Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 38 (1997), No. 2, 377-384.

On-line Packing Sequences of Segments, Cubes and Boxes

Marek Lassak
Instytut Matematyki i Fizyki ATR,
ul. Kaliskiego 7, 85-796 Bydgoszcz, Poland
e-mail: lassak@atr.bydgoszcz.pl

Abstract. We present an on-line algorithm for packing sequences of cubes of edge lengths at most 1 in a d-dimensional box, and, in particular, in the cube I_s^d of edge length s. The algorithm guarantees that if a successive cube from a sequence cannot be packed, then the volume of the unfilled part of I_s^d is at most a number of the order of magnitude $s^{d-d/(d+2)}$. As $d \to \infty$, this order of magnitude approaches the analogous order s^{d-1} in the best known off-line method. We also show a similar method for on-line packing sequences of boxes. Moreover, we prove that if D is a box with integer edge lengths and if $q \geq 2$ is an integer, then every sequence of boxes of edge lengths from the set $\{q^0, q^{-1}, q^{-2} \dots\}$ and of total volume at most $\mathrm{Vol}(D) - q[(\frac{q}{q-1})^{d-1} - 1]$ can be on-line q-adic packed in D. If all the boxes are assumed to be cubes, then the packing is possible provided the total volume is at most $\mathrm{Vol}(D)$.

MSC 1991: 52C17

We say that a sequence $C_1, C_2, \dots$ of convex bodies of Euclidean d-space E^d *can be packed* in a set $C \subset E^d$ if there are rigid motions σ_i such that the sets $\sigma_i C_i$, where $i = 1, 2, \dots$, have pairwise disjoint interiors and they are subsets of C. A packing process is called *translative* if only translations are allowed as the rigid motions.

By an *on-line packing* we mean a packing process in which we are given every C_i, where $i > 1$, only after C_{i-1} has been packed. This restriction means that each set C_i must be assigned its place before we learn the next one, and that the placement cannot be changed afterwards.

By a *box* in E^d we understand any set of the form

$$\{(x_1, \dots, x_d);\ r_j \leq x_j \leq s_j \ \text{for} \ j = 1, \dots, d\}.$$

The number $s_d - r_d$ is called the *height* of this box, and the set of points of this box satisfying $x_d = r_d$ is called the *bottom.* If $r_1, \dots, r_d, s_1, \dots, s_d$ are positive integers, then this box is called an *integer box.* If $r_1 = \ldots = r_d = 0$, this box is denoted by the symbol $B(s_1, \dots, s_d)$. The symbol I_s^d stands for the d-dimensional cube $B(s, \dots, s)$.

1. On-line q-adic packing sequences of segments, boxes and cubes

In response to a question of Kuperberg [3], the paper [2] considers the q-adic covering a segment by segments and applies it for proving a result about on-line covering of the unit cube by sequences of cubes of edge lengths at most 1. Here we propose an analogous tool for packing instead of covering.

Let $w \geq 1$ and $q \geq 2$ be integers. We say that an algorithm of packing the segment $[0, w]$ by a sequence of segments S_i of corresponding lengths $\delta_i \in \{q^0, q^{-1}, q^{-2}, \dots\}$ is *q-adic* if the rigid motions σ_i are restricted so that every segment $\sigma_i S_i$ is of the form $[c_i\delta_i, (c_i + 1)\delta_i]$ with $c_i \in \{0, \dots w\delta_i^{-1} - 1\}$.

The following on-line q-adic algorithm of *packing in the first free fitting segment* is very efficient. When we learn S_i, we find the smallest $a_i \in \{0, \dots, w{\delta_i}^{-1} - 1\}$ such that the interior of the segment $[a_i\delta_i, (a_i + 1)\delta_i]$ remains empty and we pack S_i there. If such a segment does not exist, we stop the packing process.

Lemma 1. *Let $q \geq 2$ and $w \geq 1$ be integers. Assume that we are providing the packing of a sequence of segments of lengths from the set $\{q^0, q^{-1}, q^{-2}, \dots\}$ in the segment $[0, w]$ by the method of the first free fitting segment. If the process of packing is stopped, then the total length of the segments packed up to this moment is greater than $w - 1$. Every sequence of segments of total length at most w can be packed in $[0, w]$.*

Proof. Assume that the process of packing is stopped when we wish to pack a successive segment from the sequence. Let the length of this segment be q^{-n}. Since every segment from the sequence was packed in the first free fitting segment, we have at most $q - 1$ maximal (by inclusion) free subsegments of $[0, w]$ of every length $q^{-n-1}, q^{-n-2}, \dots$ when the packing process is stopped. What is more, since a finite number of segments has been packed, the number of those maximal free subsegments is finite. Thus the sum of lengths of those maximal free subsegments is smaller than $(q - 1)(q^{-n-1} + q^{-n-2} + \dots) = q^{-n}$. Consequently, the total length of the segments packed up to this moment is greater than $w - q^{-n}$. We see that the total length of the packed segments is greater than $w - 1$. This gives the first statement of our lemma. Since the just obtained segment is of length q^{-n}, we see that the total length of segments in our sequence is greater than w. By inversion we obtain the following implication: if the total length of segments is at most w, then the process of packing does not stop after packing only a finite number of segments. This implication is nothing else but the last statement of the lemma. $\square$

Let $q \geq 2$ be an integer. By a *q-adic packing* of a sequence of boxes whose edge lengths are from the set $\{q^0, q^{-1}, q^{-2}, \dots\}$ we mean a packing in which every box $B(y_1, \dots, y_d)$ from the sequence is moved so that the following two conditions are satisfied: (a) after packing, the box is a translation of $B(v_1, \dots, v_d)$, where $v_1, \dots, v_d$ is a permutation of $y_1, \dots, y_d$,

(b) for every $j \in \{1, \ldots, d\}$, the projection of $B(v_1, \ldots, v_d)$ on the j-th coordinate axis is a segment whose both endpoints are multiples of v_j.

We will pack the boxes in the union of a number of unit cubes. This is convenient for applications. Assume for convenience that this union of w cubes has the form

$$G_w^d = \{(x_1, \ldots, x_d);\ 0 \le x_j \le 1 \text{ for } j = 1, \ldots, d-1, \text{ and } 0 \le x_d \le w\}.$$

Below we describe an on-line translative q-adic *method of strips* for q-adic packing of sequences of boxes. We apply induction in respect to the dimension.

If $d = 1$, we use the algorithm of the first free fitting segment.

Assume that the method is described in E^{d-1}, where $d \ge 2$. We give the description for E^d. For every $r \in \{0, 1, 2, \ldots\}$ we dissect G_w^d into wq^r horizontal strips of height q^{-r}.

Every box is packed in a strip of heigth equal to the heigth of this box. Every strip properly containing such a strip or properly contained in such a strip is closed for further packing. When we are given a box $B(y_1, \ldots, y_d)$ from our sequence, we take into account only the strips of height y_d which are not yet closed for packing. We consider the bottoms of those strips and the bottoms of boxes of height y_d that were earlier packed in those strips. We apply the $(d-1)$-dimensional procedure to those bottoms in order to find the proper position for packing the bottom of $B(y_1, \ldots, y_d)$. We pack $B(y_1, \ldots, y_d)$ so that its bottom moves to this position.

Lemma 2. *Every sequence of boxes of edge lengths from the set $\{q^0, q^{-1}, q^{-2}, \ldots\}$ and of total volume at most $w - q[(\frac{q}{q-1})^{d-1} - 1]$ can be packed in w unit cubes by the method of strips.*

Proof (by induction). Put $z^d(q) = q[(\frac{q}{q-1})^{d-1} - 1]$. If $d = 1$, then Lemma 2 holds true by Lemma 1. Assume that our lemma is true in E^{d-1}. Assume that the d-dimensional process of packing is stopped at a moment when we wish to pack a successive box from the sequence. Let the height of this box be q^{-b}. Denote by $\gamma(a)$ the number of open strips of height q^{-a}, where $a = 0, 1, 2, \ldots$. Since every box was packed in the "lowest" non-closed strip, by the inductive hypothesis we conclude that the total $(d-1)$-dimensional volume of the bases of the given boxes of height q^{-b} (including the last box that cannot be packed) is greater than $\gamma(b) - z^{d-1}(q)$. Thus the total volume of those boxes is greater than $q^{-b}(\gamma(b) - z^{d-1}(q))$. Similarly, we conclude that if $a \ne b$, then the total volume of the packed boxes of height q^{-a} is greater than $q^{-a}(\gamma(a) - z^{d-1}(q) - 1)$. We see that the total volume of all given boxes (including the last box that cannot be packed) is greater than $\sum_{a=0}^{\infty}[q^{-a}(\gamma(a) - z^{d-1}(q) - 1)] + q^{-b} = \sum_{a=0}^{\infty} \gamma(a) q^{-a} - (z^{d-1}(q) + 1) \sum_{a=0}^{\infty} q^{-a} + q^{-b} = w - q^{-b} - (z^{d-1}(q) + 1)\frac{q}{q-1} + q^{-b} = w - z^d(q)$. A contradiction. □

For the method of strips the estimate given in Lemma 2 is sharp. For $d = 2$ this follows from the example of the finite sequence of boxes whose first $t + 1$ boxes are $B(q^{-t}, q^0), B(q^{-t}, q^{-1}), \ldots, B(q^{-t}, q^{-t})$, and the remaining $wq^t - (q^1 + \ldots + q^t)$ boxes are of the form $B(q^0, q^{-t})$. This sequence cannot be packed even though its total area is smaller than $w - \frac{q}{q-1} + 2q^{-t}\frac{q}{q-1}$. This value tends to $w - \frac{q}{q-1} = w - q[\frac{q}{q-1} - 1]$ as t tends to ∞. A construction of an analogous example in d-space is left to the reader. The author does not know if the estimate of Lemma 2 can be improved by an application of a different on-line translative q-adic packing method. The question of possible improvements when we drop one or both the first two of those assumptions is open.

Let $u > 0$. Cubes of edge lengths of the form $u2^{-k}$, where $k \in \{0, 1, 2, \ldots\}$, are called *$u$-standard*. We will apply the 2^d-adic packing a segment by segments in order to describe a method of on-line packing of sequences of u-standard cubes in a number of cubes of edge length u, called *u-stores*. We denote those u-stores by $A_1, \ldots, A_w$ and we assume that they have pairwise disjoint interiors. We can represent every u-store A_g as $\tau_g(I_u^d)$, where τ_g is a translation. Let ϕ_k be a bijection between the family of segments of the form $Y_k(h) = [h2^{-dk}, (h+1)2^{-dk}]$, where $h = 0, \ldots, w2^{dk} - 1$, and the family of u-standard cubes

$$\tau_g\{(x_1, \ldots, x_d);\ uh_j 2^{-k} \leq x_j \leq u(h_j + 1)2^{-k} \ \text{ for } \ j = 1, \ldots, d\},$$

where $h_1, \ldots, h_d \in \{0, \ldots, 2^k - 1\}$ and $g \in \{1, \ldots, w\}$. Let ϕ_k be given for $k = 0, 1, 2, \ldots$ such that the following two conditions hold:

(i) $\phi_0([h, h+1]) = A_{h+1}$ for $h = 0, \ldots, w - 1$,

(ii) if $Y_m(a) \subset Y_n(b)$, then $\phi_m(Y_m(a)) \subset \phi_n(Y_n(b))$.

We describe the packing of a sequence of u-standard cubes applying a simultaneous auxiliary process of 2^d-adic packing a sequence of segments in the segment $[0, w]$. Assume that we are given a u-standard cube K from our sequence. If its edge length is $u2^{-k}$, then we pack a segment S of length 2^{-dk} in $[0, w]$ according to the 2^d-adic method of the first free fitting segment. Let σS denote the position of S after it is packed. Then $\phi_k(\sigma S)$ is the rigth place in $A_1 \cup \ldots \cup A_w$ for putting our cube K. We call this approach *packing in the first free fitting subcube.*

It is clear that the method described above is on-line, translative and q-adic.

Since a cube of volume $u^d 2^{-dk}$ always corresponds to a segment of length 2^{-dk} in our bijection, Lemma 1 implies the following statement.

Lemma 3. *Let $A_1, \ldots, A_w$ be u-stores with pairwise disjoint interiors. Assume that we are packing a sequence of u-standard cubes in $A_1 \cup \ldots \cup A_w$ by the method of the first free fitting subcube. If the process of packing is stopped, then the total volume of the cubes packed up to this moment is greater than $(w-1)u^d$. Every sequence of u-standard cubes of total volume at most wu^d can be packed in $A_1 \cup \ldots \cup A_w$.*

Lemma 3 remains valid for an analogous method with an arbitrary integer $q \geq 2$ instead of 2. If $u = 1$ in Lemma 3, we get the 2-adic packing of cubes.

We need the method of the first free fitting subcube for the description of the algorithm presented in Part 2. Lemma 3 is needed for the proof of Theorem 2.

We apply Lemma 2, and Lemma 3 with $u = 1$ for the on-line q-adic packing of sequences of cubes and boxes in an integer box D. For this purpose we present D as the union of unit cubes. We obtain the following theorem.

Theorem 1. *Let D be an integer box and let $q \geq 2$ be an integer. An arbitrary sequence of boxes of edge lengths from the set $\{q^0, q^{-1}, q^{-2} \ldots\}$ whose total volume is at most*

$\mathrm{Vol}(D) - q[(\frac{q}{q-1})^{d-1} - 1]$ *permits a translative on-line q-adic packing in D. If all the boxes are assumed to be cubes, the packing is possible when the total volume is at most* $\mathrm{Vol}(D)$.

2. On-line packing of sequences of cubes in a box

Below we present a method of packing sequences of cubes of edge lengths at most 1 in the box $B(s_1, \dots, s_d)$, where $s_1 \geq 1, \dots, s_d \geq 1$. Let $p_1 \geq s_1, \dots, p_d \geq s_d$ be integers. Using $p_j - 1$ hyperplanes perpendicular to the j-th coordinate axis, where $j = 1, \dots, d$, we divide the box into $\prod_{j=1}^d p_j$ congruent subboxes.

Let m be a positive integer. The number $2^{-d/m}$ is called the *factor of standardization*. We define the family of *standard cubes* as the union of the families of $2^{-t/m}$-standard cubes for $t = 0, \dots, m-1$. Every cube from our sequence of cubes is enlarged to the smallest possible standard cube. The volume increases less than $2^{d/m}$ times. It remains to explain how to pack the corresponding sequence of standard cubes.

For every $t \in \{0, \dots, m-1\}$, the subsequence of all $2^{-t/m}$-standard cubes of the sequence of standard cubes is packed in a number of $2^{-t/m}$-stores. We apply here the method of the first free fitting subcube with $u = 2^{-t/m}$. For every $t \in \{0, \dots, m-1\}$, the set of the $2^{-t/m}$-stores is successively constructed during the packing process. We open a new portion of $2^{-t/m}$-stores each time when we obtain a $2^{-t/m}$-standard cube K, and if in the existing $2^{-t/m}$-stores there is not enough empty room for packing K according to the method of the first free fitting subcube. In particular, we do so when no $2^{-t/m}$-store has been opened yet. A new portion of $2^{-t/m}$-stores is opened in any subbox not used yet for storing cubes. We find there a greatest possible box of the form $\tau(B(z_1 2^{-t/m}, \dots, z_d 2^{-t/m}))$, where $z_1, \dots, z_d$ are positive integers, and where τ is a translation. We dissect this box into $\prod_{j=1}^d z_j$ cubes of edge length $2^{-t/m}$ whose interiors are pairwise disjoint. They are the required new $2^{-t/m}$-stores. We stop the packing process if a succeding cube cannot be packed according to the described method, i.e. if no new free subbox exists when we wish to open a new portion of $2^{-t/m}$-stores.

Theorem 2. *Every sequence of cubes of edge lengths at most* 1 *and total volume at most*

$$2^{-\frac{d}{m}} \Big(\prod_{j=1}^{d} p_j - m + 1\Big) \prod_{j=1}^{d} \big(\frac{s_j}{p_j} - 1\big) \tag{1}$$

can be on-line packed in the box $B(s_1, \dots, s_d)$ by the method of dividing the box into $\prod_{j=1}^d p_j$ congruent subboxes and by using m types of stores.

Proof. Assume that we cannot pack a successive cube from the sequence. Let the smallest standard cube containing this cube be a $2^{-n/m}$-standard cube, where $n \in \{0, \dots, m-1\}$.

Let $\xi(t)$ denote the number of the subboxes into which we have packed the $2^{-t/m}$-standard cubes during the whole packing process. Let $t \in \{0, \dots, m-1\}$. Putting $u = 2^{-t/m}$ and $w = \xi(t) \prod_{j=1}^d z_j$ in Lemma 3, we see that the volume of the packed $2^{-t/m}$-standard cubes is greater than $(\xi(t) - 1) \prod_{j=1}^d (z_j 2^{-t/m})$. Since $s_j/p_j - 1 < z_j 2^{-n/m}$ for

$j = 1, \dots, d$, this volume is greater than $(\xi(t) - 1) \prod_{j=1}^{d} (s_j/p_j - 1)$. Taking all such volumes for $t = 0, \dots, m-1$ and adding the volume of the cube that cannot be packed we obtain total volume greater than $(\prod_{j=1}^{d} p_j - m + 1) \prod_{j=1}^{d} (s_j/p_j - 1)$. Since the factor of standardization is $2^{-d/m}$, the total volume of the given cubes (including the last box that cannot be packed) is greater than (1). A contradiction. □

Theorem 2 can be reformulated as follows: *if the packing process is stopped, then the total volume of the given cubes is greater than the quantity in* (1).

3. On-line packing sequences of cubes in a cube

We intend to discuss the efficiency of the method presented in the preceding part in the special case when $B(s_1, \dots, s_d)$ is the cube I_s^d divided into p^d subcubes. Our aim is to choose p and m in order to maximize, asymptotically, as $s \to \infty$, the total volume of cubes of edge lengths at most 1 that can be packed in I_s^d.

If for functions $f(s)$ and $g(s)$ we have $\lim_{s\to\infty} f(s)/g(s) = 1$, we write $f(s) \overset{s}{\sim} g(s)$. If a packing method ensures a packing of I_s^d by every sequence of boxes of edge lengths at most 1 and of total volume v_s^d such that $v_s^d \overset{s}{\sim} s^d$, as $s \to \infty$, then we take into account the number $u_s^d = s^d - v_s^d$ which estimates the volume which remains unfilled in I_s^d. We get a tool for comparing the efficiency of the packing methods which fulfil $v_s^d \overset{s}{\sim} s^d$.

The formula from Theorem 2 has now the following simpler form

$$v_s^d(m,p) = 2^{-\frac{d}{m}} \Big(p^d - m + 1\Big) \big(\frac{s}{p} - 1\big)^d = 2^{-\frac{d}{m}} \Big(1 - \frac{m-1}{p^d}\Big) \big(s - p\big)^d.$$

If we properly choose p and m with regard to d and s, then $v_s^d(m,p)$ is of the order of magnitude s^d with factor 1; for instance, $v_s^d(\lfloor\sqrt{s}\rfloor, \lfloor\sqrt{s}\rfloor) \overset{s}{\sim} s^d$. Observe that if p or m are bounded from above and if $s \to \infty$, then $v_s^d(m,p)$ cannot be of the order of magnitude s^d with factor 1. Thus we will assume that $m \to \infty$ and $p \to \infty$ provided $s \to \infty$. Hence we do not make our consideration narrower assuming that $m \ge (\log 2)ed$. This assumption means that m can be presented as $(\log 2)dk/\log k$, where $k \ge e$. Obviously, $2^{-d/m} = k^{-1/k}$. The volume of the unfilled space in I_s^d is at most $u_s^d(m,p) = s^d - v_s^d(m,p)$. We have

$$u_s^d(m,p) = s^d - k^{-1/k} \Big[1 - \frac{1}{p^d}\Big(\frac{(\log 2)dk}{\log k} - 1\Big)\Big] \big(s - p\big)^d. \tag{2}$$

If we write $k^{-1/k}$ as $1 - (1 - k^{-1/k})$ and if we take into account that $k(1 - k^{-1/k}) \overset{k}{\sim} \log k$, then by exchanging $k^{-1/k}$ for $1 - (\log k)/k$ in (2) we observe that the asymptotic behaviour of $u_s^d(m,p)$, as $s \to \infty$, depends upon the expression

$$\Big[\frac{\log k}{k} + (\log 2)d\frac{1}{p^d}\frac{k}{\log k}\Big] s^d + dps^{d-1}. \tag{3}$$

If we take $p = \lfloor \lambda s^{2/(d+2)} \rfloor$ and $k = \mu d^{1/2} p^{d/2} \log p$, where λ and μ are some constants, then the maximal order of magnitude of the components of (3) is minimized and (3) becomes a constant times $s^{d-d/(d+2)}$. We omit here a tedious calculation for finding the constants λ and μ which mimimize (3) asymptotically as $s \to \infty$. They are $\lambda = [(\log 2)d]^{1/(d+2)}$ and $\mu = \frac{1}{2}\sqrt{d/(\log 2)}$. Substituting p and k with those values of λ and μ in (3) we conclude that $u_s^d \overset{s}{\sim} cs^{d-d/(d+2)}$, where $c = (d+2)(d\log 2)^{1/(d+2)}$. Clearly, k determines $m = \lfloor d(\log 2)k/\log k \rfloor$.

From the above considerations we obtain the following theorem.

Theorem 3. *Every sequence of cubes of edge lengths at most* 1 *whose total volume is at most* $2^{-d/m}(p^d - m + 1)(\frac{s}{p} - 1)^d$ *can be on-line packed in the cube* I_s^d *by the method of dividing* I_s^d *into* p^d *congruent subcubes and by using* m *types of stores. If we properly choose* m *and* p *with regard to* d *and* s*, then this estimate is of the order of magnitude* s^d *with factor* 1*, and* $u_s^d \overset{s}{\sim} cs^{d-d/(d+2)}$*, where* $c = (d+2)(d\log 2)^{1/(d+2)}$.

Corollary. *The above method of packing a sequence of cubes of edge lengths at most* 1 *in* I_s^d *guarantees that if a succeding cube cannot be packed, then the volume of the unfilled part of* I_s^d *is at most a number of the order of magnitude* $s^{d-d/(d+2)}$ *with factor* $(d+2)(d\log 2)^{1/(d+2)}$.

We have $(d+2)(d\log 2)^{1/(d+2)} \overset{d}{\sim} d$ and $s^{d-d/(d+2)} \overset{d}{\sim} s^{d-1}$. Thus, if $s \to \infty$ and $d \to \infty$, then the estimate of the volume u_s^d that remains unfilled when applying Theorem 3 approaches the analogous estimate ds^{d-1} in the off-line packing method of Groemer [1]. The order of magnitude of u_s^d in Theorem 3 is smaller than the corresponding order of magnitude $s^{d-d/(2d-1)}$ in the method of [4] for $d > 3$, and they are equal for $d = 3$.

4. On-line packing sequences of boxes

Natural questions are if the method considered in Parts 2 and 3 can be generalized for packing boxes instead of cubes, and if the efficiency of such a generalized method is again better than the efficiency of the method from [4]. The answer to the first question is positive. But the order of magnitude of the unfilled space in I_s^d appears to be sligthly higher than that from [4]. Nevertheless, for the sake of the completness of the paper, we shortly describe this algorithm.

Let $u_1, \ldots u_d$ be positive reals. Boxes of the form $B(u_1 2^{-r_1}, \ldots, u_d 2^{-r_d})$, where $r_1, \ldots, r_d \in \{0, 1, 2, \ldots\}$, are called $(u_1, \ldots, u_d)$-*standard*. There is a homothety χ such that $\chi(B(2^{-r_1}, \ldots, 2^{-r_d})) = B(u_1 2^{-r_1}, \ldots, u_d 2^{-r_d})$ for every $r_1, \ldots, r_d \in \{0, 1, 2, \ldots\}$. According to the above homothety and to the method of strips, we define an auxiliary algorithm for on-line packing sequences of $(u_1, \ldots, u_d)$-standard boxes in a number of translations of $B(u_1, \ldots, u_d)$ (with pairwise disjoint interiors) called $(u_1, \ldots, u_d)$-*stores*. Applying Lemma 2 with $q = 2$ we obtain the following lemma analogous to Lemma 3.

Lemma 4. *Let* $A_1, \ldots, A_w$ *be* $(u_1, \ldots, u_d)$*-stores with pairwise disjoint interiors. If the process of packing of a sequence of* $(u_1, \ldots, u_d)$*-standard boxes in* $A_1 \cup \ldots \cup A_w$ *by the above auxiliary algorithm is stopped, then the total volume of the packed boxes is greater* *than* $(w-2^d+1)\prod_{j=1}^d u_j$*. Every sequence of* $(u_1,\ldots,u_d)$*-standard boxes whose total volume is at most* $(w-2^d+2)\prod_{j=1}^d u_j$ *can be packed in* $A_1\cup\ldots\cup A_w$.

Here is the promised generalized method of packing sequences of boxes of edge lengths at most 1 in $B(s_1,\ldots,s_d)$. By *standard boxes* we mean the $(2^{-t_1/m},\ldots,2^{-t_d/m})$-standard boxes, where $t_1,\ldots,t_d\in\{0,\ldots,m-1\}$. Every box is packed together with the smallest possible standard box containing it. The volume enlargement factor is smaller than $2^{d/m}$. For every $t_1,\ldots,t_d\in\{0,\ldots,m-1\}$, we pack the subsequence of all $(2^{-t_1/m},\ldots,2^{-t_d/m})$-standard boxes in a number of $(2^{-t_1/m},\ldots,2^{-t_d/m})$-stores. We apply here the auxiliary algorithm. For every $t_1,\ldots,t_d\in\{0,\ldots,m-1\}$, the $(2^{-t_1/m},\ldots,2^{-t_d/m})$-stores are constructed during the packing process. We open a new portion of them each time when we obtain a $(2^{-t_1/m},\ldots,2^{-t_d/m})$-standard box which cannot be packed in the existing $(2^{-t_1/m},\ldots,2^{-t_d/m})$-stores. We open this new portion of $(2^{-t_1/m},\ldots,2^{-t_d/m})$-stores in a new subbox. We find there a greatest box of the form $\tau(B(z_1 2^{-t_1/m},\ldots,z_d 2^{-t_d/m}))$, where $z_1,\ldots,z_d$ are positive integers, and where τ is a translation. We dissect this box into $\prod_{j=1}^d z_j$ $(2^{-t_1/m},\ldots,2^{-t_d/m})$-stores. The packing process is stopped if a box cannot be packed by the above rules.

Proposition. *We can pack arbitrary sequence of boxes of edge lengths at most* 1 *whose total volume is at most*

$$2^{-\frac{d}{m}}\ \Big[\Big(\prod_{j=1}^d p_j-m^d+1\Big)\prod_{j=1}^d\big(\frac{s_j}{p_j}-1\big)-\big(2^d-2\big)m^d\Big]$$

in the box $B(s_1,\ldots,s_d)$ *by the method of dividing this box into* $\prod_{j=1}^d p_j$ *congruent subboxes and by using* m^d *types of stores. In the case of packing in the cube* I_s^d *divided into* p^d *congruent subcubes, we can choose* m *and* p *so that this estimate is of the order of magnitude* s^d *with factor* 1 *and that* $u_s^d\overset{s}{\sim}cs^{d-d/(2d+1)}$*, where* c *is a constant.*

The proof is analogous to the proofs of Theorems 2 and 3. Instead of Lemma 3, this time we apply Lemma 4.

References

[1] Groemer, H.: *Covering and packing properties of bounded sequences of convex sets.* Mathematika **29** (1982), 18–31.

[2] Januszewski, J.; Lassak, M.; Rote, G.; Woeginger, G.: *On-line q-adic covering by the method of the n-th segment and its application to on-line covering by cubes.* Beiträge Algebra Geom. **37** (1996), 51–65.

[3] Kuperberg, W.: *Problem 74: Ein Intervallüberdeckungsspiel.* Math. Semesterber. **41** (1994), 207–210.

[4] Lassak, M.; Zhang, J.: *An on-line potato-sack theorem.* Discrete Comput. Geom. **6** (1991), 1–7.

Received June 28, 1996; revised version November 6, 1996