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The Center of the Dipper Donkin Quantized Matrix Algebra

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Abstract. We compute the center in case q is a primitive root of unity.

1. Introduction

Let $\mathbb{F}$ be a field of characteristic zero. There are several ways to quantize the matrix algebra $M_n(\mathbb{F})$. Among them, the most common is probably the algebra introduced by Faddeev, Reshetikhin, and Takhtajan in [4]. Denote that algebra by $M_n(q)$. In [3] Dipper and Donkin defined another quantized matrix algebra which has many features which are different from $M_n(q)$ e.g. the quantized determinant is not central. In [9] a two parameter quantized matrix algebra was defined for which the above two quantized matrix algebras are special cases. Even more general multi-parameter quantized algebras have been introduced by Artin, Schelter, and Tate [1], Sudbery [8], and Reshetikhin [7], independently.

To understand the structure of these quantized matrix algebras it is natural to determine their centers. In [6] the center of $M_n(q)$ was determined by finding the degree of the algebra over its center and by investigating certain quantized minors. In this paper we compute the center of the quantized matrix algebra constructed by Dipper and Donkin [3] by similar methods. Throughout, the quantum parameter is denoted by $q \in \mathbb{F}^*$. In most applications q will be a primitive mth root of unity for some positive integer m.

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The so-called Dipper Donkin quantized matrix algebra $\mathcal{M}at_n(q)$ is an associative algebra over $\mathbb{F}$ generated by elements $Z_{i,j}, i,j = 1,2,\cdots,n$ subject to the following relations:

$$\begin{aligned} Z_{i,j}Z_{s,t} &= qZ_{s,t}Z_{i,j} \text{ if } i > s \text{ and } j \leq t, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} + (q-1)Z_{s,j}Z_{i,t}, \text{ if } i > s \text{ and } j > t, \\ Z_{i,j}Z_{i,k} &= Z_{i,k}Z_{i,j} \text{ for all } i,j,k. \end{aligned} \tag{1}$$

In fact, the algebra $\mathcal{M}at_n(q)$ is a bialgebra and its structure maps are defined as follows:

$$\Delta(Z_{i,j}) = \sum_k Z_{i,k} \otimes Z_{k,j} \text{ for } i,j,k = 1,2,\cdots,n, \tag{2}$$

$$\epsilon(Z_{i,j}) = \delta_{i,j} \text{ for } i,j = 1,2,\cdots,n. \tag{3}$$

We thank the referee for an interesting remark.

2. The degree of a prime algebra

The main tool used to compute the degree of $\mathcal{M}at_n(q)$ is the theory developed in [2] by De Concini and Procesi. So let us first recall their set-up.

Let A be an algebra and let σ be an automorphism of A. A twisted derivation of A relative to σ is a linear map $D : A \longrightarrow A$ such that:

$$D(ab) = D(a)b + \sigma(a)D(b), a, b \in A. \tag{4}$$

An element a induces an inner twisted derivation $ad_\sigma a$ relative to σ by the formula:

$$ad_\sigma a(b) = ab - \sigma(b)a. \tag{5}$$

Given an automorphism σ and a twisted derivation D relative to σ we can define the twisted polynomial algebra $A_{\sigma,D}[x]$ in the indeterminate x to be the $\mathbb{F}$-module $A \otimes \mathbb{F}[x]$ thought of as formal polynomials with multiplication defined by

$$xa = \sigma(a)x + D(a). \tag{6}$$

Let A be a prime algebra (i.e. $aAb = 0$ implies $a = 0$ or $b = 0$) over $\mathbb{F}$ and let Z be the center of A. Then Z is a domain and A is a torsion free module over Z. Assume that A is a finite dimensional module over Z. Then A embeds in a finite dimensional central simple algebra $Q(A) = Q(Z) \otimes_Z A$ where $Q(Z)$ is the fraction field of Z. If $\overline{Q(Z)}$ denotes the algebraic closure of $Q(Z)$ we have that $\overline{Q(Z)} \otimes_Z A$ is the full algebra $M_d(\overline{Q(Z)})$ of $d \times d$ matrices over $\overline{Q(Z)}$. Then d is called the degree of A.

Let A be a prime algebra over $\mathbb{F}$ generated by $x_1, x_2, \cdots, x_n$ and let Z_0 be a central subalgebra of A. For each $i = 1,2,\cdots,n$, denote by A^i the subalgebra of A generated by $x_1, x_2, \cdots, x_i$ and let $Z_0^i = Z_0 \cap A^i$. We assume that the following conditions hold for each $i = 1,2,\cdots,n$:

1. $x_ix_j = b_{i,j}x_jx_i + p_{i,j}$ if $i > j$, where $b_{i,j} \in \mathbb{F}^*$ and $p_{i,j} \in A^{i-1}$.
2. A^i is a finite module over Z_0^i.
3. The formulas $\sigma_i(x_j) = b_{i,j}x_j$ for $j < i$ define an automorphism of A^{i-1} which is the identity on Z_0^{i-1}.
4. The formulas $D_i(x_j) = p_{i,j}$ for $j < i$ define a twisted derivation relative to σ_i.

It is easy to see that A then is an iterated twisted Ore extension. Let $\overline{A}$ be an associative algebra generated by $y_1, y_2, \cdots, y_n$ with defining relations $y_iy_j = b_{i,j}y_jy_i$ for $j < i$. We call this algebra the associated quasipolynomial algebra of A. In [2] the following was proved

Theorem 2.1. *Under the above assumptions the degree of A is equal to the degree of the associated quasipolynomial algebra $\overline{A}$.*

Given an $n \times n$ skew-symmetric matrix $H = (h_{i,j})_{i,j=1}^n$ over $\mathbb{Z}$ we construct the twisted polynomial algebra $\mathbb{F}_H[x_1, x_2, \cdots, x_n]$ as follows: It is the algebra generated by elements $x_1, x_2, \cdots, x_n$ with the following defining relations:

$$x_ix_j = q^{h_{i,j}}x_jx_i \text{ for } i, j = 1, 2, \cdots, n. \tag{7}$$

It can be viewed as an iterated twisted polynomial algebra with respect to any ordering of the indeterminates x_i. Given $a = (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n$ we write $x^a = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$. We say that H is the defining matrix of the algebra.

Let q be a primitive mth root of unity. We consider the matrix H as a matrix of the homomorphism $H : \mathbb{Z}^n \longrightarrow (\mathbb{Z}/m\mathbb{Z})^n$ and we denote by K the kernel of H and by h the cardinality of the image of H. The following was proved in [2]:

Theorem 2.2. (a) *The monomials x^a with $a \in K \cap \mathbb{Z}_+^n$ form a basis of the center of $\mathbb{F}_H[x_1, x_2, \cdots, x_n]$.*
(b) $\deg \mathbb{F}_H[x_1, x_2, \cdots, x_n] = \sqrt{h}$.

It is well-known that a skew-symmetric matrix over $\mathbb{Z}$ such as our matrix H can be brought into a block diagonal form by an element $W \in SL(\mathbb{Z})$. Specifically, there is a $W \in SL(\mathbb{Z})$ and a sequence of 2×2 matrices $S(m_i) = \begin{pmatrix} 0 & -m_i \\ m_i & 0 \end{pmatrix}$, $i = 1, \ldots, N$, with $m_i \in \mathbb{Z}$ for each $i = 1, \ldots, N$, such that

$$W \cdot H \cdot W^t = \begin{cases} Diag\ (S(m_1), \ldots, S(m_N), 0) \text{ with } N = \frac{n^2-1}{2}, \text{ if } n \text{ is odd} \\ Diag\ (S_1(m_1), \ldots, S(m_N)) \text{ with } N = \frac{n^2}{2}, \text{ if } n \text{ is even} \end{cases}. \tag{8}$$

Definition 2.3. *Any matrix of the form of the right-hand-side in (8) will be called a canonical form of H and will occasionally be denoted by J_H.*

Thus, a canonical form of H reduces the algebra to the tensor product of twisted Laurent polynomial algebras in two variables with commutation relation $xy = q^r yx$. By Theorem 2.2 it follows in particular that the degree of a twisted Laurent polynomial algebra in two variables is equal to $m/(m, r)$ where (m, r) is the greatest common divisor of m and r.

3. The degree of $\mathcal{M}at_n(q)$

It is well-known that $\mathcal{M}at_n(q)$ is an iterated Ore extension (cf.[1]). By Theorem 2.1, we have

$$\deg \mathcal{M}at_n(q) = \deg \overline{\mathcal{M}at_n(q)} \tag{9}$$

where $\overline{\mathcal{M}at_n(q)}$ is the associated quasipolynomial algebra of $\mathcal{M}at_n(q)$.

Let $M = \sum_{i\geq j} E_{i,j}$ be the $n\times n$ matrix where $E_{i,j}$ is the matrix unit in $M_n(\mathbb{F})$ with (i,j)-entry equal to 1 and zero elsewhere.

We define an ordering on the set of generators $Z_{i,j}$, $i,j = 1,2,\cdots,n$, of $\overline{\mathcal{M}at_n(q)}$ by lexicographic ordering on the set of indices. Let $J(n)$ be the defining matrix of $\overline{\mathcal{M}at_n(q)}$ where the order of the generators is induced by that of $\mathcal{M}at_n(q)$. It is easy to see that

$$J(n) = \begin{pmatrix} 0 & -M & -M & \cdots & \cdots & -M & -M & -M \\ M^t & 0 & -M & \cdots & \cdots & -M & -M & -M \\ M^t & M^t & 0 & \cdots & \cdots & -M & -M & -M \\ \cdots & \cdots & \cdots & & & & & \\ M^t & M^t & M^t & \cdots & \cdots & M^t & 0 & -M \\ M^t & M^t & M^t & \cdots & \cdots & M^t & M^t & 0 \end{pmatrix}. \tag{10}$$

Theorem 3.1. *The canonical form of $J(n)$ is*

(a) *$Diag(S_1, S_2, \cdots, S_{\frac{n^2}{2}})$ if n is even where $S_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $i = 1,2,\cdots,\frac{n^2}{2}$.*

(b) *$Diag(S_1, S_2, \cdots, S_{\frac{n^2-1}{2}}, 0)$ if n is odd where $S_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $i = 1,2,\cdots,\frac{n^2-1}{2}$.*

Proof. Let $H_1 = Diag(-M^{-1}, I, \cdots, I)$ where I is the $n\times n$ identity matrix. Then $H_1 \in SL(n^2,\mathbb{Z})$ and

$$J_1 = H_1 J(n) H_1^t = \begin{pmatrix} 0 & I & I & \cdots & \cdots & I & I & I \\ -I & 0 & -M & \cdots & \cdots & -M & -M & -M \\ -I & M^t & 0 & \cdots & \cdots & -M & -M & -M \\ \cdots & \cdots & \cdots & & & & & \\ -I & M^t & M^t & \cdots & \cdots & M^t & 0 & -M \\ -I & M^t & M^t & \cdots & \cdots & M^t & M^t & 0 \end{pmatrix}. \tag{11}$$

Let $H_2 = \begin{pmatrix} I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ M & I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & & & & \\ 0 & 0 & 0 & \cdots & \cdots & 0 & I \end{pmatrix}$. Then

$$J_2 = H_2 J_1 H_2^t = \begin{pmatrix} 0 & I & I & \cdots & \cdots & I & I & I \\ -I & M-M^t & 0 & \cdots & \cdots & 0 & 0 & 0 \\ -I & 0 & 0 & \cdots & \cdots & -M & -M & -M \\ \cdots & \cdots & \cdots & & & & & \\ -I & 0 & M^t & \cdots & \cdots & M^t & 0 & -M \\ -I & 0 & M^t & \cdots & \cdots & M^t & M^t & 0 \end{pmatrix}. \tag{12}$$

Let $H_3 = \begin{pmatrix} I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -I & I & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & & & & \\ 0 & -I & 0 & \cdots & \cdots & 0 & I \end{pmatrix}$. Then

$$J_3 = H_3 J_2 H_3^t = \begin{pmatrix} 0 & I & 0 & \cdots & \cdots & 0 & 0 & 0 \\ -I & M-M^t & M^t-M & \cdots & \cdots & M^t-M & M^t-M & M^t-M \\ 0 & M^t-M & 0 & \cdots & \cdots & -M & -M & -M \\ \cdots & \cdots & \cdots & & & & & \\ 0 & M^t-M & M^t & \cdots & \cdots & M^t & 0 & -M \\ 0 & M^t-M & M^t & \cdots & \cdots & M^t & M^t & 0 \end{pmatrix}. \tag{13}$$

Let $H_4 = \begin{pmatrix} I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ M^t & I & 0 & \cdots & \cdots & 0 & 0 \\ M^t-M & 0 & I & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & & & & \\ M^t-M & 0 & 0 & \cdots & \cdots & 0 & I \end{pmatrix}$. Then

$$H_4 J_3 H_4^t = \begin{pmatrix} 0 & I & 0 & \cdots & \cdots & 0 & 0 & 0 \\ -I & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & -M & -M & -M \\ \cdots & \cdots & \cdots & & & & & \\ 0 & 0 & M^t & \cdots & \cdots & M^t & 0 & -M \\ 0 & 0 & M^t & \cdots & \cdots & M^t & M^t & 0 \end{pmatrix}. \tag{14}$$

If n is even we repeat the above procedure. Then

$$J(n) \longrightarrow Diag(M_1, M_2, \cdots, M_{\frac{n}{2}}) \tag{15}$$

where $M_i = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $i = 1, 2, \cdots, \frac{n}{2}$. This proves our assertion in case n is even.

If n is odd we also repeat the above procedure. Then

$$J(n) \longrightarrow Diag(L_1, L_2, \cdots, L_{\frac{n-1}{2}}) \tag{16}$$

where

$$L_i = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{17}$$

for $i = 1, 2, \cdots, (n-3)/2$, and

$$L_{\frac{n-1}{2}} = \begin{pmatrix} 0 & -M & -M \\ M^t & 0 & -M \\ M^t & M^t & 0 \end{pmatrix}. \tag{18}$$

Similar to the above we can easily find a transformation

$$L_{\frac{n-1}{2}} \longrightarrow \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & M - M^t \end{pmatrix} \tag{19}$$

and we can find a transformation

$$M - M^t \longrightarrow Diag(N_1, N_2, \cdots, N_{\frac{n-1}{2}}, 0) \tag{20}$$

where

$$N_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, j = 1, 2, \cdots, \frac{n-1}{2}. \tag{21}$$

This proves our assertion in case n is odd. □

4. The center of $\mathcal{M}at_n(q)$

For any $B = (b_{i,j})_{i,j=1}^n \in M_n(\mathbb{Z}_+)$ we define

$$Z^B = \Pi Z_{i,j}^{b_{i,j}}, \tag{22}$$

where the factors are arranged according to lexicographic ordering. We denote the generators of $\overline{\mathcal{M}at_n(q)}$ by $X_{i,j}, i, j = 1, 2, \cdots, n$, and define the symbol X^B analogously in terms of the same ordering.

Let C be the center of $\mathcal{M}at_n(q)$ and $\overline{C}$ the center of $\overline{\mathcal{M}at_n(q)}$. For any $P \in C$ the leading term of P must be of the form $cZ^B, c \in \mathbb{F}$, for some $B = (a_{i,j})_{i,j=1}^n \in M_n(\mathbb{Z}_+)$. For any $Z_{k,l}$ the leading term of $PZ_{k,l}$ is $cq^{r_{k,l}} Z^{B+E_{k,l}}$ where $r_{k,l} = \sum_{i>k} \sum_{j \le l} a_{i,j}$. The leading term of $Z_{k,l}P$ is $cq^{l_{k,l}} Z^{B+E_{k,l}}$ where $l_{k,l} = \sum_{i<k} \sum_{j \ge l} a_{i,j}$. Since $PZ_{k,l} = Z_{k,l}P$ we get $q^{l_{k,l}} = q^{r_{k,l}}$ and this implies that cX^B is a central element of the twisted polynomial algebra $\overline{\mathcal{M}at_n(q)}$. Hence we can define a map

$$\Lambda : C \longrightarrow \overline{C} \tag{23}$$

by

$$P \mapsto cX^B \tag{24}$$

if the leading term of P is $cZ^B, c \in \mathbb{F}$.

Clearly, $\Lambda(P) = 0$ implies $P = 0$.

Theorem 4.1. (a) *If n is even and q is a primitive m-th root of unity then the center of $\mathcal{M}at_n(q)$ is generated by*

$$\{X_{i,j}^m \mid i, j = 1, 2, \cdots, n\}. \tag{25}$$

(b) *If n is odd and q is a primitive m-th root of unity then the center of $\overline{\mathcal{M}at_n(q)}$ is generated by*

$$\{X_{i,j}^m, X^A \mid i,j = 1,2,\cdots,n\} \tag{26}$$

where $A = (a_{i,j})_{i,j=1}^n \in M_n(\mathbb{Z}_+)$ is given by $a_{i,j} = 1$ if $i+j$ is even and $a_{i,j} = m-1$ if $i+j$ is odd.

Proof. Our assertions follow immediately from Theorem 3.1 and Theorem 2.2. □

Lemma 4.2. *Let $n \in \mathbb{Z}_+$ and assume that $i > s$ and $j > t$. Then*

$$Z_{i,j}Z_{s,t}^n = Z_{s,t}^n Z_{i,j} + (1-q^{-n})Z_{i,t}Z_{s,j}Z_{s,t}^{n-1}, \tag{27}$$

$$Z_{s,t}Z_{i,j}^n = Z_{i,j}^n Z_{s,t} + (1-q^n)Z_{s,j}Z_{i,t}Z_{i,j}^{n-1}. \tag{28}$$

Proof. The assertions follow from (1) by using induction on n. □

Corollary 4.3. *If q is an m-th root of unity then $Z_{i,j}^m$ is a central element for all $i,j = 1,2,\cdots,n$.*

The quantum determinant of $\mathcal{M}at_n(q)$ is defined by the following formula:

$$\det_q = \Sigma_{\sigma\in S_n}(-1)^{l(\sigma)}Z_{1,\sigma(1)}Z_{2,\sigma(2)}\cdots Z_{n,\sigma(n)} \tag{29}$$

where S_n is the symmetric group on n letters and $l()$ is the standard length function on S_n.
In [3], Dipper and Donkin proved that

Theorem 4.4.

$$\det_q = \Sigma_{\sigma\in S_n}(-q)^{-l(\sigma)}Z_{\sigma(1),1}Z_{\sigma(2),2}\cdots Z_{\sigma(n),n}. \tag{30}$$

Furthermore, the following result was obtained.

Theorem 4.5.

$$Z_{i,j}det_q = q^{i-j}det_q Z_{i,j} \text{ for } 1 \le i \text{ and } j \le n. \tag{31}$$

Let $r \le n$ be a positive integer. Given any two subsets $I = \{i_1, i_2, \cdots, i_r\}$ and $J = \{j_1, j_2, \cdots, j_r\}$ of $\{1,2,\cdots,n\}$, each having cardinality r, clearly the subalgebra of $\mathcal{M}at_n(q)$ generated by all Z_{i_a,j_b}, $a,b = 1,2,\cdots,r$, is isomorphic to $\mathcal{M}at_r(q)$. Hence we can talk about its determinant. Such a determinant is called a subdeterminant (or minor) of $\det_q$ and will be denoted by $\det_q(I,J)$. If $I = \{1,2,\cdots,n\}\setminus\{i\}$, $J = \{1,2,\cdots,n\}\setminus\{j\}$, $\det_q(I,J)$ will be denoted by $A(i,j)$.

We have [3, page 203]

Lemma 4.6. *Let $i, k \leq n$ be fixed integers. Then*

$$\delta_{i,k}\det_q = \sum_{j=1}^{n}(-1)^{k-j} Z_{i,j} A(k,j). \tag{32}$$

$$\delta_{i,k}\det_q = \sum_{j=1}^{n}(-1)^{k-j} Z_{j,i} A(j,k). \tag{33}$$

The above formulas are called the quantum Laplace expansions.

Definition 4.7. *An element $x \in \mathcal{M}at_n(q)$ is called covariant if for any $Z_{i,j}$ there exists an integer n_{ij} such that*

$$xZ_{i,j} = q^{n_{i,j}} Z_{i,j} x. \tag{34}$$

Clearly, $Z_{1,n}$ and $Z_{n,1}$ are covariant.

Let $I = \{t+1, t+2, \cdots, n\}$, $J = \{1, 2, \cdots, n-t\}$, and let $\phi_t = det_q(I, J)$. Let $I^* = \{1, 2, \cdots, t\}$, $J^* = \{n-t+1, n-t+2, \cdots, n\}$ and let $\phi_t^* = det_q(I^*, J^*)$. Then we have

Theorem 4.8. *ϕ_t and ϕ_t^* are covariant for all t.*

Proof. Let $M_t^- = \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i \leq t \text{ and } 1 \leq j \leq n-t\}$, $M_t^+ = \{(i,j) \in \mathbb{N}^2 \mid t+1 \leq i \leq n \text{ and } n-t+1 \leq j \leq n\}$, $M_t^l = \{(i,j) \in \mathbb{N}^2 \mid t+1 \leq i \leq n \text{ and } 1 \leq j \leq n-t\}$, and $M_t^r = \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i \leq t \text{ and } n-t+1 \leq j \leq n\}$. We use induction on t to prove that

$$\begin{aligned}
Z_{i,j}\phi_t &= q^{-j}\phi_t Z_{ij} \text{ if } (i,j) \in M_t^-, \\
Z_{i,j}\phi_t &= q^{n-i}\phi_t Z_{i,j} \text{ if } (i,j) \in M_t^+, \\
Z_{i,j}\phi_t &= q^{i-t-j}\phi_t Z_{i,j} \text{ if } (i,j) \in M_t^l, \text{ and} \\
Z_{i,j}\phi_t &= q^{-t}\phi_t Z_{i,j} \text{ if } (i,j) \in M_t^r.
\end{aligned} \tag{35}$$

If $t = n-1$ then $\phi_{n-1} = Z_{n,1}$ which is clearly covariant.

For $(i,j) \in M_t^l \cup M_t^r$ the assertions follow from Theorem 4.5 and the defining relations of $\mathcal{M}at_n(q)$, respectively.

Now assume that $(i,j) \in M_t^-$. If $j = n-t$ the assertion follows immediately from the defining relations of $\mathcal{M}at_n(q)$. For $j < n-t$ we have

$$Z_{i,j}\phi_t = \sum_{k=t+1}^{n} (-1)^{n-t+k} Z_{i,j} Z_{k,n-t} A_t(k, n-t) \tag{36}$$

$$= \sum_{k=t+1}^{n} (-1)^{n-t+k} [Z_{k,n-t} Z_{i,j} A_t(k,n-t) - (1-q) Z_{i,n-t} Z_{k,j} A_t(k,n-t) \tag{37}$$

where $A_t(k, n-t)$ is the subdeterminant of ϕ_t. By the inductive hypothesis and Lemma 4.6 we have

$$Z_{i,j}\phi_t = q^j \phi_t Z_{i,j} \tag{38}$$

for $(i,j) \in M_t^-$. By the same method it follows that

$$Z_{i,j}\phi_t = q^{n-i} \phi_t Z_{i,j} \tag{39}$$

for $(i,j) \in M_t^+$. Hence ϕ_t is covariant.

In a similar way it can be proved that ϕ_t^* is covariant. This completes the proof. □

Theorem 4.9. *Let q be a primitive m-th root of unity, let $\Omega_n^+ = \Pi_{t=1}^{n-1}(\phi_t)^{a_{1,t+1}}$, and let $\Omega_n^- = \Pi_{t=1}^{n-1}(\phi_t^*)^{a_{t,n}}$, with the $a_{i,j}$ as in Theorem 4.1. Then $\Omega_n = \Omega_n^+ det_q \Omega_n^-$ is a central element of $\mathcal{M}at_n(q)$ provided that n is odd.*

Proof. Obviously the leading term of Ω_n is Z^A. By Theorem 4.8 we know that Ω_n is covariant. It is easy to see that

$$Z_{i,j}\Omega_n = Z^A Z_{i,j} + \text{ lower terms.} \tag{40}$$

Hence $Z_{i,j}\Omega_n = \Omega_n Z_{i,j}$ for all $1 \leq i,j \leq n$. This completes the proof. □

Theorem 4.10. *If n is even and q is a primitive m-th root of unity then the center of $\mathcal{M}at_n(q)$ is generated by*

$$\{Z_{i,j}^m \mid i,j = 1,2,\cdots,n\}. \tag{41}$$

Proof. Let C' be the subalgebra generated by the elements $Z_{i,j}^m, i,j = 1,2,\cdots,n$. We use induction on the order of the leading term to prove that any $P \in C$ also belongs to C'. By Theorem 4.1, $\Lambda(P)$ is of the form cX^{mB} for some $c \in \mathbb{F}$ and $B \in M_n(\mathbb{Z}_+)$. Hence there exists an element $P' \in C'$ such that $\Lambda(P) = \Lambda(P')$. Hence the order of the leading term of $P - P'$ is smaller than that of leading term of P. So $P - P' \in C'$. Therefore, $P \in C'$. This completes the proof. □

Lemma 4.11. *Let I, J be two subsets of $\{1,2,\cdots,n\}$, each having cardinality s. We use $|I|$ (resp. $|J|$) to denote the sum of the elements in I (resp. in J). Then*

$$det_q(I,J)det_q = q^{|I|-|J|}det_q det_q(I,J). \tag{42}$$

Proof. The assertion follows from Theorem 4.5. □

The following theorem gives a full solution to the problem raised in Remark 1.3.4 in [3].

Theorem 4.12. *Suppose that n is odd and q is a primitive m-th root of unity. Let C' be the central subalgebra of $\mathcal{M}at_n(q)$ generated by $Z_{i,j}^m$, for all $i,j = 1,2,\cdots,n$. Then C is generated over C' by Ω_n. The minimal polynomial of Ω_n is $x^m - a$ for some $a \in C'$.*

Proof. The first statement follows as in the proof of Theorem 4.10. Hence

$$C = C' + C'\Omega_n + \cdots + C'\Omega_n^l + \cdots . \tag{43}$$

By Theorem 4.5, $det_q^m \in C$. Since $\deg \det_q < \deg \Omega_n$ if $n > 1$ we see that $det_q^m \in C'$. Similarly, $\phi_s^m, \phi_t^{*m} \in C'$.

Let $\tilde{\Omega}_n^- = \det_q \Omega_n^-$. We know that for any (i,j) there exists a $b_{i,j} \in \mathbb{Z}$ such that

$$Z_{i,j}\tilde{\Omega}_n^- = q^{b_{i,j}}\tilde{\Omega}_n^- Z_{i,j}. \tag{44}$$

We now compute the $b_{i,j}$ explicitly. For this is enough to work in the associated quasipolynomial algebra $\overline{Mat_n(q)}$ since $\tilde{\Omega}_n^-$ is covariant.

Let $D = (d_{i,j})_{i,j=1}^n$ where

$$d_{i,j} = \begin{cases} 0 \text{ if } i > j, \\ 1 \text{ if } i \le j \text{ and } i+j \text{ is even,} \\ m-1 \text{ if } i \le j \text{ and } i+j \text{ is odd.} \end{cases} \tag{45}$$

Note that the leading term of $\tilde{\Omega}_n^-$ is Z^D so when we work in the associated quasipolynomial algebra $\overline{Mat_n(q)}$ we have

$$X_{i,j}X^D = q^{b_{i,j}}X^D X_{i,j}. \tag{46}$$

It follows that

$$b_{i,j} = \begin{cases} \frac{i}{2} - \frac{j}{2} & \text{mod } m & \text{if both } i,j \text{ are even,} \\ \frac{i}{2} - \frac{j-1}{2} & \text{mod } m & \text{if } i \text{ is even and } j \text{ is odd,} \\ \frac{i-1}{2} - \frac{j}{2} & \text{mod } m & \text{if } i \text{ is odd and } j \text{ is even,} \\ \frac{i-1}{2} - \frac{j-1}{2} & \text{mod } m & \text{if both } i \text{ and } j \text{ are odd.} \end{cases} \tag{47}$$

It is easy to see that $b_{i,j} = g_i - g_j$ where $g_i = \frac{i}{2}$ if i is even $g_i = \frac{i-1}{2}$ if i is odd. Then it follows directly that for all $t = 1, \ldots, n-1$, $\tilde{\Omega}_n^- \phi_t = q^{x_t}\phi_t\tilde{\Omega}_n^-$ for some integer x_t. Therefore, $\Omega_n^m = q^x(\Omega_n^+)^m(det_q\Omega_n^-)^m$ for some $x \in \mathbb{Z}$.

By Lemma 4.11 we have

$$\begin{aligned} \phi_s\phi_t &= q^{x_{s,t}}\phi_t\phi_s \text{ and} \\ \phi_s^*\phi_t^* &= q^{y_{s,t}}\phi_t^*\phi_s^* \end{aligned} \tag{48}$$

for some integers $x_{s,t}$ and $y_{s,t}$ and this implies that $\Omega_n^m = q^w \Pi_t \phi_t^m \det_q^m \Pi_s \phi_s^{*m}$ for some integer w. Hence $\Omega_n^m \in C'$. Since Ω_n is of degree m over $Q(C')$, the fraction field of C', the minimal polynomial of Ω_n over C' is $x^m - a$ for some $a \in C'$. This completes the proof. □

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