Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 38 (1997), No. 2, 261-288.

The Projective Interpretation of the Eight 3-dimensional Homogeneous Geometries

Dedicated to the memory of Prof. Julius Strommer

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Let $(\mathbf{X}, \mathbf{G})$ be a 3-dimensional homogeneous geometry, where $\mathbf{X}$ is a simply connected Riemannian space with a maximal group $\mathbf{G}$ of isometries, acting transitively on $\mathbf{X}$ with compact point stabilizers. $\mathbf{G}$ is maximal means that no proper extension of $\mathbf{G}$ can act on the Riemannian space $\mathbf{X}$ in such a way. We recall the

Theorem. (Thurston) *Any 3-dimensional homogeneous geometry* $(\mathbf{X}, \mathbf{G})$ *above that admits a compact quotient is equivalent (equivariant) to one of the geometries* $(\mathbf{X}, \mathbf{G} = \operatorname{Isom}\mathbf{X})$ *where the space* $\mathbf{X}$ *is one of* $\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^3$, $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{H}^2\times\mathbf{R}$, $\widetilde{\mathbf{SL}_2\mathbf{R}}$, **Nil** *or* **Sol** (as described e.g. in [6, Th. 5.1., 474–476], in [7, p.358] or in [1, 310–318]).

Our purpose is to model these geometries $(\mathbf{X}, \mathbf{G})$ on the real projective sphere $\mathcal{PS}^3(\mathbf{R}) \supset \mathbf{X}$, or, especially, in the projective and affine space $\mathcal{P}^3(\mathbf{R})$, $\mathcal{A}^3(\mathbf{R})$, respectively, by special groups $\mathbf{G}$ of collineations. So we get a unified method to attack Thurston's geometrization conjecture.

Thurston guesses that each 3-dimensional compact orbifold (manifold) can be splitted (if occur) along 2-dimensional spherical ($\mathbf{S}^2$-) orbifolds (manifolds) or Euclidean ($\mathbf{E}^2$-) orbifolds (manifolds) in a unique way (up to isotopy) so that each piece (after some closing up procedure!) admits a complete, finite volume, geometric structure $\mathbf{X}/\Gamma$. Here $(\mathbf{X}, \mathbf{G})$ is one of the 8 geometries above, and Γ is a discrete subgroup of $\mathbf{G} = \operatorname{Isom}\mathbf{X}$.

I have been intending to disprove this conjecture by constructing counter examples: orbifolds and manifolds which can not wear any of the 8 metrics above. This intention has not been succeeded yet but the machinery developed here will be exploited, may be, in a future proof of the Thurston conjecture (?), (see [3], [4], [5] for further information). The point of our method is that we can‘easily recognize whether a transformation group Γ as a

* Work supported by the Hung. Nat. Found. for Sci. Research (OTKA) No. T 7351 (1993).

0138-4821/93 $ 2.50

projective collineation group (a special matrix group) leaves invariant a projective polarity and what signature of it. So we easily overview the candidates of possible geometries $\mathbf{X}$ where our group Γ can act as a subgroup in the isometry group $\mathbf{G} = \operatorname{Isom}\mathbf{X}$ of space $\mathbf{X}$. If not, we face to difficulties of various kind, left to investigate yet. In Section 10 we indicate some phenomena by presenting the Zhuk's Family 30 of simplex tilings from [4].

As a summary we give the following

TABLE

Space $\mathbf{X}$	Signature of polarity $\Pi(_\star)$ or scalar product $\langle\ ,\ \rangle$ in $\boldsymbol{V}_4$	Domain of proper points of $\mathbf{X}$ in $\mathcal{PS}^3$ ($\mathbf{V}^4(\mathbf{R})$, $\boldsymbol{V}_4$)	The group $G = \operatorname{Isom}\mathbf{X}$ as a special collineation group of $\mathcal{PS}^3$
$\mathbf{S}^3$	$(+\ +\ ++)$	$\mathcal{PS}^3$	Coll $\mathcal{PS}^3$ preserving $\Pi(_\star)$
$\mathbf{H}^3$	$(-\ +\ ++)$	$\{(\mathbf{x}) \in \mathcal{P}^3\ :\ \langle\mathbf{x},\mathbf{x}\rangle < 0\}$	Coll $\mathcal{P}^3$ preserving $\Pi(_\star)$
$\widetilde{\mathbf{SL}_2\mathbf{R}}$	$(-\ -\ ++)$ with skew line fibering	Universal covering of $\mathcal{H} :=$ $:= \{[\mathbf{x}] \in \mathcal{PS}^3\ :\ \langle\mathbf{x},\mathbf{x}\rangle < 0\}$ by fibering transformations	Coll $\mathcal{PS}^3$ preserving $\Pi(_\star)$ and fibres. See (6.17)
$\mathbf{E}^3$	$(0\ +\ ++)$	$\mathcal{A}^3 = \mathcal{P}^3 \setminus \{\omega^\infty\}$ where $\omega^\infty := (\boldsymbol{b}^0)$, $\boldsymbol{b}^0_\star = \mathbf{0}$	Coll $\mathcal{P}^3$ preserving $\Pi(_\star)$, generated by plane reflections
$\mathbf{S}^2\!\times\!\mathbf{R}$	$(0\ +\ ++)$ with O-line bundle fibering	$\mathcal{A}^3 \setminus \{O\}$ O is a fixed origin	G is generated by plane reflections and sphere inversions, leaving invariant the O-concentric 2-spheres of $\Pi(_\star)$
$\mathbf{H}^2\!\times\!\mathbf{R}$	$(0\ -\ ++)$ with O-line bundle fibering	$\mathcal{C}^+ = \{X \in \mathcal{A}^3\ :$ $\langle\overrightarrow{OX},\overrightarrow{OX}\rangle < 0$, half cone$\}$ See Sect. 8, formula (8.2)	G is generated by plane reflections and hyperboloid inversions, leaving invariant the O-concentric half-hyperboloids in the half-cone $\mathcal{C}^+$ by $\Pi(_\star)$
Sol	$(0\ -\ ++)$ with parallel plane fibering	$\mathcal{A}^3$ with parallel plane fibering	See Sect.8, formulas (8.8–9)
Nil	$(0\,0\,0\,+)$ with parallel line bundle fibering	$\mathcal{A}^3$ with a distinguished parallel plane pencil along each line	See Sect. 9, formulas (9.3–4) and (9.7)

Our treatment for $\widetilde{\mathbf{SL}_2\mathbf{R}}$ in Section 6 has some novelties in comparison with [1], [6], [7]. There we will give a new interpretation of Seifert fibre bundle in $\widetilde{\mathbf{SL}_2\mathbf{R}}$, and 4 different expressions of the Riemannian metric $(ds)^2$ in global form. We intend to publish also conforme (quadrics) interpretations for the geometries above, in order to complete our algorithmic procedure indicated in Section 10 (see Remark in Sect. 8).

1. The projective 3-sphere and 3-space

$\mathcal{PS}^3(\mathbf{R})$ (resp. $\mathcal{P}^3(\mathbf{R})$) can be modelled on the real 4-dimensional vector space $\mathbf{V}^4(\mathbf{R})$ and on its dual $\boldsymbol{V}_4(\mathbf{R})$. For $\mathbf{x}, \mathbf{y} \in \mathbf{V}^4 \setminus \{\mathbf{0}\}$

$$\mathbf{x} \overset{+}{\sim} \mathbf{y} \ \text{ stands iff there is a } c \text{ with } 0 < c \in \mathbf{R} \text{ such that } \mathbf{y} = c\mathbf{x}.$$

The *equivalence class* $[\mathbf{x}] = [\mathbf{y}]$ is a *point* of the projective 3-sphere $\mathcal{PS}^3$ (resp. $\mathbf{x} \sim \mathbf{y}$ iff $\mathbf{y} = c\mathbf{x}$ with $0 \neq c \in \mathbf{R}$ and $(\mathbf{x}) = (\mathbf{y})$ is a *point* of the projective 3-space $\mathcal{P}^3$). An *oriented plane* or its *halfspace* $[\boldsymbol{u}]$ in $\mathcal{PS}^3$ (resp. a *plane* $(\boldsymbol{u})$ in $\mathcal{P}^3$) is defined analogously by a non-zero form $\boldsymbol{0} \neq \boldsymbol{u} \in \boldsymbol{V}_4$ of the dual space $\boldsymbol{V}_4$.

Then the *incidence* of a point $(\mathbf{x})$ and the plane $(\boldsymbol{u})$, i.e. $(\mathbf{x})\,\bar{\underline{|}}\,(\boldsymbol{u})$ is defined by $\mathbf{x}\boldsymbol{u} = 0$ in $\mathcal{P}^3$ (and similarly in $\mathcal{PS}^3$). The point $[\mathbf{x}]$ lies in the open halfspace $[\boldsymbol{u}]$ in $\mathcal{PS}^3$ iff $\mathbf{x}\boldsymbol{u} > 0$.

A basis $\{\mathbf{e}_i\}$ $i = 0,1,2,3$ in $\mathbf{V}^4$ and its dual basis $\{\boldsymbol{e}^j\}$ $j = 0,1,2,3$ in $\boldsymbol{V}_4$ with $\mathbf{e} := \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ define a coordinate simplex

$$E_0[\mathbf{e}_0],\ \ E_1[\mathbf{e}_1],\ \ E_2[\mathbf{e}_2],\ \ E_3[\mathbf{e}_3]\ \text{ with a unit point } E[\mathbf{e}].$$

The duality relation $\mathbf{e}_i\boldsymbol{e}^j = \delta_i^j$ (the Kronecker symbol) implies that e.g. $E_0, E_1, E_2\,\bar{\underline{|}}\,\mathcal{E}^3(\boldsymbol{e}^3)$, and E_3 lies in the open halfspace ${}^+\mathcal{E}^3 = [\boldsymbol{e}^3]$. The normalizing unit point $E[\mathbf{e}]$ and all the closed coordinate simplex

$$\left\{[\mathbf{x}]\ :\ \mathbf{x} := \sum_{i=0}^{3} x^i\mathbf{e}_i =: x^i\mathbf{e}_i \ \text{ with } \ \sum_{i=0}^{3} x^i = 1 \ \text{ and } \ 0 \leq x^i,\ i \in \{0,1,2,3\}\right\} \tag{1.1}$$

lie in the halfspace ${}^+\mathcal{E} = [\boldsymbol{e} = \boldsymbol{e}^0 + \boldsymbol{e}^1 + \boldsymbol{e}^2 + \boldsymbol{e}^3]$. The form $\boldsymbol{u} := \boldsymbol{e}^j u_j$ takes the value $x^i u_i \in \mathbf{R}$ on the vector $\mathbf{x} := x^i\mathbf{e}_i$. (The Einstein summation convention will be applied). A *basis transformation* (with Schouten's primed index convention)

$$\mathbf{e}_{i'} = e^i_{i'}\mathbf{e}_i \ \text{ where } \ \det(e^i_{i'}) \neq 0 \tag{1.2}$$

involves the *dual basis transformation*

$$\boldsymbol{e}^{j'} = \boldsymbol{e}^j E_j^{j'} \ \text{ with } \ \delta_{i'}^{j'} = \mathbf{e}_{i'}\boldsymbol{e}^{j'} = (e^i_{i'}\mathbf{e}_i)(\boldsymbol{e}^j E_j^{j'}) = e^i_{i'}\delta_i^j E_j^{j'} = e^i_{i'}E_i^{j'}. \tag{1.3}$$

Then the *coordinate transformation of a point* $X(\mathbf{x})$ is

$$\mathbf{x} = x^{i'}\mathbf{e}_{i'} = x^i\mathbf{e}_i = x^i E_i^{i'}\mathbf{e}_{i'} \quad \Longrightarrow \quad x^{i'} = x^i E_i^{i'}; \tag{1.4}$$

the *coordinate transformation of a plane* $\omega = [\boldsymbol{u}]$ is

$$(1.5) \qquad \boldsymbol{u} = \boldsymbol{e}^{j'} u_{j'} = e^j u_j = \boldsymbol{e}^{j'} e^j_{j'} u_j \quad \Longrightarrow \quad u_{j'} = e^j_{j'} u_j,$$

and vice versa, of course, by the matrix $(e^i_{i'})$ and by its inverse $(E^{i'}_i)$ — with capital E for the better distinction.

2. Polarity and scalar product

A *polarity* in $\mathcal{PS}^3$ (and later in $\mathcal{P}^3$, respectively)

$$(2.1) \qquad (_\star) : [\boldsymbol{u}] \mapsto [\boldsymbol{u}_\star] =: [\mathbf{u}], \quad \text{i.e. the } pole \text{ of } [\boldsymbol{u}] \text{ is } [\mathbf{u}],$$

will be defined by a symmetric linear mapping

$$(2.2) \qquad (_\star) : \boldsymbol{V}_4 \to \mathbf{V}^4, \quad \boldsymbol{u} \mapsto \boldsymbol{u}_\star =: \mathbf{u} \ \text{ so that } \ \boldsymbol{u}_\star \boldsymbol{v} = \boldsymbol{v}_\star \boldsymbol{u} =: \langle \boldsymbol{u}, \boldsymbol{v} \rangle \in \mathbf{R}$$

holds for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_4$. With respect to a dual basis pair $\{\mathbf{e}_i\}$ in $\mathbf{V}^4$ resp. $\{\boldsymbol{e}^j\}$ in $\boldsymbol{V}_4$, a polarity will be expressed by

$$(2.3) \qquad \begin{array}{c} (_\star) : \boldsymbol{e}^j \mapsto \boldsymbol{e}^j_\star := \mathbf{e}^j := \Pi^{ji}\mathbf{e}_i \ \text{ with } \ \Pi^{ji} = \Pi^{ij} \\ \text{so that } \ \langle \boldsymbol{e}^j, \boldsymbol{e}^k \rangle = (\Pi^{ji}\mathbf{e}_i)\boldsymbol{e}^k = \Pi^{ji}\delta^k_i = \Pi^{jk}. \end{array}$$

In the dual basis pair $\boldsymbol{e}^{j'} = \boldsymbol{e}^j E^{j'}_j \quad \mathbf{e}_{i'} = e^i_{i'}\mathbf{e}_i$

$$(2.4) \qquad \begin{array}{c} (_\star) : \boldsymbol{e}^{j'} := \boldsymbol{e}^j E^{j'}_j \mapsto E^{j'}_j (\Pi^{ji}\mathbf{e}_i) = E^{j'}_j \Pi^{ji} E^{i'}_i \mathbf{e}_{i'} =: \Pi^{j'i'}\mathbf{e}_{i'}, \\ \text{i.e. } \ \Pi^{j'i'} = E^{j'}_j \Pi^{ji} E^{i'}_i \end{array}$$

is the formula of *coordinate transformation of a polarity.* From this we can read the properties of the *scalar product* defined in (2.2) by

$$(2.5) \qquad \langle \ , \ \rangle : \boldsymbol{V}_4 \times \boldsymbol{V}_4 \to \mathbf{R}, \quad \langle \boldsymbol{u}, \boldsymbol{v} \rangle := \boldsymbol{u}_\star \boldsymbol{v}.$$

We can define and determine the *eigen values* $\{{}^{j'}\lambda\}$ and the corresponding basis $\{\boldsymbol{e}^{j'}\}$ of *eigen forms* by $\boldsymbol{e}^{j'} \mapsto \Pi^{j'i'}\mathbf{e}_{i'} := {}^{j'}\lambda \mathbf{e}_{j'}$ (no summation here!), where the matrix

$$(2.6) \qquad (\Pi^{j'i'}) = \begin{pmatrix} {}^{0'}\lambda & & & \\ & {}^{1'}\lambda & & \\ & & {}^{2'}\lambda & \\ & & & {}^{3'}\lambda \end{pmatrix}$$

has a diagonal form. In an appropriate dual basis pair each polarity has a matrix form

$$(2.7)\qquad \begin{pmatrix} 0 &&&&&&&& \\ & \ddots &&&&&&& \\ && 0 &&&&&& \\ &&& -1 &&&&& \\ &&&& \ddots &&&& \\ &&&&& -1 &&& \\ &&&&&& +1 && \\ &&&&&&& \ddots & \\ &&&&&&&& +1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \ddots\\ 0\end{matrix}}\right\} p \\ \left.\vphantom{\begin{matrix}-1\\ \ddots\\ -1\end{matrix}}\right\} q\,; \\ \left.\vphantom{\begin{matrix}+1\\ \ddots\\ +1\end{matrix}}\right\} r \end{matrix}$$

and here the triple $(p;q;r)$ defines the *signature* of polarity $({}_\star)$, or of the scalar product $\langle\ ,\ \rangle$.

The equality

$$(2.8)\qquad 0 = \boldsymbol{u}_\star \boldsymbol{v} = \boldsymbol{v}_\star \boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{v}\rangle,$$

i.e. $(\mathbf{u})\,\overline{\underline{|}}\,(\boldsymbol{v})$ and $(\mathbf{v})\,\overline{\underline{|}}\,(\boldsymbol{u})$, means that the planes $(\boldsymbol{u})$, $(\boldsymbol{v})$ — and also the halfspaces $[\boldsymbol{u}]$ and $[\boldsymbol{v}]$ — are *orthogonal* each to the other with respect to $({}_\star)$ or to $\langle\ ,\ \rangle$.

If the polarity $({}_\star)$ is regular, i.e. $p = 0$ in the signature, then the inverse polarity

$$(2.9)\qquad (^\star)\ :\ [\mathbf{x}] \mapsto [\mathbf{x}^\star] =: [\boldsymbol{x}], \quad \text{i.e. } [\boldsymbol{x}] \text{ is the polar to } [\mathbf{x}]$$

can be defined by

$$(2.10)\qquad (^\star)\ :\ \mathbf{e}_i \mapsto \mathbf{e}_i^\star := \boldsymbol{e}_i = \boldsymbol{e}^j \pi_{ji} \quad \text{with} \quad \pi_{ji} \varPi^{ik} = \delta_j^k.$$

Then the (symmetric) scalar product (with the same symbol)

$$(2.11)\qquad \langle\ ,\ \rangle\ :\ \mathbf{V}^4 \times \mathbf{V}^4 \to \mathbf{R}$$

has also been defined by

$$(2.12)\qquad \langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}\boldsymbol{y} = \mathbf{y}\boldsymbol{x} = \langle \boldsymbol{x}, \boldsymbol{y}\rangle$$

in an obvious manner, moreover, *contra-* and *covariant coordinates of points and planes* (halfspaces) can be introduced as

$$(2.13)\qquad x^i \mathbf{e}_i \overset{\star}{\mapsto} \boldsymbol{e}^j \pi_{ji} x^i =: \boldsymbol{e}^j x_j \underset{\star}{\mapsto} x_j \varPi^{ji} \mathbf{e}_i = x^i \mathbf{e}_i$$

show.

For the projective sphere $\mathcal{PS}^3$ (resp. the projective space $\mathcal{P}^3$) any polarity by a matrix $(\varPi^{ij})$ is the same as that by a matrix $(\varPi^{ij} \cdot c)$ with any constant $0 < c \in \mathbf{R}$ (resp. $c \in \mathbf{R} \setminus \{0\}$). Thus, the signatures $(p;q;r)$ and $(p;r;q)$ lead to equivalent projective polarities.

3. Collineations

A *collineation* α of the projective sphere $\mathcal{PS}^3$ (and later of $\mathcal{P}^3$, respectively)

$$\alpha\,:\,[\mathbf{x}]\mapsto[\mathbf{y}],\quad [\boldsymbol{u}]\mapsto[\boldsymbol{v}] \tag{3.1}$$

will be defined by a pair of regular linear transformations

$$\begin{gathered}\hat{\boldsymbol{\alpha}}\,:\,\mathbf{V}^4\to\mathbf{V}^4,\ \mathbf{x}\mapsto\mathbf{y}:=\mathbf{x}\hat{\boldsymbol{\alpha}};\quad \boldsymbol{\alpha}\,:\,\boldsymbol{V}_4\to\boldsymbol{V}_4,\ \boldsymbol{u}\mapsto\boldsymbol{v}:=\boldsymbol{\alpha}\boldsymbol{u};\\ \text{so that}\quad \mathbf{x}\boldsymbol{u}=(\mathbf{x}\hat{\boldsymbol{\alpha}})(\boldsymbol{\alpha}\boldsymbol{u})=:\mathbf{y}\boldsymbol{v}\end{gathered} \tag{3.2}$$

holds for any $\mathbf{x}\in\mathbf{V}^4$ and $\boldsymbol{u}\in\boldsymbol{V}_4$. With respect to a dual basis pair $\{\mathbf{e}_i\}$ in $\mathbf{V}^4$ and $\{\boldsymbol{e}^j\}$ in $\boldsymbol{V}_4$, let the matrix (a_i^k) be defined by

$$\mathbf{e}_i\mapsto\mathbf{e}_i\hat{\boldsymbol{\alpha}}:=\mathbf{a}_i:=a_i^k\mathbf{e}_k. \tag{3.3}$$

This determines the matrix (A_l^j), defined by

$$\boldsymbol{e}^j\mapsto\boldsymbol{\alpha}\boldsymbol{e}^j:=\boldsymbol{a}^j:=\boldsymbol{e}^lA_l^j, \tag{3.4}$$

since we have required

$$\delta_i^j=\mathbf{e}_i\boldsymbol{e}^j=(\mathbf{e}_i\hat{\boldsymbol{\alpha}})(\boldsymbol{\alpha}\boldsymbol{e}^j)=(a_i^k\mathbf{e}_k)(\boldsymbol{e}^lA_l^j)=a_i^k\delta_k^lA_l^j=a_i^kA_k^j. \tag{3.5}$$

With respect to another dual basis pair $\{\mathbf{e}_{i'}\}$ in $\mathbf{V}^4$ and $\{\boldsymbol{e}^{j'}\}$ in $\boldsymbol{V}_4$ by $\mathbf{e}_{i'}=e_{i'}^i\mathbf{e}_i$, $\boldsymbol{e}^{j'}=\boldsymbol{e}^jE_j^{j'}$ with $e_{i'}^iE_i^{j'}=\delta_{i'}^{j'}$, the matrix $(a_{i'}^{k'})$, defined by $\mathbf{e}_{i'}\mapsto\mathbf{e}_{i'}\hat{\boldsymbol{\alpha}}:=a_{i'}^{k'}\mathbf{e}_{k'}$, will be expressed by

$$\mathbf{e}_{i'}\mapsto(e_{i'}^i\mathbf{e}_i)\hat{\boldsymbol{\alpha}}=e_{i'}^ia_i^k\mathbf{e}_k=e_{i'}^ia_i^kE_k^{k'}\mathbf{e}_{k'},\quad\text{i.e.}\quad a_{i'}^{k'}=e_{i'}^ia_i^kE_k^{k'}. \tag{3.6}$$

Then the analogous matrix $(A_{l'}^{j'})$, defined by $\boldsymbol{e}^{j'}\mapsto\boldsymbol{e}^{l'}A_{l'}^{j'}$, will be

$$A_{l'}^{j'}=e_{l'}^lA_l^jE_j^{j'} \tag{3.7}$$

according to the requirement $a_{i'}^{k'}A_{k'}^{j'}=\delta_{i'}^{j'}$. If

$$\alpha\left(\hat{\boldsymbol{\alpha}}(a_i^j);\ \boldsymbol{\alpha}(A_s^r)\right)\quad\text{and}\quad\beta\left(\hat{\boldsymbol{\beta}}(b_j^k);\ \boldsymbol{\beta}(B_t^s)\right)$$

are two collineations, then their product is expressible as the formula

$$\alpha\beta\left(\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}(a_i^jb_j^k);\ \boldsymbol{\beta}\boldsymbol{\alpha}(B_t^sA_s^r)\right) \tag{3.8}$$

shows with matrices according to the above composition rules. The transformation $\alpha(\hat{\boldsymbol{\alpha}}(a_i^j);\ \boldsymbol{\alpha}(A_r^s))$ of $\mathcal{PS}^3$ (resp. of $\mathcal{P}^3$) can equivalently be expressed as

$$\alpha\left(\hat{\boldsymbol{\alpha}}c\,(a_i^j\cdot c);\ \frac{1}{c}\boldsymbol{\alpha}\left(\frac{1}{c}\cdot A_r^s\right)\right) \tag{3.9}$$

with any constant $0 < c \in \mathbf{R}$ in the case of $\mathcal{PS}^3$ (resp. $c \in \mathbf{R} \setminus \{0\}$ for $\mathcal{P}^3$).

An important example is a *central axial collineation* $\sigma(\alpha, C)$ with a pointwise fixed axis plane $\alpha[\boldsymbol{a} = \boldsymbol{e}^j a_j]$, with the planewise fixed centre $C[\mathbf{c} = c^i \mathbf{e}_i]$ and with the characteristic form value defined by $k(\mathbf{c}\boldsymbol{a}) \in \mathbf{R}$. $\sigma(\alpha; C)$ is smoothly introduced for any point of $\mathcal{PS}^3$ and $\mathcal{P}^3$, respectively, as follows

$$\hat{\boldsymbol{\sigma}}(\alpha, C) \,:\, \mathbf{x} \mapsto \mathbf{x}\hat{\boldsymbol{\sigma}} =: \mathbf{y} = \mathbf{x} - k(\mathbf{x}\boldsymbol{a})\mathbf{c} \tag{3.10}$$

where for the constant $k \neq 0$, $1 - k(\mathbf{c}\boldsymbol{a}) \neq 0$ are assumed. For the planes this will be

$$\boldsymbol{\sigma} \,:\, \boldsymbol{u} \mapsto \boldsymbol{\sigma u} =: \boldsymbol{v} = \boldsymbol{u} - \boldsymbol{a}(\mathbf{c}\boldsymbol{u})K \tag{3.11}$$

with the following requirement for the constant $K \in \mathbf{R}$

$$\begin{aligned} \mathbf{x}\boldsymbol{u} = \mathbf{y}\boldsymbol{v}, \quad \text{i.e.} \quad \mathbf{x}\boldsymbol{u} &= [\mathbf{x} - k(\mathbf{x}\boldsymbol{a})\mathbf{c}]\,[\boldsymbol{u} - \boldsymbol{a}(\mathbf{c}\boldsymbol{u})K] = \\ &= \mathbf{x}\boldsymbol{u} - k(\mathbf{x}\boldsymbol{a})(\mathbf{c}\boldsymbol{u}) - (\mathbf{x}\boldsymbol{a})(\mathbf{c}\boldsymbol{u})K + k(\mathbf{x}\boldsymbol{a})(\mathbf{c}\boldsymbol{a})(\mathbf{c}\boldsymbol{u})K \Leftrightarrow \\ &\Leftrightarrow (\mathbf{x}\boldsymbol{a})(\mathbf{c}\boldsymbol{u})\,[k(\mathbf{c}\boldsymbol{a})K - k - K] = 0 \quad \text{for any } \mathbf{x} \in \mathbf{V}^4 \text{ and } \boldsymbol{u} \in \boldsymbol{V}_4, \\ &\text{i.e.} \quad (\mathbf{c}\boldsymbol{a}) = \frac{1}{K} + \frac{1}{k} \quad \text{or} \quad K = \frac{k}{k(\mathbf{c}\boldsymbol{a}) - 1}. \end{aligned} \tag{3.12}$$

In particular, $\sigma(\alpha; C)$ will be involutive iff $k(\mathbf{c}\boldsymbol{a}) = 2$ and $K = k$. Then $\sigma(\alpha; C)$ is called *reflection in the axis plane* $\alpha[\boldsymbol{a}]$ (and *in the centre* $C(\mathbf{c})$ in $\mathcal{P}^3$). Note that the pairs $(\mathbf{c}; \boldsymbol{a})$ and $(\lambda\mathbf{c}; \boldsymbol{a}\frac{1}{\lambda})$ define the same central axial collineation in $\mathcal{P}^3$ if $\lambda \in \mathbf{R} \setminus \{0\}$, and a reflection can always be achived by $(\mathbf{c}\boldsymbol{a}) = 2$, $k = K = 1$.

4. Projective metric 3-sphere and 3-space

Now we consider all collineations of $\mathcal{PS}^3$ (or those of $\mathcal{P}^3$) leaving invariant a fixed polarity in the formulas (2.2), (2.3) by the matrix Π^{ji} with $\boldsymbol{e}^j_\star = \Pi^{ji}\mathbf{e}_i$ in the dual basis pair $\{\mathbf{e}_i\}$, $\{\boldsymbol{e}^j\}$. This is a subgroup of the full collineation group of $\mathcal{PS}^3$ (resp. $\mathcal{P}^3$). The requirements can easily be expressed for any such collineation $\alpha(\hat{\boldsymbol{\alpha}}; \boldsymbol{\alpha})$ by (3.3–4) and (2.3) as follows

$$\begin{aligned} (\boldsymbol{e}^i_\star)\hat{\boldsymbol{\alpha}} &\overset{!}{=} \frac{1}{c}(\boldsymbol{\alpha}\boldsymbol{e}^i)_\star, \quad \text{i.e. equivalently} \\ (\Pi^{ij}\mathbf{e}_j)\hat{\boldsymbol{\alpha}} = \Pi^{ij} a^l_j \mathbf{e}_l &\overset{!}{=} \frac{1}{c}(\boldsymbol{e}^k A^i_k)_\star = \frac{1}{c} \cdot A^j_k \Pi^{kl}\mathbf{e}_l; \\ \Pi^{ij} a^l_j = \frac{1}{c} \cdot A^i_k \Pi^{kl} \quad &\Leftrightarrow \quad c \cdot \Pi^{ij} = A^i_k \Pi^{kl} A^j_l. \end{aligned} \tag{4.1}$$

Then we may reformulate: Our collineation α leaves invariant the polarity $(_\star)$ iff it preserves the scalar product $\langle\ ,\ \rangle$ at least up to a factor of proportionality. In fact we can write

$$\begin{aligned} \langle \alpha\boldsymbol{e}^i, \alpha\boldsymbol{e}^j \rangle &= \big((\boldsymbol{e}^r A^i_r)_\star(\boldsymbol{e}^r A^j_s)\big) = \big((A^i_r \Pi^{rt}\mathbf{e}_t)(\boldsymbol{e}^s A^j_s)\big) = \\ &= A^i_r \Pi^{rt}\delta^s_t A^j_s = A^j_r \Pi^{rs} A^j_s = c \cdot \Pi^{ij} = c \cdot \langle \boldsymbol{e}^i, \boldsymbol{e}^j \rangle \end{aligned} \tag{4.2}$$

with a proportionality factor c depending on the representant $\boldsymbol{\alpha}(A_i^j)$ of the transformation α; $0 < c \in \mathbf{R}$ for $\mathcal{PS}^3$ and $c \in \mathbf{R} \setminus \{0\}$ for $\mathcal{P}^3$, respectively.

The most important example again is the reflection $\sigma([\boldsymbol{a}]; [\boldsymbol{a}_\star])$ in any axis plane $[\boldsymbol{a}]$ and in its pole $[\boldsymbol{a}_\star]$ as the centre. In particular, the formulas (3.10) and (3.11) become to

$$\sigma([\boldsymbol{a}];\ [\boldsymbol{a}_\star] = [\mathbf{a}])\ :\ \mathbf{x} \mapsto \mathbf{x}\hat{\boldsymbol{\sigma}} =: \mathbf{y} := \mathbf{x} - \frac{2(\mathbf{x}\boldsymbol{a})}{(\mathbf{a}\boldsymbol{a})}\mathbf{a}, \quad \text{resp.}$$
$$\boldsymbol{u} \mapsto \boldsymbol{\sigma}\boldsymbol{u} := \boldsymbol{v} =: \boldsymbol{u} - \frac{2(\mathbf{a}\boldsymbol{u})}{(\mathbf{a}\boldsymbol{a})}\boldsymbol{a}, \quad \text{where } \mathbf{a}\boldsymbol{a} \neq 0 \tag{4.3}$$

is assumed. Then $c = 1$ stands in (4.2).

Besides reflections there are other collineations leaving invariant a fixed polarity $(_\star)$, e.g. the so-called dilatations.

A *dilatation* $\delta(\varrho, D)$ is defined as a central axial collineation with a radical axis plane $\varrho[\boldsymbol{r}]$, i.e. $\boldsymbol{r}_\star = 0$ or $\boldsymbol{r} \in \mathrm{Ker}(_\star)$, and a centre $D[\mathbf{d}]$. Dilatations may appear if the polarity $(_\star)$ is not regular. In fact, a dilatation above preserves $(_\star)$ or $\langle\ ,\ \rangle$ as by (2.2), (3.11)

$$\langle \boldsymbol{u} - \boldsymbol{r}(\mathbf{d}\boldsymbol{u})K;\ \boldsymbol{v} - \boldsymbol{r}(\mathbf{d}\boldsymbol{v})K \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle \quad \text{for} \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_4 \setminus \{\boldsymbol{0}\} \tag{4.4}$$

shows. Any fixed projective polarity $(_\star)$ or the corresponding scalar product $\langle\ ,\ \rangle$ will be called also a *projective metric.* The collineation group, leaving $(_\star)$ invariant, is the *similarity group* of the metric. Its subgroup, generated by reflections in (4.3), is called also the isometry group of $(_\star)$ or $\langle\ ,\ \rangle$.

5. Infinitesimal metrics for $\mathbf{S}^3$ and $\mathbf{H}^3$

Putting together the facts in sections 2–4 we intend to define an infinitesimal distance metric and arc length in any projective metric sphere (or space) above, i.e. if a polarity $(_\star)$ or a scalar product $\langle\ ,\ \rangle$ is fixed in $\mathcal{PS}^3$ (or in $\mathcal{P}^3$).

First we remark that the scalar product $\langle\ ,\ \rangle$ by (2.2)–(2.5) induces also the scalar product of the vectors in the subspace $\mathrm{Im}(_\star) \subset \mathbf{V}^4$, as we have already expressed in (2.12) where $\mathrm{Im}(_\star) = \mathbf{V}^4$. Then it is necessary to define the *arc length square*

$$(\mathrm{d}s)^2 = \frac{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{x} \rangle - \langle \mathbf{x}, \mathrm{d}\mathbf{x} \rangle^2}{K \cdot \langle \mathbf{x}, \mathbf{x} \rangle^2} \tag{5.1}$$

for infinitesimally near different points $[\mathbf{x}]$ and $[\mathbf{x} + \mathrm{d}\mathbf{x}]$ of $\mathcal{PS}^3$ as in the classical surface theory (see e.g. in [2]).

More precisely, also later on, any smooth curve given by

$$\mathbf{R} \supset [t_0, t_1] \ni t \mapsto \mathbf{x}(t) \in \mathbf{V}^4$$
$$\text{with } \ \langle \mathbf{x}(t), \mathbf{x}(t) \rangle \neq 0, \quad \dot{\mathbf{x}}(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} \neq \mathbf{0}, \quad \dot{\mathbf{x}}(t) \not\sim \mathbf{x}(t) \tag{5.2}$$

shall have the *arc length*

$$s(t_0,t_1) = \int\limits_{t_0}^{t_1} \sqrt{\frac{\langle \mathbf{x}(t),\mathbf{x}(t)\rangle\langle \dot{\mathbf{x}}(t),\dot{\mathbf{x}}(t)\rangle - \langle \mathbf{x}(t),\dot{\mathbf{x}}(t)\rangle^2}{K\langle \mathbf{x}(t),\mathbf{x}(t)\rangle^2}}\,\mathrm{d}t. \tag{5.3}$$

One can choose the curve with some projective freedom

$$\mathbf{x}(t) \overset{+}{\sim} c(t)\cdot \mathbf{x}(t), \quad 0 < c(t) \in C^1[t_0,t_1] \tag{5.4}$$

(i.e. $c(t)$ is an arbitrary positive smooth function according to (5.2)).

The curvature constant K makes $(\mathrm{d}s)^2$ positive in (5.1) or in (5.3) and in an appropriate domain to be described later on.

This machinery works well in the case of the sphere $\mathbf{S}^3$, where the scalar product $\langle\ ,\ \rangle$ has the signature $(+\ +\ ++)$, and the group $\mathbf{G}$ of isometries of $\mathbf{S}^3$ consists of all collineations of $\mathcal{PS}^3$ which preserve the scalar product $\langle\ ,\ \rangle$ with 1's in the diagonal form (2.6), (2.7). Then we can choose in (5.1) the usual sphere

$$\langle \mathbf{x},\mathbf{x}\rangle = 1, \quad K = 1, \quad \text{and} \quad \langle \mathbf{x},\mathrm{d}\mathbf{x}\rangle = 0$$

leads to a positive definite quadratic form in the 3-dimensional tangential space to each point of $\mathbf{S}^3$.

We have also a "good" arc length by (5.1) in the hyperbolic space $\mathbf{H}^3$, where $\langle\ ,\ \rangle$ has the signature $(-\ +\ ++)$. The group $\mathbf{G}$ of isometries of $\mathbf{H}^3$ consists of all collineations of $\mathcal{PS}^3$ (now, more of $\mathcal{P}^3$) which preserve the scalar product $\langle\ ,\ \rangle$ with $(-1,1,1,1)$ in the diagonal form (2.6–7). To represent the proper points of $\mathbf{H}^3$ we can choose in (5.1) the imaginary sphere

$$\langle \mathbf{x},\mathbf{x}\rangle = -1, \quad K = -1. \quad \text{Then} \quad \langle \mathbf{x},\mathrm{d}\mathbf{x}\rangle = 0$$

leads to positive definite quadratic form in the tangential 3-space at each proper point of $\mathbf{H}^3$. This is easily seen from the inertia law, since the 3-subspace of $\mathbf{V}^4$, orthogonal to any $\mathbf{x}$ with $\langle \mathbf{x},\mathbf{x}\rangle < 0$, has a positive definite scalar product induced from $\mathbf{V}^4$ having had signature $(-\ +\ ++)$. Other coordinate systems of $\mathbf{H}^3$ can be introduced (see [2]), these lead to other variants of (5.1).

6. Projective interpretation for $\widetilde{\mathbf{SL_2R}}$

We get a more difficult problem if we require a collineation group $\mathbf{G}$, acting on $\mathcal{PS}^3$ (or on $\mathcal{P}^3$), that preserves a global polarity $(_\star) : \boldsymbol{V}_4 \to \mathbf{V}^4$ and a positive definite infinitesimal quadratic form in each point of a "large" arcwise simply connected 3-dimensional $\mathbf{G}$-orbit $\mathbf{X}$ in $\mathcal{PS}^3$ (or in $\mathcal{P}^3$). Then $(\mathbf{X},\mathbf{G})$ will be a 3-dimensional homogeneous geometry in our sense. The maximality of $\mathbf{G}$, related to the properties above, can be easily achieved. It turns out that all the Thurston geometries can be obtained in such a way; and the natural requirements of the theorem, recalled in the introduction, do not allow another possibility. This byproduct is left to an other publication. It remains open, whether this method can be refined to a proof of the Thurston theorem.

As we saw in Section 5, the groups leaving invariant the polarities of signature $(++++)$ and $(-+++)$ led just to the sphere $\mathbf{S}^3$ resp. to the hyperbolic space $\mathbf{H}^3$, represented in a simply connected domain of $\mathcal{PS}^3$, resp. more in that one of $\mathcal{P}^3$.

One might think that the collineation group $\mathbf{G}_\star$ leaving invariant the polarity $(_\star)$, i.e. the scalar product of signature $(--++)$, say $(-1,-1,+1,+1)$ in the diagonal form (2.6–7), satisfies our requirements, too, by the formula (5.1). But this is not true, because then we do not get a positive definite arc length square and, consequently, no point has compact stabilizer under $\mathbf{G}_\star$ in $\mathcal{PS}^3$ or in $\mathcal{P}^3$.

At the origin $E_0(1;0;0;0)$ of the unparted hyperboloid solid we obtain from (5.1) in coordinates

$$(6.1)\qquad \begin{aligned}(\mathrm{d}s)^2 &= \frac{1}{K}\left\{(-1)\left[-(\mathrm{d}x^0)^2-(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2\right]-\left[(-1)(\mathrm{d}x^0)\right]^2\right\} = \\ &= \frac{1}{K}\left\{(\mathrm{d}x^1)^2-(\mathrm{d}x^2)^2-(\mathrm{d}x^3)^2\right\}\end{aligned}$$

as an example. Imagine our hyperboloid solid in the usual Euclidean coordinate simplex: the origin $E_0(\mathbf{e}_0)$, and the ideal points of the axes $E_1^\infty(\mathbf{e}_1)$, $E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$, with unit point $E(\mathbf{e}=\mathbf{e}_0+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$. This will model $\widetilde{\mathbf{SL}_2\mathbf{R}}$ in our projective interpretation, if we choose an appropriate subgroup $\mathbf{G}<\mathbf{G}_\star$ as isometry group.

Consider first the one parameter *screw* collineation group

$$(6.2)\qquad \mathbf{S}(\varphi)\,:\,\begin{pmatrix}\mathbf{e}_0\\ \mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}\mapsto\begin{pmatrix}\cos\varphi & \sin\varphi & & \\ -\sin\varphi & \cos\varphi & & \\ & & \cos\varphi & -\sin\varphi \\ & & \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\mathbf{e}_0\\ \mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}$$

acting in $\mathcal{P}^3$ for $-\frac{\pi}{2}<\varphi\le\frac{\pi}{2}\,(\mathrm{mod}\,\pi)$ and in $\mathcal{PS}^3$ for $-\pi<\varphi\le\pi\,(\mathrm{mod}\,2\pi)$. This leaves our $(--++)$-polarity and the hyperboloid solid

$$(6.3)\qquad \mathcal{H}\,:\,\langle\mathbf{x},\mathbf{x}\rangle := -x^0x^0-x^1x^1+x^2x^2+x^3x^3<0$$

invariant, and the $\mathbf{S}(\varphi)$-orbits are mutually skew lines. Indeed, the orbit through a point $X(x^0,x^1,x^2,x^3)$ by (6.2) is the unique line with the following Plücker 2-vector

$$(6.4)\qquad \begin{aligned}&(x^0x^0+x^1x^1)\mathbf{e}_0\wedge\mathbf{e}_1+(x^0x^3+x^1x^2)\mathbf{e}_0\wedge\mathbf{e}_2+(-x^0x^2+x^1x^3)\mathbf{e}_0\wedge\mathbf{e}_3+\\ &(-x^0x^2+x^1x^3)\mathbf{e}_1\wedge\mathbf{e}_2-(x^0x^3+x^1x^2)\mathbf{e}_1\wedge\mathbf{e}_3-(x^2x^2+x^3x^3)\mathbf{e}_2\wedge\mathbf{e}_3+\end{aligned}$$

up to a constant factor [2].

The plane $H_0^2 := E_0E_2E_3$ $(x^1=0)$ intersects the hyperboloid $\mathcal{H}$ and represents the hyperbolic plane $\mathbf{H}^2$ with absolute figure

$$(6.5)\qquad -x^0x^0+x^2x^2+x^3x^3=0,\quad x^1=0$$

as a base for the line bundle (6.4). A line of (6.4) intersects H_0^2 at the point $Z(z^i\mathbf{e}_i)$

$$(6.6)\qquad z^0=x^0x^0+x^1x^1;\quad z^1=0;\quad z^2=x^0x^2-x^1x^3;\quad z^3=x^0x^3+x^1x^2.$$

The universal covering space $\tilde{\mathcal{H}}$ of our hyperboloid $\mathcal{H}$, constructed by extending the fixed point free action of $\mathbf{S}(\varphi)$ for any $\varphi \in \mathbf{R}$, models our space

$$\tilde{\mathcal{H}} =: \mathbf{X} := \widetilde{\mathbf{SL}_2\mathbf{R}} := \widetilde{T_1\mathbf{H}^2}.$$

Every bundle line by (6.2)–(6.4), that "growes up" at a point of the hyperbolic plane H_0^2, will represent the revolving unit tangent vector at that point of $\mathbf{H}^2$ as defined for the model $\widetilde{T_1\mathbf{H}^2}$ above [1], [6], [7] (see our Remark later).

This *isometry group* $\mathbf{G}$ of our space $\tilde{\mathcal{H}}$ will be defined by the collineations leaving invariant our line bundle and our polarity in (6.2–4), both. In the basis $\{\mathbf{e}_i\}$ any element of $\mathbf{G}$ can be described by a matrix

$$\begin{aligned}
(a_i^j) \text{ with } & a_1^0 = \mp a_0^1, \quad a_1^1 = \pm a_0^0, \quad a_1^2 = \pm a_0^3, \quad a_1^3 = \mp a_0^2; \\
& a_3^0 = \pm a_2^1, \quad a_3^1 = \mp a_2^0, \quad a_3^2 = \mp a_2^3, \quad a_3^3 = \pm a_2^2; \\
& -(a_0^0)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 = -1; \\
& -(a_2^0)^2 - (a_2^1)^2 + (a_2^2)^2 + (a_2^3)^2 = +1; \\
& -a_0^0 a_2^0 - a_0^1 a_2^1 + a_0^2 a_2^2 + a_0^3 a_2^3 = 0; \\
& -a_0^0 a_2^1 + a_0^1 a_2^0 - a_0^2 a_2^3 + a_0^3 a_2^2 = 0; \\
& \det(a_i^j) = 1.
\end{aligned} \tag{6.7}$$

Indeed, conjugacy by the matrices of $\mathbf{G}^+$, defined by the upper signs, maps any matrix of $\mathbf{S}(\varphi)$ onto itself, by the other ones of $\mathbf{G} \setminus \mathbf{G}^+$ onto the inverse.

The following subset $\mathbf{T} < \mathbf{G}^+$, represented by matrices

$$\begin{gathered}
\mathbf{T} : (t_i^j) := \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix}, \\
\text{with inverse} \\
(t_i^j)^{-1} = (T_j^k) = \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix},
\end{gathered} \tag{6.8}$$

indeed makes a group which acts transitively on the points of $\mathbf{X} := \tilde{\mathcal{H}}$ (by the former extension); (t_i^j) maps the origin $E_0(1;0;0;0)$ onto $X(x^0;x^1;x^2;x^3)$.

Surprisingly, the correspondence, on the base of (6.7–8)

$$x^3 + x^0 =: a, \quad x^1 + x^2 =: b, \quad -x^1 + x^2 =: c, \quad -x^3 + x^0 =: d \tag{6.9}$$

as a coordinate transformation, implies just an isomorphism

$$\begin{gathered}
\mathbf{T} \to \mathbf{SL}_2\mathbf{R}, \quad (x^0;x^1;x^2;x^3) \mapsto \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad bc - ad < 0; \\
\text{where } (z^0;z^1) \mapsto (z^0;z^1) \begin{pmatrix} d & b \\ c & a \end{pmatrix} \Leftrightarrow \frac{z^1}{z^0} =: z \mapsto \frac{az+b}{cz+d}, \quad z \in \mathbf{C} \cup \infty
\end{gathered} \tag{6.10}$$

is a projective mapping of the complex projective line in homogeneous coordinates.

Thus we get an equivariance $\tilde{\mathcal{H}} \to \widetilde{\mathbf{SL}_2\mathbf{R}}$ in a very geometric manner. Comparing with (6.2) we get — it seems to be — a new interpretation of the fibre bundle in $\widetilde{\mathbf{SL}_2\mathbf{R}}$.

Taking (6.9) and its inverse we obtain the fibre

$$\begin{pmatrix} d & b \\ c & a \end{pmatrix} \mapsto \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad \varphi \in \mathbf{R} \tag{6.11}$$

by a coordinate transformation (3.7) for every element $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ of $\mathbf{SL}_2\mathbf{R}$ in (6.10).

Remark. Compare our formula (6.11) with the less natural interpretation in [1], [6], [7], where the rotations of the unit tangent vectors $\widetilde{T_1\mathbf{H}^2}$ at the points of the hyperbolic plane $\mathbf{H}^2$ at the same time, have not been explained yet. Now, considering differentials of complex valued function in (6.10), we obtain

$$(\mathrm{d}z^0; \mathrm{d}z^1) \mapsto (\mathrm{d}z^0; \mathrm{d}z^1) \begin{pmatrix} d & b \\ c & a \end{pmatrix} \mapsto (\mathrm{d}z^0; \mathrm{d}z^1) \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} d & b \\ c & a \end{pmatrix} \tag{6.12}$$

a mapping chain of the complex tangents as an explanation, sophisticated enough.

The differential $(\mathrm{d}\mathbf{x} = \mathrm{d}x^j \mathbf{e}_j)$ at the point $X(\mathbf{x} = x^j \mathbf{e}_j)$ is sent by (T_j^k) onto $(\mathrm{d}\bar{\mathbf{x}} = \mathrm{d}\bar{x}^k \mathbf{e}_k)$ at E_0. The stabilizer $\mathbf{G}_0$ of E_0 in $\mathbf{G}^+$, by (6.7), has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_2^2 & a_2^3 \\ 0 & 0 & a_3^2 & a_3^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega & \sin\omega \\ 0 & 0 & -\sin\omega & \cos\omega \end{pmatrix}, \quad -\pi < \omega \le \pi \ (\mathrm{mod}\, 2\pi). \tag{6.13}$$

It is compact, as rotations about the fibre. Hence we have the $\mathbf{G}_0$-invariant positive definite quadratic differential form

$$(\mathrm{d}\bar{s})^2 = (\mathrm{d}\bar{x}^1)^2 + (\mathrm{d}\bar{x}^2)^2 + (\mathrm{d}\bar{x}^3)^2, \quad \mathrm{d}\bar{x}^0 = 0; \tag{6.14}$$

as arc length square at E_0. Then the global quadratic differential form, by (T_j^k) and by a projective normalizing factor, will be

$$\begin{aligned} (\mathrm{d}s)^2 = \Big\{ & \left[-(\mathrm{d}x^0)x^1 + (\mathrm{d}x^1)x^0 - (\mathrm{d}x^2)x^3 + (\mathrm{d}x^3)x^2\right]^2 + \\ & + \left[-(\mathrm{d}x^0)x^2 - (\mathrm{d}x^1)x^3 + (\mathrm{d}x^2)x^0 + (\mathrm{d}x^3)x^1\right]^2 + \\ & + \left[-(\mathrm{d}x^0)x^3 + (\mathrm{d}x^1)x^2 - (\mathrm{d}x^2)x^1 + (\mathrm{d}x^3)x^0\right]^2 \Big\} \cdot \\ & \cdot \frac{1}{\left[-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2\right]^2} \end{aligned} \tag{6.15}$$

as the arc length square at any $X(\mathbf{x} = x^i\mathbf{e}_i)$. The coordinate transformation (6.9) serves us

$$
\begin{aligned}
(\mathrm{d}s)^2 = \{ & [(\mathrm{d}a)c - (\mathrm{d}b)a + (\mathrm{d}c)d - (\mathrm{d}d)b]^2 + \\
& + [-(\mathrm{d}a)c + (\mathrm{d}b)d + (\mathrm{d}c)a - (\mathrm{d}d)b]^2 + \\
& + [(\mathrm{d}a)d + (\mathrm{d}b)c - (\mathrm{d}c)b - (\mathrm{d}d)a]^2 \} \cdot \\
& \cdot \frac{1}{4(bc - ad)^2}
\end{aligned} \tag{6.16}
$$

the corresponding arc length square in $\widetilde{\mathbf{SL}_2\mathbf{R}}$.

Our group $\mathbf{G}$, as the 4-parameter isometry group of $\mathbf{X} = \tilde{\mathcal{H}}$, has a very economic parametrization by (6.7) as follows

$$
\mathbf{G} : (a_i^j) = \begin{pmatrix}
\operatorname{ch} r \cos\psi & \operatorname{ch} r \sin\psi & \operatorname{sh} r \cos\alpha & \operatorname{sh} r \sin\alpha \\
\mp \operatorname{ch} r \sin\psi & \pm \operatorname{ch} r \cos\psi & \pm \operatorname{sh} r \sin\alpha & \mp \operatorname{sh} r \cos\alpha \\
\operatorname{sh} r \cos(\alpha - \omega) & \operatorname{sh} r \sin(\alpha - \omega) & \operatorname{ch} r \cos(\psi + \omega) & \operatorname{ch} r \sin(\psi + \omega) \\
\pm \operatorname{sh} r \sin(\alpha - \omega) & \mp \operatorname{sh} r \cos(\alpha - \omega) & \mp \operatorname{ch} r \sin(\psi + \omega) & \pm \operatorname{ch} r \cos(\psi + \omega)
\end{pmatrix} \tag{6.17}
$$
$$
\psi \in \mathbf{R}, \quad -\pi < \omega \leq \pi \,(\operatorname{mod} 2\pi), \quad 0 \leq r, \quad -\pi < \alpha \leq \pi \,(\operatorname{mod} 2\pi).
$$

Such a matrix (a_i^j) expresses a rotation by ω at the origin in formula (6.13) (for the lower sign it is combined by a halfturn, reversing the previous rotation axis, if occurs), then comes a transformation by (6.8), mapping the origin E_0 onto

$$
\begin{gathered}
X(x^0 = \operatorname{ch} r \cos\psi; \quad x^1 = \operatorname{ch} r \sin\psi; \quad x^2 = \operatorname{sh} r \cos\alpha; \quad x^3 = \operatorname{sh} r \sin\alpha), \\
\text{with } \langle \mathbf{x}, \mathbf{x} \rangle = -\operatorname{ch}^2 r + \operatorname{sh}^2 r = -1,
\end{gathered} \tag{6.18}
$$

according to the first row of (6.17). Our (6.2) will be obtained from (6.17) by choosing $\varphi = \psi$, $r = 0$, $\omega = -2\varphi$.

Our formula (6.18) yields a parametrization of $\mathbf{X} = \tilde{\mathcal{H}}$, and a new very simple arc length square from (6.15)

$$
\begin{aligned}
(\mathrm{d}s)^2 = & \{(\mathrm{d}\psi)\operatorname{ch}^2 r + (\mathrm{d}\alpha)\operatorname{sh}^2 r\}^2 + \\
& + \{[(\mathrm{d}\psi) + (\mathrm{d}\alpha)]\operatorname{ch} r \operatorname{sh} r \sin(\psi - \alpha) + (\mathrm{d}r)\cos(\psi - \alpha)\}^2 + \\
& + \{[(\mathrm{d}\psi) + (\mathrm{d}\alpha)]\operatorname{ch} r \operatorname{sh} r \cos(\psi - \alpha) + (\mathrm{d}r)[-\sin(\psi - \alpha)]\}^2 = \\
& = (\mathrm{d}\psi)^2 \operatorname{ch}^2 r(\operatorname{ch}^2 r + \operatorname{sh}^2 r) + 2(\mathrm{d}\psi)(\mathrm{d}\alpha)[2\operatorname{ch}^2 r \operatorname{sh}^2 r] + \\
& + (\mathrm{d}\alpha)^2 \operatorname{sh}^2 r[\operatorname{ch}^2 r + \operatorname{sh}^2 r] + (\mathrm{d}r)^2.
\end{aligned} \tag{6.19}
$$

The skew line through $X(\mathbf{x} = x^i\mathbf{e}_i)$ intersect the hyperbolic base plane H_0^2 by (6.6) in the point with inhomogeneous coordinates

$$
\frac{z^1}{z^0} = 0; \quad \frac{z^2}{z^0} = \operatorname{th} r \cos(\alpha + \psi); \quad \frac{z^3}{z^0} = \operatorname{th} r \sin(\alpha + \psi). \tag{6.20}
$$

This shows the connection of (6.17) with the polar coordinates ($\vartheta := \alpha + \psi;\ r$) of the base plane H_0^2. With $\mathrm{d}\psi = \mathrm{d}\varphi,\ \ \mathrm{d}\alpha = \mathrm{d}\vartheta - \mathrm{d}\varphi$ by formulas (6.18–19) we get

$$(6.21) \qquad (\mathrm{d}s)^2 = (\mathrm{d}r)^2 + (\mathrm{d}\vartheta)^2 \operatorname{ch}^2 r \operatorname{sh}^2 r + [\mathrm{d}\varphi + (\mathrm{d}\vartheta) \operatorname{sh}^2 r]^2$$

a useful formula by the polar coordinate (ϑ, r) of the hyperbolic base plane and the fibre coordinate φ, which serves interesting comparison with the corresponding formulas of $\mathbf{H}^3$, $\mathbf{H}^2\times\mathbf{R}$ and **Nil**.

Thus, we have obtained just a Seifert fibre space $\tilde{\mathcal{H}}$ equivariant to $\widetilde{\mathbf{SL_2R}}$ with some — it seems to be — new projective interpretations.

7. The Euclidean space $\mathbf{E}^3$ and $\mathbf{S}^2\times\mathbf{R}$ in $\mathcal{P}^3$

Now, we turn to discussing degenerated polarities, first the Euclidean projective metric space $\mathbf{E}^3$ in $\mathcal{P}^3$. The polarity $(_\star)$ then has the canonical form

$$(7.1) \qquad \boldsymbol{e}^0_\star = \mathbf{0}, \quad \boldsymbol{e}^1_\star = \mathbf{e}_1, \quad \boldsymbol{e}^2_\star = \mathbf{e}_2, \quad \boldsymbol{e}^3_\star = \mathbf{e}_3$$

in the dual basis pair $\{\boldsymbol{e}^j\}$ and $\{\mathbf{e}_i\}$ as in Section 2.

a) Any collineation $\alpha(\hat{\boldsymbol{\alpha}}; \boldsymbol{\alpha})$, leaving $(_\star)$ invariant, has $\boldsymbol{\alpha}$ in the matrix form (4.1) for the form space $\boldsymbol{V}_4$, specialized by

$$(7.2) \qquad \boldsymbol{\alpha} : (\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3) \mapsto (\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3) \begin{pmatrix} 1 & A_0^1 & A_0^2 & A_0^3 \\ 0 & A_1^1 & A_1^2 & A_1^3 \\ 0 & A_2^1 & A_2^2 & A_2^3 \\ 0 & A_3^1 & A_3^2 & A_3^3 \end{pmatrix},$$

$$A^\varrho_\sigma \delta^{\sigma\tau} A^\omega_\tau = c \cdot \delta^{\varrho\omega} \quad \text{(with Kronecker } \delta \text{ symbol) and } 0 < c \in \mathbf{R}, \text{ as usual.}$$

Throughout, the greek indices run over 1, 2, 3. This formula expresses the Euclidean similarities in projective terms for the planes. This involves in $\mathbf{V}^4$, i.e. for the points of $\mathcal{P}^3$

$$(7.3) \qquad \hat{\boldsymbol{\alpha}} : \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a_0^1 & a_0^2 & a_0^3 \\ 0 & a_1^1 & a_1^2 & a_1^3 \\ 0 & a_2^1 & a_2^2 & a_2^3 \\ 0 & a_3^1 & a_3^2 & a_3^3 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \qquad \begin{matrix} \text{by } a^\sigma_\varrho \delta_{\sigma\tau} a^\tau_\omega = \delta_{\varrho\omega} \cdot \frac{1}{c} \\ \text{with the preceding constant } c. \end{matrix}$$

Any proper point X of $\mathbf{E}^3$ is represented in $\mathcal{P}^3$ by the $\mathbf{V}^4$-class

$$(7.4) \qquad (\mathbf{x} = x^i \mathbf{e}_1), \quad x^0 \neq 0 \text{ and } x^1/x^0 =: x,\ \ x^2/x^0 =: y,\ \ x^3/x^0 =: z$$

are the Cartesian coordinates, i.e.

$$(7.5) \qquad \mathbf{x}\boldsymbol{e}^0 = x^0 \neq 0, \text{ e.g. } x^0 = 1$$

is the assumption for the proper points of $\mathbf{E}^3$. The points of the ideal plane to $\mathbf{E}^3$ satisfy just the equation

$$\mathbf{x}\boldsymbol{e}^0 = x^0 = 0, \text{ then the form } \boldsymbol{e}^0 \in \boldsymbol{V}_4, \tag{7.6}$$

i.e. the 1-dimensional kernel $(\boldsymbol{e}^0)$ of the polarity, describes the ideal plane in $\mathcal{P}^3$, which is excluded. So we get the space $\mathbf{X} = \mathbf{E}^3$. Thus we introduce, in general, the affine space $\mathcal{A}^3 := \mathcal{P}^3 \setminus (\boldsymbol{e}^0)$, too.

An ordered point pair $\overrightarrow{XY}$, with $x^0 = y^0$ by (7.4), can be considered as a difference vector of $\mathcal{A}^3$ or, in particular, of $\mathbf{E}^3$, and it belongs to the subspace $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle =: \mathbf{V}^3 \subset \mathbf{V}^4$. Thus the polarity (7.1) induces the positive definite scalar product for the vectors of $\mathbf{E}^3$ as mentioned at the beginning of Section 5 on the base of formulas (2.2–5).

The infinitesimal arc length square is defined in $\mathbf{E}^3$ by

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \tag{7.7}$$

according to (7.4) as usual. This is invariant under the isometry group $\mathbf{G}$ of $\mathbf{E}^3$, when $c = 1$ in (7.2), (7.3), according to Section 4. In projective coordinates (7.7) can be written

$$\begin{gathered}(\mathrm{d}s)^2 = \Big\{\big[(\mathrm{d}x^1)x^0 - (\mathrm{d}x^0)x^1\big]^2 + \big[(\mathrm{d}x^2)x^0 - (\mathrm{d}x^0)x^2\big]^2 + \big[(\mathrm{d}x^3)x^0 - (\mathrm{d}x^0)x^3\big]^2\Big\} \cdot \frac{1}{(x^0)^4} \\ \text{at any proper point } X(\mathbf{x} = x^i \mathbf{e}_i), \quad x^0 \neq 0.\end{gathered} \tag{7.8}$$

From this we obtain (7.7) by choosing $x^0 = \mathit{const}$, $\mathrm{d}x^0 = 0$, but (7.8) is more general, in analogy to (5.1), (6.8). Any point has $O(2, \mathbf{R})$ as the stabilizer in the isometry group $\mathbf{G}$ which is conjugated to that of the origin $(1; 0; 0; 0)$, i.e. to (7.3) with zero translation: $a_0^1 = a_0^2 = a_0^3 = 0$.

b) Second, we discuss the projective model of $\mathbf{S}^2 \times \mathbf{R}$. To the fixed polarity (7.1), we choose a new isometry group $\mathbf{G}$, acting transitively on the space $\mathbf{X} := \mathbf{S}^2 \times \mathbf{R}$, obtained by excluding the ideal points of the plane $(\boldsymbol{e}^0)$, furthermore, the point $(\mathbf{e}_0)$ from $\mathcal{P}^3$. Each $\alpha(\hat{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \in \mathbf{G}$ shall have the form

$$\begin{gathered}\boldsymbol{\alpha} : (\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3) \mapsto (\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3) \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & A_1^1 & A_1^2 & A_1^3 \\ 0 & A_2^1 & A_2^2 & A_2^3 \\ 0 & A_3^1 & A_3^2 & A_3^3 \end{pmatrix}, \\ \text{where } A_\sigma^\varrho \delta^{\sigma\tau} A_\tau^\omega = \delta^{\varrho\omega} \text{ and } 0 < \lambda \in \mathbf{R}; \\ \hat{\boldsymbol{\alpha}} : \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & a_1^1 & a_1^2 & a_1^3 \\ 0 & a_2^1 & a_2^2 & a_2^3 \\ 0 & a_3^1 & a_3^2 & a_3^3 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{matrix}\text{by } a_\varrho^\sigma \delta_{\sigma\tau} a_\omega^\tau = \delta_{\varrho\omega} \\ \text{with the preceding} \\ \text{“similarity” constant } \lambda.\end{matrix}\end{gathered} \tag{7.9}$$

The dilatations, with axis plane $(\boldsymbol{e}^0)$ and centre $(\mathbf{e}_0)$, in the sense of Section 3, have the matrix forms

$$\sigma : \begin{pmatrix} \frac{1}{\lambda} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad 0 < \lambda \in \mathbf{R}. \tag{7.10}$$

They constitute the centre subgroup $\mathbf{K}$ of $\mathbf{G}$ at (7.9), and are isomorphic to the additive group $\mathbf{R}(+)$. Any point stabilizer is conjugate to that of the north pole $(1;0;0;1)$, i.e. to

$$\text{(7.11)} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & \mp\sin\omega & \pm\cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{matrix} \text{as } O(2,\mathbf{R}), \\ \text{i.e. a compact subgroup} \\ \text{of } \mathbf{G}. \end{matrix}$$

We remind to the model of $\mathbf{E}^3$ in $\mathcal{P}^3$ before. Our $\mathbf{G}$ can be interpreted as isometries of a unit sphere in $\mathbf{E}^3$, combined by the dilatation from the centre $E_0(\mathbf{e}_0)$ with similarity constant λ. This last one can be considered as a translation in the third, i.e. $\mathbf{R}$-direction by $\ln\lambda$. So is the equivariance to $\mathbf{S}^2\times\mathbf{R}$.

To derive the $\mathbf{G}$-invariant projective term of the arc length square in our model, we can follow the analogy of (5.1) with

$$\text{(7.12)} \qquad \mathbf{x} = \frac{x^\tau}{x^0}\mathbf{e}_\tau, \quad \mathrm{d}\mathbf{x} = \left[(\mathrm{d}x^\tau x^0 - \mathrm{d}x^0 x^\tau)\mathbf{e}_\tau\right]\frac{1}{x^0x^0}$$

and the scalar product $\langle \mathbf{e}_\tau, \mathbf{e}_\omega\rangle = \delta_{\tau\omega}$ by the Kronecker symbol. Thus we obtain our desired formula

$$\text{(7.13)} \qquad \begin{aligned} (\mathrm{d}s)^2 &= \frac{\langle \mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} \\ &= \frac{\left[(\mathrm{d}x^1)x^0 - (\mathrm{d}x^0)x^1\right]^2 + \left[(\mathrm{d}x^2)x^0 - (\mathrm{d}x^0)x^2\right]^2 + \left[(\mathrm{d}x^3)x^0 - (\mathrm{d}x^0)x^3\right]^2}{(x^0)^2\left[(x^1)^2 + (x^2)^2 + (x^3)^2\right]}. \end{aligned}$$

This can be specialized by a usual parametrization, e.g.

$$\text{(7.14)} \qquad \begin{aligned} &x^0 := 1, \quad x^1 := e^t\cos\varphi\sin\vartheta, \quad x^2 := e^t\sin\varphi\sin\vartheta, \quad x^3 := e^t\cos\vartheta; \\ &(\mathrm{d}s)^2 = (\mathrm{d}t)^2 + (\mathrm{d}\varphi)^2\sin^2\vartheta + (\mathrm{d}\vartheta)^2, \qquad t\in\mathbf{R}, \quad -\pi<\varphi\le\pi, \quad 0\le\vartheta\le\pi, \end{aligned}$$

where φ, ϑ are the polar coordinates of the sphere $\mathbf{S}^2$, t is the fibre coordinate.

Remark. We have not obtained the full isometry group of $\mathbf{S}^2\times\mathbf{R}$ yet. For this we should extend the stabilizer of the north pole $(1;0;0;1)$ in (7.11) by the sphere inversion — well known in $\mathbf{E}^3$ — fixing pointwise the unit sphere

$$\text{(7.15)} \qquad -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0,$$

which is no more a collineation but an involutive biquadratic mapping of $\mathcal{P}^3 \setminus (\boldsymbol{e}^0) \setminus (\mathbf{e}_0)$. We obtain in formulas

$$\text{(7.16)} \qquad \begin{gathered} \mathbf{x}\left(\frac{x^1}{x^0}; \frac{x^2}{x^0}; \frac{x^3}{x^0}\right) \mapsto \mathbf{y}\left(\frac{y^1}{y^0}; \frac{y^2}{y^0}; \frac{y^3}{y^0}\right) \\ \mathbf{y} = \frac{\mathbf{x}}{\langle\mathbf{x},\mathbf{x}\rangle} \quad \text{i.e.} \quad y^\nu = \frac{x^0y^0\cdot x^\nu}{(x^1)^2+(x^2)^2+(x^3)^2} \quad \nu = 1,2,3 \\ x^0 \ne 0, \quad (x^1;x^2;x^3) \ne (0;0;0) \end{gathered}$$

This expresses a mapping of $\mathbf{S}^2\times\mathbf{R}$ that fixes pointwise $\mathbf{S}^2$ but reverses each $\mathbf{R}$-fibre, i.e. the mapping is a reflection.

8. $\mathbf{H}^2\times\mathbf{R}$ and Sol in $\mathcal{P}^3$

We turn to the polarity given in $\mathcal{P}^3$ by

$$(\star)\ :\ \boldsymbol{e}^0_\star = \mathbf{0},\quad \boldsymbol{e}^1_\star = -\mathbf{e}_1,\quad \boldsymbol{e}^2_\star = \mathbf{e}_2,\quad \boldsymbol{e}^3_\star = \mathbf{e}_3 \tag{8.1}$$

of signature $(0\ -\ ++)$, expressed in the dual basis pair $\{\boldsymbol{e}^j\},\ \{\mathbf{e}_i\}$.

a) First we model $\mathbf{H}^2\times\mathbf{R}$ similarly as $\mathbf{S}^2\times\mathbf{R}$ in Section 7. We choose the group $\mathbf{G}$ with $\alpha(\hat{\boldsymbol{\alpha}};\boldsymbol{\alpha})$, as typical isometry of $\mathbf{H}^2\times\mathbf{R}$, fixing the open half cone solid $\mathcal{C}^+$ to $(\star)$ as follows

$$\mathcal{C}^+ := \left\{X(\mathbf{x}=x^i\mathbf{e}_i)\in\mathcal{P}^3\ :\ -(x^1)^2+(x^2)^2+(x^3)^2<0<x^0,x^1\right\} =: \mathbf{H}^2\times\mathbf{R}. \tag{8.2}$$

The fixed apex $E_0(\mathbf{e}_0)$ and the ideal points of the plane $(\boldsymbol{e}^0)$ are excluded. Thus we obtain, e.g. for the points of $\mathcal{C}^+ =: \mathbf{H}^2\times\mathbf{R}$:

$$\hat{\boldsymbol{\alpha}}\ :\ \begin{pmatrix}\mathbf{e}_0\\ \mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \mapsto \begin{pmatrix}\frac{1}{\lambda} & 0 & 0 & 0\\ 0 & a_1^1 & a_1^2 & a_1^3\\ 0 & a_2^1 & a_2^2 & a_2^3\\ 0 & a_3^1 & a_3^2 & a_3^3\end{pmatrix}\begin{pmatrix}\mathbf{e}_0\\ \mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix},\qquad \begin{matrix}\text{by } a_\varrho^\sigma\chi_{\sigma\tau}a_\omega^\tau=\chi_{\varrho\omega}\\ \text{with } 0<\lambda\in\mathbf{R}\end{matrix} \tag{8.3}$$

as a similarity constant. Here we have defined

$$(\chi_{\sigma\tau}) := \begin{pmatrix}-1 & & \\ & 1 & \\ & & 1\end{pmatrix}\qquad \begin{matrix}\text{for the Minkowskian scalar product in}\\ \langle\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\rangle =: \mathbf{V}^3;\end{matrix} \tag{8.4}$$

and the elements $\sigma(\hat{\boldsymbol{\sigma}};\boldsymbol{\sigma})$, at the formula (7.10), constitute the centre subgroup $\mathbf{K}$ of $\mathbf{G}$ isomorphic to $\mathbf{R}(+)$ again. Any point stabilizer is conjugate to that of $(1;1;0;0)\in\mathcal{C}^+$, i.e. to $O(2,\mathbf{R})$.

The group $\mathbf{G}$ at (8.3) makes the positive definite arc length square again by (5.1) and (7.12) but with $(\chi_{\sigma\tau})$ in $\langle\ ,\ \rangle$

$$\begin{aligned}
(\mathrm{d}s)^2 &:= \frac{1}{\langle\mathbf{x},\mathbf{x}\rangle}\left\{2\langle\mathbf{x},\mathrm{d}\mathbf{x}\rangle^2-\langle\mathbf{x},\mathbf{x}\rangle\langle\mathrm{d}\mathbf{x},\mathrm{d}\mathbf{x}\rangle\right\} =\\
&= \frac{1}{(x^0)^2\left[-(x^1)^2+(x^2)^2+(x^3)^2\right]^2}\cdot\\
&\cdot\Big\{(\mathrm{d}x^0)^2\left[-(x^1)^2+(x^2)^2+(x^3)^2\right]^2+\\
&+2\mathrm{d}x^0\mathrm{d}x^1\left[x^0x^1\left[-(x^1)^2+(x^2)^2+(x^3)^2\right]\right]+\\
&+2\mathrm{d}x^0\mathrm{d}x^2\left[-x^0x^2\left[-(x^1)^2+(x^2)^2+(x^3)^2\right]\right]+\\
&+2\mathrm{d}x^0\mathrm{d}x^3\left[-x^0x^3\left[-(x^1)^2+(x^2)^2+(x^3)^2\right]\right]+\\
&+(\mathrm{d}x^1)^2(x^0)^2\left[(x^1)^2+(x^2)^2+(x^3)^2\right]+\\
&+2\mathrm{d}x^1\mathrm{d}x^2(x^0)^2\left[-2x^1x^2\right]+\\
&+(2\mathrm{d}x^1\mathrm{d}x^3(x^0)^2[-2x^1x^3]+\\
&+(\mathrm{d}x^2)^2(x^0)^2\left[(x^1)^2+(x^2)^2-(x^3)^2\right]+\\
&+(2\mathrm{d}x^2\mathrm{d}x^3(x^0)^2[2x^2x^3]+\\
&+(\mathrm{d}x^3)^2(x^0)^2\left[(x^1)^2-(x^2)^2+(x^3)^2\right]\Big\}
\end{aligned} \tag{8.5}$$

for any $X(\mathbf{x}=x^i\mathbf{e}_i)\in\mathcal{C}^+=\mathbf{H}^2\times\mathbf{R}$, in the first formula by $\mathbf{x}=\frac{x^\tau}{x^0}\mathbf{e}_\tau$. This becomes simpler in special coordinates, e.g.

$$\begin{gathered} x^0=1;\quad x^1=e^t\operatorname{ch}r;\quad x^2=e^t\operatorname{sh}r\cos\alpha;\quad x^3=e^t\operatorname{sh}r\sin\alpha;\\ t\in\mathbf{R},\quad 0\le r,\quad -\pi<\alpha\le\pi\quad\text{yield}\\ (\mathrm{d}s)^2=(\mathrm{d}t)^2+(\mathrm{d}r)^2+\operatorname{sh}^2r(\mathrm{d}\alpha)^2, \end{gathered}\tag{8.6}$$

where $(\alpha;r)$ are the polar coordinates of the hyperbolic plane $\mathbf{H}^2$, t is the fibre coordinate.

Remark. As at the end of Section 7, our group $\mathbf{G}$ by (8.3) has not been the full isometry group of $\mathbf{H}^2\times\mathbf{R}$ yet. To obtain this, we should extend the stabilizer of $(1;1;0;0)$ again by the inversion fixing pointwise the "unit" two parted hyperboloid through $(1;1;0;0)$ with equation $\langle\mathbf{x},\mathbf{x}\rangle=-1$, i.e. with

$$(x^0)^2-(x^1)^2+(x^2)^2+(x^3)^2=0,\quad 0<x^0.$$

This inversion is an involutive mapping of $\mathcal{C}^+$, now is given by

$$\begin{gathered}\mathbf{x}\left(\frac{x^1}{x^0};\ \frac{x^2}{x^0};\ \frac{x^3}{x^0}\right)\to\mathbf{y}\left(\frac{y^1}{y^0};\ \frac{y^2}{y^0};\ \frac{y^3}{y^0}\right)\\ \mathbf{y}=\frac{\mathbf{x}}{-\langle\mathbf{x},\mathbf{x}\rangle},\quad\text{i.e.}\quad y^\tau=\frac{x^0\cdot y^0\cdot x^\tau}{(x^1)^2-(x^2)^2-(x^3)^2}\quad \tau=1,2,3;\end{gathered}$$

represents the reflection of $\mathbf{H}^2\times\mathbf{R}$ that fixes the base $\mathbf{H}^2$ but reverses each $\mathbf{R}$-fibre. In a future paper we shall discuss how to linearize the isometries of $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$, respectively, as collineations of $\mathcal{PS}^4$ but restricted to a quadrics of signatures $(-++++)$, resp. $(--+++)$.

b) Second, we model the **Sol** geometry in the Minkowski plane-time above, where the group $\mathbf{G}$ will be chosen otherwise. Now our polarity $(_\star)$ of signature $(0\ -\ ++)$ will be written in a new dual basis pair

$$\begin{gathered}\{\boldsymbol{f}^j\},\quad\{\mathbf{f}_i\},\quad \mathbf{f}_i\boldsymbol{f}^j=\delta_i^j;\quad\text{defined by}\\ \boldsymbol{f}^0=\boldsymbol{e}^0,\quad \boldsymbol{f}^1=(\boldsymbol{e}^1+\boldsymbol{e}^2)\frac{\sqrt2}{2},\quad \boldsymbol{f}^2=(-\boldsymbol{e}^1+\boldsymbol{e}^2)\frac{\sqrt2}{2},\quad \boldsymbol{f}^3=\boldsymbol{e}^3;\\ \mathbf{f}_0=\boldsymbol{e}_0,\quad \mathbf{f}_1=(\mathbf{e}_1+\mathbf{e}_2)\frac{\sqrt2}{2},\quad \mathbf{f}_2=(-\mathbf{e}_1+\mathbf{e}_2)\frac{\sqrt2}{2},\quad \mathbf{f}_3=\mathbf{e}_3;\\ (_\star):\ \boldsymbol{f}^0_*=\mathbf{0},\quad \boldsymbol{f}^1_*=\mathbf{f}_2,\quad \boldsymbol{f}^2_*=\mathbf{f}_1,\quad \boldsymbol{f}^3_*=\mathbf{f}_3.\end{gathered}\tag{8.7}$$

We see that $\langle\boldsymbol{f}^1,\boldsymbol{f}^1\rangle=\langle\boldsymbol{f}^2,\boldsymbol{f}^2\rangle=0$.

Define first the group $\mathbf{G}^+$ by the elements $\alpha(\hat{\boldsymbol{\alpha}},\boldsymbol{\alpha})$, where

$$\hat{\boldsymbol{\alpha}}:\begin{pmatrix}\mathbf{f}_0\\\mathbf{f}_1\\\mathbf{f}_2\\\mathbf{f}_3\end{pmatrix}\mapsto\begin{pmatrix}1&a_0^1&a_0^2&a_0^3\\0&a_1^1&0&0\\1&0&a_2^2&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\mathbf{f}_0\\\mathbf{f}_1\\\mathbf{f}_2\\\mathbf{f}_3\end{pmatrix}\quad\begin{gathered}\text{with}\\ a_1^1=e^{-a_0^3},\quad a_2^2=e^{a_0^3},\\ \text{hence }\det(a_i^j)=1.\end{gathered}\tag{8.8}$$

$\mathbf{G}^+$ acts transitively on the Minkowski plane-time $\mathcal{A}^3 = \mathcal{P}^3 \setminus (\boldsymbol{e}^0) =:$ **Sol** with points $X(\mathbf{x} = x^i\mathbf{f}_i)$, $x^0 \neq 0$ or $x^0 = 1$, $x^1 = x$, $x^2 = y$, $x^3 = z$. Any point stabilizer is trivial in $\mathbf{G}^+$. But it can be extended to a dihedral group $\mathbf{D}(4)$ of order 8, to obtain the full isometry group $\mathbf{G}$ of **Sol**. E.g. at the origin $(1;0;0;0)$ this $\mathbf{D}(4)$ is generated now by a plane reflection and a halfturn

$$\begin{aligned}&\mathbf{f}_0 \mapsto \mathbf{f}_0, \quad \mathbf{f}_1 \mapsto -\mathbf{f}_1, \quad \mathbf{f}_2 \mapsto \mathbf{f}_2, \quad \mathbf{f}_3 \mapsto \mathbf{f}_3, \quad \text{resp.}\\ &\mathbf{f}_0 \mapsto \mathbf{f}_0, \quad \mathbf{f}_1 \mapsto \mathbf{f}_2, \quad \mathbf{f}_2 \mapsto \mathbf{f}_1, \quad \mathbf{f}_3 \mapsto -\mathbf{f}_3.\end{aligned} \tag{8.9}$$

All the transformations of $\mathbf{G}$ preserves the plane fibre structure of **Sol**. A fibre is an $(x;y)$ plane for a fixed z of the base.

To derive the arc length square for our **Sol**, we follow the general method. By choosing appropriate $\hat{\boldsymbol{\alpha}}$ in (8.8) we can achieve

$$(1,x,y,z)\begin{pmatrix}1 & -xe^z & -ye^{-z} & -z\\ 0 & e^z & 0 & 0\\ 0 & 0 & e^{-z} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = (1,0,0,0). \tag{8.10}$$

Thus any differential form at $X(\mathbf{x})$ is mapped onto one at the origin $F_0(\mathbf{f}_0)$. So we get in affine term

$$\begin{aligned}(\mathrm{d}s)^2 &= (\mathrm{d}x)^2e^{2z} + (\mathrm{d}y)^2e^{-2z} + (\mathrm{d}z)^2\\ &\text{or}\\ (\mathrm{d}s)^2 &= \Big\{\big[(\mathrm{d}x^1)x^0 - (\mathrm{d}x^0)x^1\big]^2 e^{2x^3/x^0} +\\ &\quad + \big[(\mathrm{d}x^2)x^0 - (\mathrm{d}x^0)x^2\big]^2 e^{-2x^3/x^0} +\\ &\quad + \big[(\mathrm{d}x^3)x^0 - (\mathrm{d}x^0)x^3\big]^2\Big\}\frac{1}{(x^0)^4}\end{aligned} \tag{8.11}$$

in projective term.

9. The Nil space in $\mathcal{P}^3$

Let the polarity $(\star)$ is of signature $(0\,0\,0+)$ given by

$$(\star)\ :\ \boldsymbol{e}^0_* = \mathbf{0}, \quad \boldsymbol{e}^1_* = \mathbf{0}, \quad \boldsymbol{e}^2_* = \mathbf{0}, \quad \boldsymbol{e}^3_* = \mathbf{e}_3 \tag{9.1}$$

in the dual basis pair $\{\boldsymbol{e}^j\}$, $\{\mathbf{e}_i\}$. The group $\mathbf{G}_*$, leaving $(\star)$ invariant, has elements $\alpha(\hat{\boldsymbol{\alpha}}(a^j_i);\boldsymbol{\alpha}(A^k_j))$ with particular matrix entries

$$a^0_3 = a^1_3 = a^2_3 = 0 \neq a^3_3, \quad A^0_3 = A^1_3 = A^2_3 = 0 \neq A^3_3, \tag{9.2}$$

i.e. $E_3(\mathbf{e}_3)$ is common fixed point. $\mathbf{G}_*$ acts transitively on the other points of $\mathcal{P}^3$. Any point stabilizer assumed to be compact, hence also that of the origin $E_0(1,0,0,0)$. Therefore,

$$\hat{\boldsymbol{\alpha}}(\omega)\ :\ \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\omega & \sin\omega & 0\\ 0 & \mp\sin\omega & \pm\cos\omega & 0\\ 0 & 0 & 0 & \pm 1\end{pmatrix} \tag{9.3}$$

can be chosen for this stabilizer. At the same time the invariant plane $(\boldsymbol{e}^0)$ as ideal plane has been distinguished

$$\mathbf{Nil} := \mathcal{P}^3 \setminus (\boldsymbol{e}^0) = \mathcal{A}^3.$$

The cyclic normal subgroup $\mathbf{K}(z)$ describes the fibre bundle by

$$\text{(9.4)} \qquad \mathbf{K}(z) : \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \text{and} \quad \mathbf{L}(x,y,z) : \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

characterizes the Heisenberg group as general non-commutative translations for the final isometry group $\mathbf{G}$ of $\mathbf{Nil}$. The usual group operation

$$\text{(9.5)} \qquad (x;y;z) \star (x';y';z') = (x+x';\ y+y';\ z+yx'+z')$$

reads from the right multiplication of $\mathbf{L}(x,y,z)$ for the points of $\mathcal{A}^3$.

The invariant plane pencil of the stabilizers of the points on the line $E_0E_3^\infty$, by (9.3) has the form $\{(\boldsymbol{e}^0 p + \boldsymbol{e}^3) : p \in \mathbf{R}\}$ with $p=0$ for $E_0(\mathbf{e}_0)$. It will be mapped onto

$$\text{(9.6)} \qquad \begin{pmatrix} 1 & -x & -y & xy-z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p+xy-z \\ 0 \\ -x \\ 1 \end{pmatrix} \qquad \begin{array}{c} \text{at the points of } XE_3^\infty \\ \text{with } X(1,x,y,z) \text{ if } p=0. \end{array}$$

The final isometry group $\mathbf{G}$ of $\mathbf{Nil}$ has 4 parameters and two components

$$\text{(9.7)} \quad \alpha(\omega,x,y,z) : \begin{pmatrix} 1 & x & y & z \\ 0 & \cos\omega & \sin\omega & x\sin\omega \\ 0 & \mp\sin\omega & \pm\cos\omega & \pm x\cos\omega \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \qquad \begin{array}{c} \text{for points} \\ X\left(x=\frac{x^1}{x^0};\ y=\frac{x^2}{x^0};\ z=\frac{x^3}{x^0};\right) \\ \text{of } \mathbf{Nil} = \mathcal{P}^3 \setminus (\boldsymbol{e}^0). \end{array}$$

(The upper signs belong to the identity component.) This general matrix is a product of those in (9.3) and (9.4). The stabilizer of a general point $X(x,y,z)$ can be expressed by conjugacy, i.e. by the product of matrices in (9.6–7). The axis of these rotations will be the fibre XE_3, but the invariant plane pencil is no more parallel with the (x,y)-plane, as (9.6) says.

The infinitesimal arc length square is well known in affine coordinates

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + [\mathrm{d}z - (\mathrm{d}y)x]^2,$$

this will be

$$\text{(9.8)} \qquad (\mathrm{d}s)^2 = \Big\{ \big[(\mathrm{d}x^1)x^0 - (\mathrm{d}x^0)x^1\big]^2 + \big[(\mathrm{d}x^2)x^0 - (\mathrm{d}x^0)x^2\big]^2 + \\ + \big[(\mathrm{d}x^3)x^0 - (\mathrm{d}x^2)x^1\big]^2 \Big\} \cdot \frac{1}{(x^0)^2}$$

in projective coordinates $X(x^0 \neq 0;\ x^1; x^2; x^3)$.

Summarizing, we have obtained **Nil** as a non-trivial fibre bundle over the Euclidean (x, y)-plane with some novelties analogously as $\widetilde{\mathbf{SL_2R}}$ in Section 6.

Remark. The fibering procedure at the $\mathbf{SL_2R}$ geometry (formula (6.2)) can be applied in the space $\mathbf{S}^3$ analogously (Hopf fibering). Moreover, $\mathbf{E}^2\times\mathbf{R}$ would have some importance, too. But these homogeneous geometries would be not maximal. Namely $\mathbf{S}^3$ and $\mathbf{E}^3$ contain these spaces, respectively, with richer groups of isometries.

10. The simplex tilings of I. K. Zhuk in spaces $\mathbf{S^3}$, $\mathbf{H^3}$, Nil and $\widetilde{\mathbf{SL_2R}}$

As an application we illustrate our method, how to find metric realizations for combinatorial tilings $(\mathcal{T},\ \Gamma(a, 2b))$ in different spaces, depending on the natural parameters $3 \leq a$, $2 \leq b$. This will be our new version of the so-called Zhuk tilings [8], discussed in details and with figures in [3], [4] Section 3. In [8] $\Gamma(a, 2b) = K_{mn}$ holds, i.e. $m = a,\ \ n = b$.

We start with a 3-simplex $A_0A_1A_2A_3$ as a combinatorial incidence structure of its vertices $A_i(\mathbf{a}_i)\ \ i \in \{0, 1, 2, 3\}$, edges $A_iA_j\ \ i < j$, faces $b^i(\boldsymbol{b}^i) = A_jA_kA_l\ \ \{i, j, k, l\} = \{0, 1, 2, 3\}$ and its solid.

We have just introduced the basis vectors $\{\mathbf{a}_i\}$ for $\mathbf{V}^4(\mathbf{R})$ and the dual basis $\{\boldsymbol{b}^j\}$ for the dual space $\boldsymbol{V}_4$ of linear forms with $\mathbf{a}_i\boldsymbol{b}^j = \delta_i^j$; and so the real projective sphere $\mathcal{PS}^3$, furthermore the real projective space $\mathcal{P}^3$ (analogously as in Section 1, but with slightly modified denotations) have been defined.

Let the tiling $\mathcal{T}$ and the group $\Gamma(a, 2b)$, by the "face pairing"

$$r_0 : A_1A_2A_3 \to A_1A_3A_2; \quad r_1 : A_2A_3A_0 \to A_2A_0A_3; \quad r : A_0A_1A_3 \to A_0A_1A_2 \tag{10.1}$$

of the simplex $A_0A_1A_2A_3$, be introduced. First combinatorially, the pairing maps the vertices, edges, faces as introduced, preserving the incidences. The solid of $A_0A_1A_2A_3$ will be mapped onto the neighbouring simplices along the corresponding faces. For instance

$$\begin{gathered} A_0^r = A_0 \quad A_1^r = A_1, \quad A_3^r = A_2, \quad A_2^r \text{ and } A_3 \text{ are separated by } A_0A_1A_2 \\ A_0^{r^{-1}} = A_0, \ldots, A_2^{r^{-1}} = A_3; \quad A_3^{r^{-1}} \text{ and } A_2 \text{ are separated by } A_0A_1A_3. \end{gathered} \tag{10.2}$$

In the basis — dual basis pair $\{\mathbf{a}_i\}$ — $\{\boldsymbol{b}^j\}$ we have, e.g., the matrix presentations for the mapping r (and its inverse r^{-1}, too)

$$r((r_i^j), (R_j^k)); \quad \text{with} \quad r_i^jR_j^k = \delta_i^k; \quad \mathbf{a}_i \mapsto r_i^j\mathbf{a}_j, \quad \text{i.e.}$$

$$(r_i^j) : \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s & t & A & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}, \quad \text{with} \quad \begin{gathered} A = 2\cos\frac{2\pi}{a}, \\ 3 \leq a \in \mathbf{N}, \\ \text{i.e. } -1 \leq A < 2, \end{gathered} \quad \text{and} \tag{10.3}$$

$$(R_j^k) : (\,\boldsymbol{b}^0 \quad \boldsymbol{b}^1 \quad \boldsymbol{b}^2 \quad \boldsymbol{b}^3\,) \mapsto (\,\boldsymbol{b}^0 \quad \boldsymbol{b}^1 \quad \boldsymbol{b}^2 \quad \boldsymbol{b}^3\,) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ s & t & -1 & A \end{pmatrix}.$$

Here the parameter a, i.e. $A = 2\cos\frac{2\pi}{a}$, is fixed for the group $\Gamma(a, 2b)$ as a rotational order. The relation

$$r^a = 1 \tag{10.4}$$

holds, indeed. The line of A_0A_1 is pointwise fixed. The parameters s and t will be fixed later by the other relations and some projective freedom. In such a way we attempt to define for our group

$$\begin{gathered}\Gamma(a, 2b) = (r_0, r_1, r\ -\ 1 = r_0^2 = r_1^2 = r^a = (rr_0)^b = r_1rr_1r_0r_1r^{-1})\\ 3 \le a \in \mathbf{N}, \quad 2 \le b \in \mathbf{N},\end{gathered} \tag{10.5}$$

generated by the combinatorial half-turns r_0, r_1 and by the combinatorial rotation r in (10.1), a true projective representations, i.e. the relations in (10.5) will be fulfilled, and only the group theoretical consequences of these relations hold. This will be the extension of the so-called Poincaré algorithm to the projective sphere $\mathcal{PS}^3$ in the sense of [5], not detailed more.

We further continue describing our example. The half-turns

$$r_0 : \begin{pmatrix}\mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{pmatrix} \mapsto \begin{pmatrix}-1 & x & y & y\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{pmatrix}, \quad \begin{matrix}\text{and}\\ \text{similarly}\end{matrix} \quad r_1 : \begin{pmatrix}0 & 0 & 0 & 1\\ v & -1 & u & v\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \tag{10.6}$$

with some simplifications, indeed describe involutive transformations for (10.1). The transformation r_0r has to be a rotation of order $2 \le b \in \mathbf{N}$, which fixes pointwise A_1A_2. Thus we obtain, say in the dual basis $\{\boldsymbol{b}^j\}$,

$$\begin{gathered}r_0r : (\boldsymbol{b}^0 \quad \boldsymbol{b}^1 \quad \boldsymbol{b}^2 \quad \boldsymbol{b}^3) \mapsto (\boldsymbol{b}^0 \quad \boldsymbol{b}^1 \quad \boldsymbol{b}^2 \quad \boldsymbol{b}^3)\begin{pmatrix}-1 & x & y & y\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -s & sx+t & sy+A & sy-1\end{pmatrix}\\ \text{with} \quad B := 2\cos\frac{2\pi}{b} = sy-2, \quad \text{for } 2 \le b; \text{ and } b = 2 \text{ iff } s = y = 0.\end{gathered} \tag{10.7}$$

The equation $r_1r_0r_1 = r^{-1}r_1r$, equivalent to the 5th relation in (10.5), yields the following matrix equation in the dual basis $\{\boldsymbol{b}^j\}$

$$\begin{gathered}\begin{pmatrix}0 & 0 & 1 & 0\\ v+us & -1+ut & uA+v & -u\\ 1 & 0 & 0 & 0\\ tv+A+(tu-1)s & (tu-2)t & (tu-1)A+s+tv & 1-tu\end{pmatrix} =\\ = \begin{pmatrix}0 & 0 & 1 & 0\\ (vx-1)v+vy+u & 1-vx & (vx-1)u+vy+v & v(vx-2)\\ 1 & 0 & 0 & 0\\ xv+y & -x & xy+y & -1+vx\end{pmatrix}.\end{gathered} \tag{10.8}$$

a) First, we consider the cases

$$2 < b, \quad B + 2 = sy > 0. \tag{10.9}$$

Then we obtain a compatibility criterion for tu

$$B + 2 = tu(A + tu - 1)^2, \quad A = 2\cos\frac{2\pi}{a} > -2, \quad B = 2\cos\frac{2\pi}{b} > -2 \tag{10.10}$$

and the solution, with some freedom of a basis change

$$t = 1 = v, \quad x = 2 - u, \quad s = A + u - 1, \quad y = u(A + u - 1). \tag{10.11}$$

Zhuk [8] derived equivalent conditions by absolute trigonometry only for the spherical space $\mathbf{S}^3$ and the hyperbolic space $\mathbf{H}^3$.

To specify the representation above, we look for a polarity (sections 2–4) $(_\star) : \boldsymbol{b}^i \mapsto b^{ij}\mathbf{a}_j$, invariant under r, r_0, r_1 above. This plane $\to$ point correspondence, expressing the orthogonality relations, will be equivalently given by the quadratic form over the dual space $\boldsymbol{V}_4$

$$\begin{aligned}
\xi_i b^{ij}\xi_j := \xi_0\xi_0 - \xi_0(2-u)\xi_1 - \xi_0 u(A+u-1)\xi_2 - \xi_0 u(A+u-1)\xi_3 + \\
+\xi_1\xi_1 - \xi_1 u\xi_2 - \xi_1 u\xi_3 + \xi_2 u\xi_2 - \xi_2 Au\xi_3 + \xi_3 u\xi_3 = \\
= \left[\xi_0 - \frac{1}{2}(2-u)\xi_1 - \frac{1}{2}u(A+u-1)\xi_2 - \frac{1}{2}u(A+u-1)\xi_3\right]^2 + \\
+\frac{1}{4}(4-u)u\left[\xi_1 - \frac{(2-u)(A+u-1)+2}{4-u}\xi_2 - \frac{(2-u)(A+u-1)+2}{4-u}\xi_3\right]^2 + \\
+\frac{u(2-A-u)(A+2)}{4-u}\left[\xi_2 - \frac{2A+u}{2(2-A-u)}\xi_3\right]^2 + \frac{u(A+2)(4-4A-3u)}{4(2-A-u)}\xi_3\xi_3.
\end{aligned} \tag{10.12}$$

Example 10.1. $3 = a = b$, i.e. $A = B = -1$, and the solution of (10.10)

$$u = c_{33} = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2, \quad x = -\frac{\sqrt{5}-1}{2} = 2\cos\frac{3\pi}{5},$$
$$s = \frac{\sqrt{5}-1}{2} = 2\cos\frac{2\pi}{5}, \quad y = \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5},$$

lead to a unique quadratic form of signature $(++++)$ in (10.12). Thus by Section 4, we get the spherical projective metric space $\mathbf{S}^3$, and our simplex has the face angles by (10.12)

$$\beta^{01} = \frac{3\pi}{5}, \quad \beta^{02} = \frac{\pi}{3} = \beta^{03}, \quad \beta^{12} = \frac{\pi}{5} = \beta^{13}, \quad \beta^{23} = \frac{2\pi}{3}.$$

$$\text{E.g.} \quad \cos\beta^{01} = \frac{-b^{01}}{\sqrt{b^{00}\cdot b^{11}}} = \frac{2-c_{33}}{2\sqrt{1\cdot 1}} = \frac{1-\sqrt{5}}{4} \quad \Rightarrow \quad \beta^{01} = \frac{3\pi}{5}$$

serves the face angle at the simplex edge A_2A_3.

We can check that our representation is true, indeed. The face angle $\beta^{23} = \frac{2\pi}{3}$ at A_0A_1 provides $a = 3$ surrounding simplices in $\mathcal{T}$, $\beta^{02} + \beta^{03} = \frac{2\pi}{3}$ at A_1A_3 and A_1A_2 provides $2b = 6$ simplices in $\mathcal{T}$, and $b = 3$ is just the rotational order of the stabilizer of these edges in Γ. These parameters are put in the symbol $\Gamma(a, 2b)$. The angle sum at the 6 Γ-equivalent half-edges in A_0A_2, A_2A_3 and A_3A_0 amounts $2\beta^{13} + 2\beta^{01} + 2\beta^{12} = 2\pi$. Thus the 6 image simplices exactly surround these edges in the tiling $\mathcal{T}$, the edge stabilizer is trivial here as the 5th relation dictates.

We could compute any metric date of our simplex $A_0A_1A_2A_3$, e.g. for the distances $\mathrm{d}(A_jA_k)$ we have

$$\cos[\mathrm{d}(A_jA_k)] = \frac{a_{jk}}{\sqrt{a_{jj} \cdot a_{kk}}} \quad \text{by} \quad (a_{jk}) := (b^{ij})^{-1} \quad \text{from (10.12).}$$

Surprisingly, the other double root of the equation (10.10) does not give any geometric realization.

Examples 10.2–10.8. The parameter pairs as follows

2. $a = 4$, $b = 3$, i.e. $A = 0$, $B = -1$; $1 = u(u-1)^2$, $u = c_{43} := 1.754\,877\,667$;

3. $a = 3$, $b = 4$, i.e. $A = -1$, $B = 0$; $2 = u(u-2)^2$, $u = c_{34} := 2.839\,286\,755$;

4. $a = 5$, $b = 3$, then **5.** $a = 3$, $b = 5$, analogously lead to $(-\;+\;++)$ signatures in (10.12). These interesting Zhuk tilings are metrically realized in the hyperbolic space $\mathbf{H}^3$. All data can be computed from (10.10–12) uniquely.

6. $a = 6$, $b = 3$ leads to the $\mathbf{H}^3$-tiling with regular simplex of face angle $\frac{\pi}{3}$, i.e. all vertices are ideal (at the absolute).

7. $a = 3$, $b = 6$ and **8.** $a = b = 4$ provide also $\mathbf{H}^3$-tilings with simplices of ideal vertices. The face angles are

$$\mathbf{7}: \qquad \beta^{01} = \beta^{23} = \frac{2\pi}{3}, \quad \beta^{02} = \beta^{03} = \beta^{12} = \beta^{13} = \frac{\pi}{6}, \quad \text{resp.}$$

$$\mathbf{8}: \qquad \beta^{01} = \beta^{23} = \frac{\pi}{2}, \quad \beta^{02} = \beta^{03} = \beta^{12} = \beta^{13} = \frac{\pi}{4}, \quad \text{uniquely.}$$

b) Second, let us consider the cases (not discussed by Zhuk, in general)

$$\text{(10.13)} \qquad \begin{gathered} b = 2, \quad 0 = B + 2 = s = y, \quad 3 \le a \quad -2 \le A = 2\cos\frac{2\pi}{a} < 2 \\ \text{and the solution of (10.8)} \\ t = 1 = v, \quad u = 1 - A, \quad x = 1 + A. \end{gathered}$$

It turns out that the invariant polarity, or equivalenty, the invariant quadratic form over $\boldsymbol{V}_4$ can be obtained from (10.12) just by substitution $u = 1 - A$. The coefficients in the diagonal form of (10.12) will then be

$$\text{(10.14)} \qquad 1; \quad \frac{1}{4}(3+A)(1-A); \quad \frac{(1-A)(A+2)}{3+A}; \quad \frac{1}{4}(A-1)^2(A+2).$$

i) For $3 \le a < 6$ i.e. $-1 \le A < 1$ the signature by (10.14) is $(+\,+\,++)$ and we get $\mathbf{S}^3$-tilings with interesting Hopf–Seifert fibrations.
ii) For $a = 6$, i.e. $A = 1$ the signature is $(0\,0\,0\,+)$ and by Section 9 we get a simplex tiling $\mathcal{T}$ in the **Nil** space with group $\Gamma(a = 6,\ 2b = 4)$ and Seifert-fibration.
iii) For $6 < a$, i.e. $1 < A < 2$, the signature by (10.14) is $(-\,-\,++)$, and we get simplex tilings in the space $\widetilde{\mathbf{SL_2R}}$ by Section 6, again with Seifert-fibrations.

In general, for the parameters $(a; b) = (a; 2)$, $3 \le a$

$$\Gamma(a;\ 2b = 4) = \big(r_0, r_1, r\ -\ 1 = r_0^2 = r_1^2 = r^a = (rr_0)^2 = r_1rr_1r_0r_1r^{-1}\big) \tag{10.15}$$

is the presentation, specialized from (10.5). As a consequence, the transformation

$$t := (r_1r_0)^3 = (r_1rr_0)^2 \tag{10.16}$$

then generates an invariant cyclic subgroup $\langle t\rangle$ in Γ and the factor group $\Gamma/\langle t\rangle$ has a presentation

$$\bar{1} = \bar{r}^a = \bar{s}^3 = \bar{h}^2 = \bar{r}\bar{s}\bar{h} \quad \text{with} \quad s := r_0r_1, \quad h := r_1rr_0. \tag{10.17}$$

Thus we get Seifert fibred orbifolds over the 2-orbifolds of signature

$$(+, 0;\ [2, 3, a];\ \{\ \}), \tag{10.18}$$

This is a 2-sphere with 3 cone points of orders 2, 3, a, respectively. The 2-orbifolds are of $\mathbf{S}^2$ for $a = 3, 4, 5$; of $\mathbf{E}^2$ for $a = 6$; of $\mathbf{H}^2$ for $6 < a$, respectively.

The skew transformation t by (10.16), acting on points, is described by

$$t\ :\ \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \mapsto \begin{pmatrix} -1 & A{+}1 & 0 & 0 \\ (A{-}2)(A{+}1) & -A^3{+}3A{+}1 & 0 & 0 \\ -(A{+}1) & A(A{+}1) & A & -A^2+1 \\ -(A{+}1) & A{+}1 & A^2{-}1 & -A(A^2{-}2) \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{10.19}$$

with conjugated complex eigenvalues:

$$\begin{gathered} \lambda = \frac{1}{2}(3A - A^3 \pm \mathrm{i}(A^2-1)\sqrt{4 - A^2} =: \cos\psi \pm \mathrm{i}\sin\psi. \\ \text{Now} \quad \cos\psi := \cos\frac{2\pi}{a}\left(3 - 4\cos^2\frac{2\pi}{a}\right) = -\cos\frac{6\pi}{a} = \cos\left(\pi - \frac{6\pi}{a}\right), \\ \text{and} \quad \psi = \pi\frac{a-6}{a} \quad \text{can be taken.} \end{gathered} \tag{10.20}$$

Examples 10.9–10.11. For $a = 3, 4, 5$ we have spherical simplices with face angles β^{ij} at the edges A_kA_l $(\{i, j, k, l\} = \{0, 1, 2, 3\})$, obtained from (10.12) with $u = 1 - A = 1 - 2\cos\frac{2\pi}{a}$ as follows

$$\beta^{23} = \frac{2\pi}{a}, \quad \beta^{02} = \beta^{03} = \frac{\pi}{2}, \quad \cos\beta^{01} = \frac{1}{2} + \cos\frac{2\pi}{a},$$

$$\cos\beta^{12} = \cos\beta^{13} = \frac{1}{2}\sqrt{1 - 2\cos\frac{2\pi}{a}}; \quad \text{i.e. for } a = 3,\ 4,\ 5 \text{ we get}$$

$$\beta^{01} = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5} \quad \text{and} \quad \beta^{12} = \beta^{13} = \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{5}, \quad \text{respectively.}$$

The skew transformation t has ψ as the corresponding angle parameter:

$$a=3\,:\,\psi=\pi,\quad \text{i.e.}\quad t=-\mathrm{id},\quad \text{the order}\quad |\Gamma|=2\cdot 12=24;$$
$$a=4\,:\,\psi=\frac{\pi}{2},\quad |\Gamma|=4\cdot 24=96,$$
$$a=5\,:\,\psi=\frac{\pi}{5},\quad |\Gamma|=10\cdot 60=600,$$

and we get interesting Hopf–Seifert fibre orbifolds of spherical type.

Example 10.12. If $a=6$, i.e. $A=1$, our **Nil**-orbifold fibers over the Euclidean 2-orbifold

$$(+,0;\,[2,3,6];\,\{\ \})=\mathbf{E}^2/\mathbf{p6}$$

as a special case of (10.18), **p6** is a plane crystallographic group of $\mathbf{E}^2$. Now (10.20) provides $\lambda=1$ as eigen value. Our polarity by (10.12) is of signature $(0\,0\,0\,+)$ now

$$(_\star)\,:\,(\,\boldsymbol{b}^0\quad \boldsymbol{b}^1\quad \boldsymbol{b}^2\quad \boldsymbol{b}^3\,)\mapsto \begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}\mathbf{a}_0\\\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix}.$$

Moreover, our generators r_0, r_1 and r have the common invariant plane $\boldsymbol{b}^0\cdot 1+\boldsymbol{b}^1\cdot 1+\boldsymbol{b}^2\cdot 1+\boldsymbol{b}^3\cdot 1=:\boldsymbol{b}^{0'}$, chosen as ideal plane $\omega^\infty=(\boldsymbol{b}^{0'})$ and they have the common fixed point $-\mathbf{a}_0+\mathbf{a}_1=:\mathbf{a}_{1'}$ the pole at our polarity as a new ideal point $A_{1'}^\infty$ for our new coordinate simplex. We define the basis change

$$(\,\boldsymbol{b}^{0'}\quad \boldsymbol{b}^{1'}\quad \boldsymbol{b}^{2'}\quad \boldsymbol{b}^{3'}\,)=(\,\boldsymbol{b}^0\quad \boldsymbol{b}^1\quad \boldsymbol{b}^2\quad \boldsymbol{b}^3\,)\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}\mathbf{a}_{0'}\\\mathbf{a}_{1'}\\\mathbf{a}_{2'}\\\mathbf{a}_{3'}\end{pmatrix}=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{pmatrix}\begin{pmatrix}\mathbf{a}_0\\\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix}.$$

Then we can express our generators and the fibre transformation t for the points of our affine **Nil** space

$$\mathcal{A}^3:=\{[\mathbf{x}]\in\mathcal{PS}^3\,:\,\mathbf{x}\boldsymbol{b}^{0'}=:x^{0'}=1\}=:\mathbf{Nil}$$

$$r_0\,:\,\begin{pmatrix}\mathbf{a}_{0'}\\\cdots\\\mathbf{a}_{1'}\\\mathbf{a}_{2'}\\\mathbf{a}_{3'}\end{pmatrix}\mapsto\begin{pmatrix}1&:&2&0&0\\\cdots&\cdot&\cdots&\cdots&\cdots\\0&:&-1&0&0\\0&:&-2&0&1\\0&:&-2&1&0\end{pmatrix}\begin{pmatrix}\mathbf{a}_{0'}\\\cdots\\\mathbf{a}_{1'}\\\mathbf{a}_{2'}\\\mathbf{a}_{3'}\end{pmatrix},\quad r_1\,:\,\begin{pmatrix}1&:&0&0&1\\\cdots&\cdot&\cdots&\cdots&\cdots\\0&:&-1&0&0\\0&:&0&1&-1\\0&:&0&0&-1\end{pmatrix},$$

$$r\,:\,(R_{j'}^{k'})=\begin{pmatrix}1&:&0&0&0\\\cdots&\cdot&\cdots&\cdots&\cdots\\0&:&1&0&0\\0&:&1&1&-1\\0&:&0&1&0\end{pmatrix},\quad t\,:\,\begin{pmatrix}1&:&2&0&0\\\cdots&\cdot&\cdots&\cdots&\cdots\\0&:&1&0&0\\0&:&0&1&0\\0&:&0&0&1\end{pmatrix}.$$

The fibre transformation t is just a translation in **Nil** by the vector $2{\cdot}\mathbf{a}_{1'}$. Now we can see that r is a 6-fold rotation about A_0A_1, furthermore r_1r_0 is a 3_1-screw motion about the axis $(1', x^{1'}, \frac{1'}{3}, \frac{1'}{3})$ with $\frac{2'}{3}\mathbf{a}_{1'}$ translation, and r_1rr_0 is a 2_1-screw motion about the axis $(1', x^{1'}, 0', \frac{1'}{2})$ with $1'{\cdot}\mathbf{a}_{1'}$ translation.

Examples 10.13–10.∞. If $6 < a$, i.e. $1 < A$, we get an $\widetilde{\mathbf{SL_2R}}$-orbifold for each a, which fibers over the corresponding hyperbolic 2-orbifold (10.18). Our invariant polarity by (10.12) is of signature $(- \; - + +)$ from (10.14). Now the fibering skew transformation t by (10.19) has the eigen values by (10.20). We can proceed the above computations on the projective model of $\widetilde{\mathbf{SL_2R}}$, i.e. the covering space $\widetilde{\mathcal{H}}$ of the unparted hyperboloid solid

$$\mathcal{H} = \{[\mathbf{x}] \in \mathcal{PS}^3 \, : \, \langle \mathbf{x}, \mathbf{x} \rangle := x^i a_{ij} x^j > 0\} \quad \text{in } \mathcal{PS}^3$$
$$\text{where } \; (a_{ij}) \; \text{ is the inverse of } \; (b^{ij}) \; \text{ by (10.12)},$$

and the action of the cyclic group $\langle t \rangle$ is extended as universal covering transformations of $\mathcal{H}$. We know from Section 6 that we obtain the universal covering space $\widetilde{\mathcal{H}}$, fibered by $\mathbf{R}$-lines over the hyperbolic plane $\mathbf{H}^2$. The only modification is that we have chosen "$>$" in the definition of $\mathcal{H}$, since A_0, A_1, A_2, A_3 lie in this domain now (compare with (6.3)). From (10.19) we could express a new basis $\{\mathbf{a}_{i'}\}$ where the fibering transformation

$$t \; : \; \begin{pmatrix} \mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \\ \mathbf{a}_{3'} \end{pmatrix} \mapsto \begin{pmatrix} \cos\psi & \sin\psi & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \\ \mathbf{a}_{3'} \end{pmatrix}$$

acts on $\widetilde{\mathcal{H}}$. The parameter $\psi = \pi \dfrac{a-6}{a}$ by (10.20) shows (e.g. for $a = 11$) that the universal covering $\widetilde{\mathcal{H}}$ is necessary, indeed. Further computations would be to express r_1r_0 and r_1rr_0 in the new basis $\{\mathbf{a}_{i'}\}$, related to t above, like in the previous Example 12.

I thank István Prok for typesetting the text in TEX.

I am indebted to Professor Benno Klotzek and to the other geometrician colleagues in Potsdam for the useful conversations and for their hospitality in October 1995. This work has been made in the framework of our collaboration.

The author thanks the Referee for his/her advices.

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Received February 13, 1996; revised version October 8, 1996