $2\mathrm{G} \subset S_1$. Guided by the geometric structure of G, the set L_1 can be divided into two classes, the *vertices*

$$\mathbb{V} := \{p \in \mathbb{L} \mid \text{all coordinates even}\} = 2\mathbb{L} \subset \mathbb{L}_1$$

and the *edges* (represented by their midpoints)

$$\mathbb{E} := \{p \in \mathbb{L} \mid \text{two coordinates even and one odd}\} \subset \mathbb{L}_1$$

(with three subclasses $\mathbb{E}_x$, $\mathbb{E}_y$, and $\mathbb{E}_z$, according to the odd coordinate).

Step 2: Embedding of the knot

Every knot (or more precisely, every knot that has a piecewise linear representation) can be embedded into $2\mathrm{G} \subset S_1$ by a closed path P consisting of points of $\mathbb{L}_1$ and line segments (each parallel to an axis) between these points. Note that, among the lattice points along such a path, vertices and edges necessarily alternate.

This path P will be the basis for the construction of the desired knotted spacefiller K, a polyhedral torus homotopic to P. Consider all congruent copies $\mathrm{P} + p$ of P obtained by adding a point $p \in \mathbb{V} = 2\mathbb{L}$. Obviously, these translates cover G, they are, however, far from disjoint. It is therefore necessary to modify P to obtain an (equivalent) knot Q (homotopic to P, both in $\mathbb{R}^3$ and in S_1) for which the translates are pairwise disjoint.

The path (or knot) P can be described by the sequence $v_0, e_1, v_1, e_2, \ldots e_n, v_n = v_0$ of its vertices and edges. In order to facilitate the construction, set $i(x) = n$ and assume that $e_n = (-1, 0, 0)$, $v_n = v_0 = (0, 0, 0)$, $e_1 = (1, 0, 0)$, and that there are indices $i(y)$ and $i(z)$ such that both $e_{i(y)}$ and $e_{i(y)+1}$ belong to the class $\mathbb{E}_y$, and both $e_{i(z)}$ and $e_{i(z)+1}$ belong to the class $\mathbb{E}_z$. For example, an admissible representation of the trivial knot would be (omitting the edges):

$$\begin{aligned} v_0 = (0,0,0) - (2,0,0) - (2,2,0) - (2,4,0) - (2,4,2) - (2,4,4) - (0,4,4) - \\ - (-2,4,4) - (-2,2,4) - (-2,0,4) - (-2,0,2) - (-2,0,0) - (0,0,0) = v_{12} \end{aligned}$$

Step 3: Modification of the knot

Since the translations by elements of $\mathbb{V}$ act transitively on the vertices and on each of the classes $\mathbb{E}_x$, $\mathbb{E}_y$, $\mathbb{E}_z$ of edges, in order to construct Q it is sufficient to consider the four cubes C, C_x, C_y, and C_z (where $C_x = C + (1, 0, 0)$, etc.): If the knots $\mathrm{Q} - v_i$ $(i = 1, \ldots, n)$ do not intersect in these four cubes, then all the translates of Q are pairwise disjoint.

To each vertex v_i assign a point $p_i = (x_i, y_i, z_i) \in C$ such that (for $i = 1, \ldots, n$) no coordinate occurs twice. For technical reasons (whose purpose will become clear later) some additional assumptions on the p_i are convenient (and can obviously be satisfied): To $v_0 = v_n$ assign $p_0 = p_n = p_{i(x)} = (0, 0, 0)$ and choose $x_{i(y)}$, $z_{i(y)}$, $x_{i(z)}$, and $y_{i(z)}$ such that their absolute values are less than that of all the other coordinates. Furthermore, assume that p_i has been chosen such that (if applicable) the right angle determined by the two lines meeting in p_i is intersected by the coordinate axis which is orthogonal to the plane determined by them. (Applied to the sample path of Step 2 this means that, e.g., for p_1

the coordinate x_1 has to be chosen from the interval $(0, 1/2)$ and the coordinate y_1 has to be chosen from $(-1/2, 0)$.)

When constructing Q, the segment $C \cap (\mathrm{P} - v_i)$ — which consists of two line segments (parallel to two of the axes) meeting in (0,0,0) — is replaced by a corresponding segment $C \cap (\mathrm{Q} - v_i)$ which is formed by two line segments (parallel to the original ones, and having the same directions) meeting in p_i. Therefore, for every point of $C \cap (\mathrm{Q} - v_i)$ at least two coordinates coincide with those of p_i. By the choice of these coordinates (as pairwise distinct) this means that (for $1 \leq i \leq n$) these segments are pairwise disjoint.

For each edge $e_i \in \mathbb{E}_x$ either $e_i - v_{i-1} \in C_x$ or $e_i - v_i \in C_x$ (depending on the orientation). To obtain the segment $C_x \cap (\mathrm{Q} - v_{i-1})$, in the first case, the segment $C_x \cap (\mathrm{P} - v_{i-1})$ has to be replaced by some path leading from $(1/2, y_{i-1}, z_{i-1})$ to $(3/2, y_i, z_i)$. A possible choice is the following path obtained by joining five points by four straight line segments $(1/2, y_{i-1}, z_{i-1})$ – $(1 + x_i, y_{i-1}, z_{i-1})$ – $(1 + x_i, y_i, z_{i-1})$ – $(1 + x_i, y_i, z_i)$ – $(3/2, y_i, z_i)$. In the second case, the replacement $C_x \cap (\mathrm{Q} - v_i)$ of $C_x \cap (\mathrm{P} - v_i)$ is obtained similarly by joining the points $(1/2, y_i, z_i)$ – $(1 + x_i, y_i, z_i)$ – $(1 + x_i, y_{i-1}, z_i)$ – $(1 + x_i, y_{i-1}, z_{i-1})$ – $(3/2, y_{i-1}, z_{i-1})$. In most cases, these paths do not intersect by the choice of coordinates (just as in the case of vertices): For each point of the segment corresponding to e_i (in C_x) (at least) two of the coordinates coincide with those of either $p_i + (1, 0, 0)$ or $p_{i-1} + (1, 0, 0)$. Therefore, the segments corresponding to two edges are disjoint if they do not have a common endpoint (vertex). However, if a vertex v_i occurs twice then both e_i and e_{i+1} belong to $\mathbb{E}_x$ and, if these edges are positively oriented, the last part of the segment corresponding to e_i and the first part of the segment corresponding to e_{i+1} (or vice-versa, if they are negatively oriented) have the same x-y-coordinates and may intersect: The line segment $(1 + x_i, y_i, z_i)$ – $(3/2, y_i, z_i)$ (from $\mathrm{Q} - v_{i-1}$ corresponding to e_i) intersects the line segment $(1/2, y_i, z_i)$ – $(1 + x_{i+1}, y_i, z_i)$ (from $\mathrm{Q} - v_i$ corresponding to e_{i+1}) if $x_i < x_{i+1}$ (for positively oriented edges), and similarly, $(1/2, y_i, z_i)$ – $(1 + x_i, y_i, z_i)$ (from $Q - v_i$ corresponding to e_i) intersects $(1 + x_{i+1}, y_i, z_i)$ – $(3/2, y_i, z_i)$ (from $Q - v_{i+1}$ corresponding to e_{i+1}) if $x_i > x_{i+1}$ (for negatively oriented edges). But when constructing the segments corresponding to the edges in E_x the actual values of the x_i are not important — it is only used that they are distinct. (In fact, they could be replaced by arbitrarily chosen values.) Therefore, this exceptional case can easily be handled by exchanging x_i and x_{i+1}. Similarly, when three or more consecutive edges belong to E_x then, for the corresponding segments, the x_i can be permuted to produce (descending or ascending) order.

For a unified notation, write $(1/2, y_l(i), z_l(i))$ – $(x(i), y_l(i), z_l(i))$ – $(x(i), y_r(i), z_l(i))$ – $(x(i), y_r(i), z_r(i))$ – $(3/2, y_r(i), z_r(i))$ to denote the path segment $\mathrm{Q} - v(i)$ (where $v(i) = v_i$ or $v(i) = v_{i-1}$) corresponding to $e_i \in \mathbb{E}_x$.

(The edges in $\mathbb{E}_y$ and $\mathbb{E}_z$ are, of course, treated similarly.)

Step 4: Thickening of the knot

For the final step divide the cubes C, C_x, C_y, and C_z into sufficiently small cubes, i.e., cubes with a side length less than half the minimal distance occurring between the coordinates x_i, y_i, z_i, $-1/2$, and $1/2$. This guarantees that each small cube meets at most one of the knot segments $Q - v_i$.

Now construct $K_0 - v_i$ by taking the union of all small cubes that meet $\mathrm{Q} - v_i$. The result is a polyhedral torus K_0 which is homotopic to Q and P and whose translates (by the elements of $\mathbb{V}$) still are pairwise disjoint. 'Blowing up' (or thickening) K_0 so that the resulting body and its translates tile S_1 will complete the construction and provide a knotted polyhedron K as desired. This task can be achieved as follows:

Step 4a: Edge segments

Consider first C_x. If $(1/2, y_l(i), z_l(i))$ belongs to a segment $Q - v(i)$ then add to $K_0 - v(i)$ all cubes meeting the line segment from $(1/2, y_l(i), z_l(i))$ to $(1/2, y_l(i), \pm 1/2)$ (the sign depending on the sign of $z_l(i)$). Then add all the remaining cubes meeting the face $x = 1/2$ of $\mathbb{E}_x$ — a 'two-sided comb' extending from (and including) the line $(1/2, -1/2, z_{i(x)}) - (1/2, 1/2, z_{i(x)})$ — to $\mathrm{Q} - v_{i(x)}$. This works since the coordinates $y_l(i)$ (taken from the y_i) are all different, and the coordinates $z_l(i)$ (from the z_i), for $i \neq i(x)$, all have absolute values greater than $z_{i(x)}$. Proceed analogously with the opposite face ($x = 3/2$).

These two face layers (and analogously constructed ones in C_y and C_z) serve to hide the interior of C_x (or C_y and C_z, respectively). By their construction, and because of the assumptions on the coordinates of p_i, the segments (of K_0), corresponding to two edges of Q which meet orthogonally in v_i, are extended only outwards or sidewards, but not inwards, and therefore they will not touch when the polyhedron K is put together.

Next consider the values $x(i)$ (for edges in $\mathbb{E}_x$), i.e., those values of the x-coordinate where, with the help of two orthogonal line segments, a change from one line parallel to the x-axis to another one takes place. For each $e_j \in \mathbb{E}_x$ $(j \neq i)$ the path segment of $\mathrm{Q} - v(j)$ passes either through $(x(i), y_l(j), z_l(j))$ or $(x(i), y_r(j), z_r(j))$. Accordingly, add to $K_0 - v(j)$ all cubes meeting the line segment $(x(i), y_l(j), z_l(j)) - (x(i), y_l(j), \pm 1/2)$ (or the line segment $(x(i), y_r(j), z_r(j)) - (x(i), y_r(j), \pm 1/2)$, respectively) where the sign is chosen such that this line does not meet the two orthogonal line segments (of $\mathrm{Q} - v(i)$ at $x = x(i)$). Add all remaining cubes meeting the square cross section determined (in $\mathbb{E}_x$) by $x = x(i)$ (a simply connected set) to $K_0 - v(i)$.

So far, from the small cubes of C_x two layers at two faces and some layers (less than n) between them have been assigned to segments $K_0 - v(i)$. In the sections between these layers $\mathrm{Q} - v(i)$ (if in C_x) is a line segment parallel to the x-axis (and therefore $K_0 - v(i)$ is a prism parallel to this axis). In each section, extend these finitely many prisms (less than n, but at least one because of $e_{i(x)}$) to layers (parallel to the x-z-plane and of appropriately chosen width) which fill that section. (Since all the coordinates y_i are different, no conflict occurs.)

The resulting segments $K - v(i)$ all are simply connected and form a tiling of C_x. Proceed analogously for C_y (extension again parallel to the z-axis) and C_z (extension parallel to the y-axis).

Step 4b: Vertex segments

Now consider C and use a construction similar to the previous one: The segment of $\mathrm{Q} - v_0 = \mathrm{Q}$ (corresponding to v_0) passing through C coincides with the x-axis since $p_0 = (0, 0, 0)$. If a segment of $\mathrm{Q} - v_i$ intersects the x-y-plane in $(x_i, y_i, 0)$ then add to $K_0 - v_i$ all cubes meeting the line segment from $(x_i, y_i, 0)$ to $(x_i, \pm 1/2, 0)$ where the sign of $1/2$ equals the sign of y_i. (Again, no conflict occurs, since the x_i are all different.) Then add all remaining cubes meeting the square with $z = 0$ to $K_0 - v_0$. In other words: Add to K_0 prisms parallel to the y-axis (in both directions) either until a face of C is reached, or until another segment $K_0 - v_i$ is met. In the latter case, add cubes to this segment (in the same way), outwards and parallel to the y-axis, until the face of C is reached. The result of this procedure is a square layer (at $z = 0$) of small cubes, where each cube is assigned to one of the segments $K_0 - v_i$. Now, similarly, add more cubes to the extended segments $K_0 - v_i$: To every cube of the square layer just constructed add cubes to form prisms parallel to the z-axis either until a face is reached, or until a cube of another segment $K_0 - v_j$ is met — in the latter case add the remaining cubes to this segment.

Since any right angle formed by a segment contains a coordinate axis these angles are enlarged only outwards or sidewards, but never inwards — therefore the construction does not create loops, and the result is a tiling of C by simply connected polyhedra $K - v_i$. Moreover, for the same reason no loops will occur (at an edge cube) when, finally, the segments are glued together to form the polyhedron K.

The result

In the end, each one of the small cubes in $C \cup C_x \cup C_y \cup C_z$ is assigned to one of the segments $K_0 - v_i$ to form a segment $K - v_i$, therefore the n segments $K - v_i$ form a tiling of this set, and

$$K = \bigcup_i \big((K - v_i) + v_i\big) \supset \mathrm{Q}$$

is a knotted prototile with the desired properties: Because of its construction this polyhedron belongs to the same homotopy class as Q (and P), and the family of its translates (by the elements of $\mathbb{V}$) forms a tiling of S_1.

4. Remarks.

(a) *(Step 1)* The construction does not depend on P being a knot. It may as well start from an 1-dimensional manifold of higher genus, and may even posess more than one connected component (e.g., it may be a link). In this way, by starting from P and some (disjoint) congruent copies of P, some additional symmetries for the resulting tiling may be obtained.

(b) *(Step 2)* Both the choice of the embedded path P and the type of modifications leading to Q influence how the copies of K are linked in the final tiling.

(c) *(Step 1)* Instead of S_1 and S_2 the construction can also be based on the 'linked double-layers' used in [3], or some similar divisions of 3-space.

5. References.

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