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Oriented Matroids in Terms of Order Functions

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Abstract. An oriented matroid of rank r is described as a pair $\mathcal{M} = (E, \chi)$ where E is a nonempty set and χ is a chirotope of rank r on E. Relating the chirotope to an order function $\omega : (H, a, b) \mapsto (H|ab)$ according to the rule $(H|ab) = \chi(P, a) \cdot \chi(P, b)$ where H is a hyperplane, P is a base of H and a and b are points with $a, b \in E \setminus H$, ω is shown to be harmonic and strict. The presented theorem gives a geometric characterization of oriented matroids in terms of order functions.

1. Introduction

Oriented matroids appear in varied shapes in different branches of mathematics. They are seen, for example, under the aspect of signed bases, of collections of covectors, as arrangements of pseudospheres, or collections of cocircuits. The concept of an oriented matroid generalizes combinatorial properties of point and vector configurations, directed graphs, arrangements of hyperplanes, convex polytopes, or linear programs, see [2, 3, 6].

The definition in terms of chirotopes, which is used in this paper, has been developed over a long period. An idea of base orientation was presented for the first time in [16]. Later the connection to oriented matroids was discovered, see [27]. The term *chirotope* was coined by Dress and Dreiding to reflect geometric aspects of chirality in organic chemistry. It turned out that chirotopes are equivalent to oriented matroids. At the same time Bokowski independently initiated a similar approach from a geometric point of view, especially using the Grassmann–Plücker relations. In the next years the concept of chirotopes was used by several authors to investigate and describe oriented matroids [7, 8, 11, 13]. Further developments, compare [26, 3, 6], finally led to the definition which is used here.

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Starting from the description of oriented matroids in terms of chirotopes, we will develope their geometric characterization. The motivation comes from the following observation:

Let E be a finite set of points in $\mathbb{R}^n$ forming a combinatorial geometry M of dimension d on E (i.e. a simple matroid of rank $r = d + 1$). The oriented matroid $\mathcal{M}$ of the points can be given by a list of signs that encodes the orientation of all ordered bases of M. If $B = (x_0, \ldots, x_d) \in E^{d+1}$ is some ordered base this sign can be computed with the help of the function $\chi : (x_0, \ldots, x_d) \mapsto sgn\, det(\overrightarrow{x_0x_1} \ldots \overrightarrow{x_0x_n})$ which is called the *chirotope* on E. There is a natural connection between orientation given by the function *sgn det* and the geometric order: Let H be a hyperplane spanned by the points $p_0, \ldots, p_{d-1}$. Points $a, b \in E \setminus H$ are in the same side respectively in distinct sides of H, if $\chi(p_0, \ldots, p_{d-1}, a) = \chi(p_0, \ldots, p_{d-1}, b)$ respectively $\chi(p_0, \ldots, p_{d-1}, a) = -\chi(p_0, \ldots, p_{d-1}, b)$. Thus there is a side-dividing or *order function* $\omega : (H, a, b) \mapsto \chi(p_0, \ldots, p_{d-1}, a) \cdot \chi(p_0, \ldots, p_{d-1}, b)$ which describes orientation and hence the oriented matroid.

This idea will be transferred to matroids and oriented matroids in general, which are not necessarily representable as point configurations. The concept of order functions was introduced by Sperner [31], who developed the theory for desarguesian affine and projective spaces. Karzel transferred it to non-desarguesian spaces, see [20, 21] and compare [23, 24] for various models and concepts of geometric ordering. There exist also several geometric settings and descriptions of orientation, see [4, 5, 10, 15, 28], but no attention was paid there to discrete or finite structures.

Transferring the theory of order functions to matroids, or, more precisely, to combinatorial geometries, discovers new aspects of oriented matroids. Differently to the idea of the Topological Representation Theorem of Folkman and Lawrence [14] oriented matroids themselves are seen here as geometries: They are incidence structures equipped with an order function of certain kind. For the rank 3 case this relationship is pointed out in [19]. The investigations there base on the algebraic methods from [12], essentially utilizing the theory of Tutte groups. Independently from this results we prove that, for matroids of any rank, there is a bijection between chirotopes and special order functions. The method developed here discovers new aspects of describing the relations between matroids and oriented matroids and provides new tools for investigating the structure of oriented matroids in general as well as various examples. So it would be interesting to look for new ideas concerning the cube problem of Las Vergnas [9] by using order functions.

In the second section of this paper we start with the definitions of matroids and of oriented matroids in terms of chirotopes and continue to compile further facts. Next we present the basic ideas about order functions on (simple) matroids and derive characteristic properties of such order functions which are generated by chirotopes. Section four concerns with the reverse approach, the generation of a chirotope from a certain order function. Combining both directions we get our final result, which gives a characterization of oriented matroids in terms of order functions.

2. Preliminaries – Definitions and facts

Matroids, as introduced in 1935 by Whitney, can be considered as an abstraction of linear dependences of vector configurations in finite-dimensional vector spaces, see [29, 33]. This motivates the following definition of matroids in terms of bases:

Definition 1. *A matroid is a pair $M = (E, \mathcal{B})$, where E is a nonempty set and $\mathcal{B}$ is a nonempty family of finite subsets of E (called bases of M) that satisfies:*
(B) *For all $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, there exists an $f \in B_2 \setminus B_1$ such that $(B_1 \setminus e) \cup f \in \mathcal{B}$.*

It follows that all bases of M do have the same cardinality r, which is called the *rank* of the matroid. In this paper we always assume $r \geq 3$.

The information about the matroid structure can be encoded in a function $\mu : E^r \to \{0, 1\}$ with certain properties by setting $\mu(x_1, \ldots, x_r) = 1$ if $\{x_1, \ldots, x_r\} \in \mathcal{B}$ and otherwise $\mu(x_1, \ldots, x_r) = 0$. Thus a matroid equivalently is quoted as $M = (E, \mu)$.

Oriented matroids can be considered as a combinatorial model of the determinant values on vector configurations. This motivates the following definition in terms of chirotopes, see [9]:

Definition 2. *Let E be a set and $r \in \mathbb{N}$ with $3 \leq r \leq |E|$. A chirotope of rank r on E is a map $\chi : E^r \to \{-1, 0, +1\}$ that satisfies*
(C1) *χ is an alternating sign map, i.e. for any permutation $\pi \in \Sigma_r$ and $(x_1, \ldots, x_r) \in E^r$ we have $\chi(x_{\pi(1)}, \ldots, x_{\pi(r)}) = sgn(\pi)\chi(x_1, \ldots, x_r)$.*
(C2) *The map $|\chi| : E^r \to \{0, 1\}$ defines a matroid $(E, |\chi|)$.*
(C3) *For every $P \in E^{r-2}$ and $a, b, c, d \in E \setminus P$, the set*
$\{\chi(P, a, b)\chi(P, c, d), -\chi(P, a, c)\chi(P, b, d), \chi(P, a, d)\chi(P, b, c)\}$
either contains $\{+1, -1\}$ or it equals $\{0\}$.

An *oriented matroid* of rank r is a pair $\mathcal{M} = (E, \chi)$ where E is a nonempty set and χ is a chirotope of rank r on E. We call a matroid on E *orientable* if there exists a chirotope on E.

The underlying matroid $M = (E, |\chi|)$ always is simple (i.e. it has no loops and no parallel elements) and therefore it is called the *underlying combinatorial geometry* of $\mathcal{M}$. The elements of E are called *points.*

Next we collect some facts and notations on matroids, which will be used in the following. Additional explanations and proofs can be found, e.g., in [1, 29, 30, 32].
For a matroid $M = (E, \mathcal{B})$, let $cl : \begin{cases} 2^E \to 2^E \\ A \to \overline{A} \end{cases}$ be the closure operator of M. It follows that a subset X of E is closed if and only if $\overline{X} = X$. The submatroid $M_X = (X, \mathcal{B}_X)$ generated by $X \subseteq E$ (sometimes called the reduction of M on X) is referred to as *subspace* generated by X. Setting $r(X) := |B|$ for $B \in \mathcal{B}_X$ we get the rank function of M_X.

A subspace H of rank $r(M) - 1$ is called a *hyperplane* of M and $\mathcal{H}$ denotes the set of all hyperplanes of M. The members of the family $\mathcal{B}^o := \{(x_1, \ldots, x_r) \in E^r : \{x_1, \ldots, x_r\} \in \mathcal{B}\}$ will be called *ordered bases.*

Finally we present a conclusion of the chirotope condition (C3), which can be seen as a special oriented version of the Steinitz Exchange Theorem:

Proposition 2.1. *Let $\mathcal{M} = (E, \chi)$ be an oriented matroid of rank r, $P \in E^{r-2}$ and $x_1, x_2, a, b \in E \setminus P$ such that $(P, x_1, a), (P, x_2, a), (P, x_1, b), (P, x_2, b) \in \mathcal{B}^o$ but $(P, x_1, x_2) \notin \mathcal{B}^o$. Then $\chi(P, x_1, a)\chi(P, x_1, b) = \chi(P, x_2, a)\chi(P, x_2, b)$.*

Proof.
Set $\varepsilon_1 := \chi(P, x_1, a)\chi(P, x_2, b)$, $\varepsilon_2 := \chi(P, x_1, x_2)\chi(P, a, b)$ and $\varepsilon_3 := \chi(P, x_1, b)\chi(P, a, x_2)$. Because of $(P, x_1, a), (P, x_2, b) \in \mathcal{B}^o$ we have $\varepsilon_1 \neq 0$ and from $(P, x_1, x_2) \notin \mathcal{B}^o$ we deduce $\varepsilon_2 = 0$. Applying (C3) we get $\varepsilon_1 = -\varepsilon_3$. Thus $\chi(P, x_1, a)\chi(P, x_2, b) = \chi(P, x_1, b)\chi(P, x_2, a)$, which obviously yields our proposition. □

3. Order functions in oriented matroids

The theory of order functions provides new tools for the investigation of orientation. The basic properties of order functions derive from the idea, that in the 3-dimensional euclidean space each line is devided by each of its points into two halflines, each plane by each of its lines into two halfplanes and the space by each plane into two halfspaces. We transfer the basic characteristics of order functions from [24] to matroids:

Definition 3. *Let M be a simple matroid of rank r on E ($r \geq 3$), $\mathcal{H}$ the family of hyperplanes of M and $(\mathcal{H} \times E \times E)^* = \{(H, a, b) \in (\mathcal{H} \times E \times E) : a, b \notin H\}$.*

A mapping $\omega : \begin{cases} (\mathcal{H} \times E \times E)^* \longrightarrow \{1, -1\} \\ \quad (H, a, b) \qquad \longrightarrow (H|ab) \end{cases}$

is called an order function on M if it fulfills the following condition:

(O) $\forall H \in \mathcal{H}\ \forall a, b, c \in E \setminus H\ : (H|ab)(H|bc) = (H|ac)$

Conclusions.
1. $\forall H \in \mathcal{H}\ \forall a \in E \setminus H : (H|aa) = 1$
2. $\forall H \in \mathcal{H}\ \forall a, b \in E \setminus H : (H|ab) = (H|ba)$
3. $\forall H \in \mathcal{H}\ \forall a, b \in E \setminus H : (H|ab)(H|ab) = 1$

Remarks. Order functions as defined here coincide with the derivated order functions of first kind from [31]. Halfordered planes, ordered planes and ordered spaces, respectively, which describe richer structures of geometric ordering (compare [24, 25]) can be defined in terms of order functions (see [17]).

Let $\mathcal{M} = (E, \chi)$ an oriented matroid of rank r with underlying combinatorial geometry M. We consider some subspace $M_Y = (Y, \mathcal{B}_Y)$ of M with $Y \subseteq E$ and $r(Y) := k$ ($k \leq r$). A hyperplane of M_Y then is a subspace of M_Y (and hence of M) of rank $k-1$. We introduce the following notations: If $B = (b_1, \dots, b_k) \in \mathcal{B}^o_Y$ is some ordered base of M_Y, then the related unordered base is written as $B^* = \{b_1, \dots, b_k\} \in \mathcal{B}_Y$. Ordered bases $A, B \in \mathcal{B}^o_Y$ are called *bases with common hyperplane* H of M_Y (shorthand $A \asymp B$ or, more precisely, $A \asymp_H B$), if there exists $A' \subseteq A^*$ and $B' \subseteq B^*$ with $|A'| = k - 1 = |B'|$ fulfilling $\overline{A'} = \overline{B'} = H$.

Proposition 3.1. *Let $A, B \in \mathcal{B}^o_Y$ be two ordered bases of a subspace M_Y of rank k. Then there exist ordered bases $X_1, \dots, X_k \in \mathcal{B}^o_Y$ such that $A \asymp X_1, X_1 \asymp X_2, \dots, X_k \asymp B$.*

Proof. Assume $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$ and write singletons $\{x\}$ shortly by x. We take $X_0 := A$ and argue by moving i step by step from 0 to $k-1$ and thereby replacing $a_{i+1} \in X^*_i$ by a suitable $b_{l_{i+1}} \in B^*$ and defining X_{i+1} and the common hyperplane H_{i+1}.

Fix some $i \in \{0, \ldots, k-1\}$ and assume $X_i = (b_{l_1}, \ldots, b_{l_i}, a_{i+1}, \ldots, a_k) \in \mathcal{B}_Y^o$.
First case: $a_{i+1} \in X_i^* \cap B^*$. Then there exists $b_{l_{i+1}} \in B^*$ such that $b_{l_{i+1}} = a_{i+1}$.
Second case: $a_{i+1} \in X_i^* \setminus B^*$. Because the matroid M_Y obeys the base exchange property (B) (compare definition 1), there exists an element $b_{l_{i+1}} \in B^* \setminus X_i^*$ such that $(X_i^* \setminus a_{i+1}) \cup b_{l_{i+1}} \in \mathcal{B}_Y$.
We put $X_{i+1} = (b_{l_1}, \ldots, b_{l_i}, b_{l_{i+1}}, a_{i+2}, \ldots, a_k)$ and $H_{i+1} = \overline{X_i^* \setminus a_{i+1}}$. Hence $X_{i+1}^* = (X_i^* \setminus a_{i+1}) \cup b_{l_{i+1}} = (A^* \setminus \{a_1, \ldots, a_{i+1}\}) \cup \{b_{l_1}, \ldots, b_{l_{i+1}}\}$ and H_{i+1} is a common hyperplane of X_i^* and X_{i+1}^*.
Finally we have $X_k^* = (A^* \setminus \{a_1, \ldots, a_k\}) \cup \{b_{l_1}, \ldots, b_{l_k}\} = B^*$ and $H_k = \overline{X_k^* \setminus a_k} = \overline{b_{l_1}, \ldots, b_{l_k}}$ is a common hyperplane of X_{k-1} und B. □

Remark. In general, the ordered bases $X_1, \ldots, X_k$ depend on the choice of the b_{l_i} and therefore are not uniquely determined.
Whenever sequences $A \asymp X_1, X_1 \asymp X_2, \ldots, X_k \asymp B$ occur in the following they are assumed to be produced according to the given order of $A = (a_1, \ldots, a_k)$.

Next we repeat that any permutation $\pi \in \Sigma_k$ acts at an ordered base $B = (x_1, \ldots, x_k) \in \mathcal{B}_Y^o$ by the rule $\pi(B) = (x_{\pi(1)}, \ldots, x_{\pi(k)})$. Thus, if $A, B \in \mathcal{B}_Y^o$ and $A \asymp X_1, \ldots, X_k \asymp B$ as we got in proposition 3.1, there exists a permutation π obeying $\pi(B) = X_k$.

Proposition 3.2. *For any hyperplane H of $\mathcal{M}$, points $a, b \in E \setminus H$ and any two ordered bases P and Q of H we have $\chi(P, a)\chi(P, b) = \chi(Q, a)\chi(Q, b)$.*

Proof. By applying proposition 3.1 to $P = (p_1, \ldots, p_{r-1})$ and $Q = (q_1, \ldots, q_{r-1}) \in \mathcal{B}_H^o$ we get ordered bases X_i $(i = 1, \ldots, r-1)$ satisfying $P \asymp X_1, \ldots, X_{r-1} \asymp Q$. Hence $(P, a) \asymp (X_1, a), (X_1, a) \asymp (X_2, a), \ldots, (X_{r-1}, a) \asymp (Q, a)$ and $(P, b) \asymp (X_1, b), \ldots, (X_{r-1}, b) \asymp (Q, b)$, respectively. Thereby the bases in each pair differ in at most one element (at position i) and we have $\pi(X_{r-1}, a) = (Q, a)$ and $\pi(X_{r-1}, b) = (Q, b)$ assuming $\pi \in \Sigma_k$ to be that permutation whose restriction to H maps X_{r-1} onto Q and lets the r-th position unchanged. Now it is straightforward to see from proposition 2.1 that $\chi(P, a)\chi(P, b) = \chi(X_1, a)\chi(X_1, b)$, $\chi(X_1, a)\chi(X_1, b) = \chi(X_2, a)\chi(X_2, b), \ldots, \chi(X_{r-2}, a)\chi(X_{r-2}, b) = \chi(X_{r-1}, a)\chi(X_{r-1}, b)$. Because χ is an alternating map, we have $\chi(X_{r-1}, a)\chi(X_{r-1}, b) = sgn(\pi)\chi(Q, a)sgn(\pi)\chi(Q, b)$. This implies $\chi(P, a)\chi(P, b) = \chi(Q, a)\chi(Q, b)$ finishing the proof. □

Now we relate an order function ω_χ to the given chirotope χ on E. For a hyperplane H and two points $a, b \in E \setminus H$ we define: $(H|ab) := \chi(p_1, \ldots, p_{r-1}, a)\chi(p_1, \ldots, p_{r-1}, b)$. Proposition 3.2 shows, that the value $(H|ab)$ does not depend on the choice of the points $p_1, \ldots, p_{r-1}$ from H.

Proposition 3.3. *Let χ be a chirotope on E, and define $\omega_\chi(H, a, b) := (H|ab)$ for some hyperplane H and $a, b \in E \setminus H$. Then ω_χ is an order function on the underlying combinatorial geometry M.*

Proof.
1. For each hyperplane $H = \overline{p_1, \ldots, p_{r-1}}$ and points $a, b \in E \setminus H$ the sets $\{p_1, \ldots, p_{r-1}, a\}$ and $\{p_1, \ldots, p_{r-1}, b\}$ are bases of E. Thus $\chi(p_1, \ldots, p_{r-1}, a) \neq 0$ and $\chi(p_1, \ldots, p_{r-1}, b) \neq 0$ and therefore we get $\omega_\chi : (\mathcal{H} \times E \times E)^* \longrightarrow \{1, -1\}$.

2. For each hyperplane $H = \overline{p_1, \ldots, p_{r-1}}$ and $a, b, c \in E \setminus H$ we get:

$$\begin{aligned}(H|a,b)\cdot(H|b,c) &= \chi(p_1,\ldots,p_{r-1},a)\cdot\chi(p_1,\ldots,p_{r-1},b)\cdot\chi(p_1,\ldots,p_{r-1},b)\cdot\chi(p_1,\ldots,p_{r-1},c)\\ &= \chi(p_1,\ldots,p_{r-1},a)\cdot\chi(p_1,\ldots,p_{r-1},c)\\ &= (H|a,c), \quad \text{which proves (O).}\end{aligned}$$ □

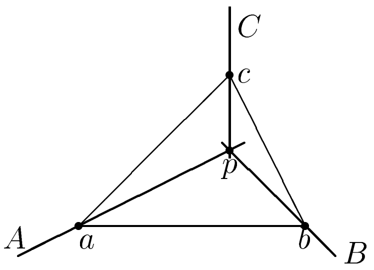
We call ω_χ the *order function generated by* χ.

For the rest of the chapter let (E, χ) be an oriented matroid of rank r and ω_χ the order function generated by χ. Generation of ω_χ from a chirotope causes special properties.

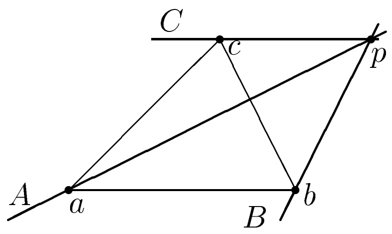
Proposition 3.4. *Each order function generated from a chirotope satiesfies:*

(OC1) *For any three different hyperplanes A, B, C with $r(A \cap B \cap C) = r - 2$ and $a, b, c \in E$ with $a \in A \backslash (B \cup C)$, $b \in B \backslash (A \cup C)$, $c \in C \backslash (A \cup B)$, we have $(A|bc)(B|ac)(C|ab) = -1$.*

Illustration: (plane case; $r(M) = 3$, $A \cap B \cap C = \{p\}$)



$(A|bc) = -1$, $(B|ac) = -1$, $(C|ab) = -1$



$(A|bc) = -1$, $(B|ac) = +1$, $(C|ab) = +1$

Remark. Following [31] order functions with property (OC1) are called *harmonic.*

Proof of proposition 3.4. Because of $r(A \cap B \cap C) = r(E) - 2 = r - 2$ there exist points $p_1, \ldots, p_{r-2} \in A \cap B \cap C$ such that $A = \overline{\{p_1, \ldots, p_{r-2}, a\}}$, $B = \overline{\{p_1, \ldots, p_{r-2}, b\}}$ and $C = \overline{\{p_1, \ldots, p_{r-2}, c\}}$. Since A, B, C are three distinct hyperplanes, the sets $\{p_1, \ldots, p_{r-2}, a, b\}$, $\{p_1, \ldots, p_{r-2}, a, c\}$ and $\{p_1, \ldots, p_{r-2}, b, c\}$ form bases of E and we conclude:

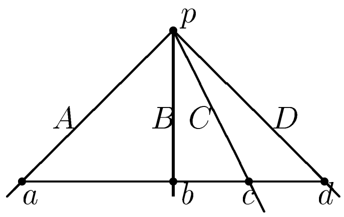
$$\begin{aligned}(A|bc)(B|ac)(C|ab) = {}& \chi(p_1,\ldots,p_{r-2},a,b)\cdot\chi(p_1,\ldots,p_{r-2},a,c)\\ &\cdot\chi(p_1,\ldots,p_{r-2},b,a)\cdot\chi(p_1,\ldots,p_{r-2},b,c)\\ &\cdot\chi(p_1,\ldots,p_{r-2},c,a)\cdot\chi(p_1,\ldots,p_{r-2},c,b)\\ = {}& -1\end{aligned}$$

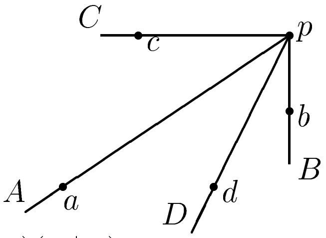
since by (C1), χ is alternating. □

Proposition 3.5. *Each order function generated from a chirotope satiesfies:*

(OC2) *For any four different hyperplanes A, B, C, D with $r(A \cap B \cap C \cap D) = r - 2$ and $a, b, c, d \in E \setminus (A \cap B \cap C \cap D)$ with $a \in A$, $b \in B$, $c \in C$, $d \in D$, exactly one of the values $(A|bc)(D|bc)$, $(B|ac)(D|ac)$, or $(C|ab)(D|ab)$ equals -1.*

Illustration: (plane case; $r(M) = 3$, $A \cap B \cap C \cap D = \{p\}$)





$(A|bc)(D|bc) = +1 \cdot +1 = +1$
$(B|ac)(D|ac) = -1 \cdot +1 = -1$
$(C|ab)(D|ab) = +1 \cdot +1 = +1$

$(A|bc)(D|bc) = -1 \cdot -1 = +1$
$(B|ac)(D|ac) = +1 \cdot +1 = +1$
$(C|ab)(D|ab) = +1 \cdot -1 = -1$

Remark. Property (OC2) is similar to a property which was used in [22] to describe order structures in projective spaces. Joussen [18] named such order functions *strict*.

Proof of proposition 3.5. From (OC1) we deduce that at least one of the values $(A|bc)(D|bc)$, $(B|ac)(D|ac)$, $(C|ab)(D|ab)$ equals -1. Assume $(A|bc)(D|bc) = -1$ and put $A \cap B \cap C \cap D = \overline{p_1, \ldots, p_{r-2}}$ and $W := (p_1, \ldots, p_{r-2})$. Then (W, a, b), (W, a, c), (W, b, c), (W, a, d), (W, b, d) and (W, c, d) form ordered bases of M. We put $\varepsilon_1 := \chi(W, a, b) \cdot \chi(W, c, d)$, $\varepsilon_2 := -\chi(W, a, c) \cdot \chi(W, b, d)$ and $\varepsilon_3 := \chi(W, a, d) \cdot \chi(W, b, c)$ and this gives $(A|bc)(D|bc) = -\varepsilon_1 \cdot \varepsilon_2$, $(B|ac)(D|ac) = -\varepsilon_1 \cdot \varepsilon_3$ and $(C|ab)(D|ab) = -\varepsilon_2 \cdot \varepsilon_3$. Our assumption $(A|bc)(D|bc) = -1$ implies $\varepsilon_1 \cdot \varepsilon_2 = 1$ and so we have $\varepsilon_1 = \varepsilon_2$. Applying (C3) we get $\varepsilon_3 = -\varepsilon_1 = -\varepsilon_2$ and thus $(B|ac)(D|ac) = 1$ and $(C|ab)(D|ab) = 1$. □

4. Orientation generated by an order function

In the previous section we proved that an order function always is harmonic and strict, if it comes from a chirotope. Now we reverse the direction of asking questions. We start from a matroid, equipped with a harmonic and strict order function and produce a chirotope and therefore an orientation from it.

Let M be a combinatorial geometry of rank r on E und let ω be an order function on M fulfilling (OC1) and (OC2). First we make the following observation:

Proposition 4.1 (Modular hyperplane relation). (H*) *If F, G, H are three distinct hyperplanes with $r(F \cap G \cap H) = r - 2$ and a and b are points with $a, b \in H$ und $a, b \notin F \cup G$, then $(F|ab) = (G|ab)$.*

Remark. If $r(M) = 3$, so (H*) corresponds to the 'line relation' (Geradenrelation) for order functions in an exchange space; compare [24].

Proof. Let $D := F \cap G \cap H$, which is a subspace of rank $r - 2$ of M. Then there exist points $f \in F \setminus (H \cup G)$ and $g \in G \setminus (H \cup F)$ such that $F = \overline{D \cup \{f\}}$ and $G = \overline{D \cup \{g\}}$. As, moreover, $\overline{D \cup \{a\}} = H = \overline{D \cup \{b\}}$, by applying (OC1) we conclude $(F|ag)(G|af)(H|gf) = -1$ and $(F|bg)(G|bf)(H|gf) = -1$. This implies $(F|ag)(G|af) = (F|bg)(G|bf)$ and together with (O) we finally get $(F|ab) = (G|ab)$. □

We regard the family $\mathcal{B}^o$ of ordered bases of M and introduce an equivalence relation $\sim$ with exactly two classes.

Let $A = (a_1, \ldots, a_r), B = (b_1, \ldots, b_r) \in \mathcal{B}^o$ be ordered bases. There exist ordered bases $X_1, \ldots, X_r \in \mathcal{B}^o$ and hyperplanes $H_1, \ldots, H_r \in \mathcal{H}$ produced according to proposition 3.1 such that $A \asymp_{H_1} X_1,\ X_1 \asymp_{H_2} X_2, \ldots,\ \ X_{r-1} \asymp_{H_r} X_r$ and $\pi(B) = X_r$, where $\pi \in \Sigma_r$ is a suitable permutation. We define
$[A, X_1, \ldots, X_r, B] := sgn(\pi)(H_1|a_1 b_{\pi(1)})(H_2|a_2 b_{\pi(2)}) \cdots (H_r|a_r b_{\pi(r)}) \in \{1, -1\}$.
Then we observe: If $B = \pi(A)$, so $X_r = A$ and $(H_i|a_i b_{\pi(i)}) = (H_i|a_i a_i) = 1$ for all $i = 1, \ldots, r$. Thus $[A, X_1, \ldots, X_r, B] = [A, X_1, \ldots, X_r, \pi(A)] = sgn(\pi)(H_1|a_1 a_1) \cdots (H_r|a_r a_r) = sgn(\pi)$.

Proposition 4.2. *For some fixed number $i \in \{1, \ldots, r-1\}$, let $A = (a_1, \ldots, a_r), B = (b_1, \ldots, b_r)$ and $B' = (b'_1, \ldots, b'_r) = (b_1, \ldots, b_{i-1}, b_{i+1}, b_i, b_{i+2}, \ldots, b_r) \in \mathcal{B}^o$ be ordered bases with $A \asymp X_1,\ X_1 \asymp X_2, \ldots, X_{r-1} \asymp X_r = B$ and $A \asymp Y_1, Y_1 \asymp Y_2, \ldots, Y_{r-1} \asymp Y_r = B'$. Then $[A, X_1, \ldots, X_r, B] = -[A, Y_1, \ldots, Y_r, B']$.*

Proof. Since $X_1, \ldots, X_r, Y_1, \ldots, Y_r$ are produced according to proposition 3.1, we have
$A \asymp_{H_1} X_1, X_1 \asymp_{H_2} X_2, \ldots, X_{r-1} \asymp_{H_r} X_r = B$ where
$H_k = \overline{X^*_{k-1} \setminus a_k}$ and $X_k = (b_1, \ldots, b_k, a_{k+1}, \ldots, a_r)$ for $k = 1, \ldots, r$,
and $A \asymp_{G_1} Y_1, Y_1 \asymp_{G_2} Y_2, \ldots, Y_{r-1} \asymp_{G_r} Y_r = B'$
where $G_k = \overline{Y^*_{k-1} \setminus a_k}$ for $k = 1, \ldots, r$, $\ Y_k = X_k$ if $k = 1, \ldots, i-1$,
$Y_i = (b_1, \ldots, b_{i-1}, b_{i+1}, a_{i+1}, \ldots, a_r)$, $\ Y_{i+1} = (b_1, \ldots, b_{i-1}, b_{i+1}, b_i, a_{i+2}, \ldots, a_r)$ and
$Y_k = (b_1, \ldots, b_{i-1}, b_{i+1}, b_i, \ldots, b_k, a_{k+1}, \ldots, a_r)$ if $i+2 \leq k \leq r$.
If $k \in \{1, \ldots, r\} \setminus \{i, i+1\}$, then clearly holds $Y^*_k = X^*_k$, $G_k = H_k$ and $(H_k|a_k b_k) = (G_k|a_k b'_k)$.
We obtain
$[A, X_1, \ldots, X_r, B][A, Y_1, \ldots, Y_r, B']$
$= (H_i|a_i b_i)(H_{i+1}|a_{i+1} b_{i+1})(G_i|a_i b'_i)(G_{i+1}|a_{i+1} b'_{i+1})$
$= (H_i|a_i b_i)(H_{i+1}|a_{i+1} b_{i+1})(G_i|a_i b_{i+1})(G_{i+1}|a_{i+1} b_i)$.
We see: $H_i = \overline{X^*_{i-1} \setminus a_i}$, $X^*_i = (X^*_{i-1} \setminus a_i) \cup b_i$, $H_{i+1} = \overline{X^*_i \setminus a_{i+1}} = \overline{((X^*_{i-1} \setminus a_i) \cup b_i) \setminus a_{i+1}}$ and moreover $G_i = \overline{Y^*_{i-1} \setminus a_i} = \overline{X^*_{i-1} \setminus a_i} = H_i$, $Y^*_i = (Y^*_{i-1} \setminus a_i) \cup b'_i = (X^*_{i-1} \setminus a_i) \cup b_{i+1}$ and $G_{i+1} = \overline{Y^*_i \setminus a_{i+1}} = \overline{((X^*_{i-1} \setminus a_i) \cup b_{i+1}) \setminus a_{i+1}}$. This yields:
$[A, X_1, \ldots, X_r, B][A, Y_1, \ldots, Y_r, B'] =$
$\quad = (H_i|a_i b_i)(\overline{((X^*_{i-1} \setminus a_i) \cup b_i) \setminus a_{i+1}}|a_{i+1} b_{i+1})(H_i|a_i b_{i+1})(\overline{((X^*_{i-1} \setminus a_i) \cup b_{i+1}) \setminus a_{i+1}}|a_{i+1} b_i)$
$\quad = (H_i|b_i b_{i+1})(\overline{X^*_{i-1} \setminus a_i \setminus a_{i+1} \cup b_i}|a_{i+1} b_{i+1})(\overline{X^*_{i-1} \setminus a_i \setminus a_{i+1} \cup b_{i+1}}|a_{i+1} b_i)$
$\quad \overset{(OC1)}{=} (H_i|b_i b_{i+1})(-1)(\overline{((X^*_{i-1} \setminus a_i) \setminus a_{i+1}) \cup a_{i+1}}|b_i b_{i+1})$
$\quad = (-1)(H_i|b_i b_{i+1})(H_i|b_i b_{i+1}) = -1$,
and this completes the proof. □

Proposition 4.3. *Let $A, B, X_1, \ldots, X_r, Y_1, \ldots, Y_r \in \mathcal{B}^o$ be ordered bases such that $A \asymp_{H_1} X_1$, $\ldots, X_{r-1} \asymp_{H_r} X_r$ and $A \asymp_{G_1} Y_1, \ldots, Y_{r-1} \asymp_{G_r} Y_r$ where $H_1, \ldots, H_r, G_1, \ldots, G_r \in \mathcal{H}$ are hyperplanes and $\pi, \sigma \in \mathcal{S}_r$ are the related permutations satisfying $\pi(B) = X_r$ and $\sigma(B) = Y_r$. Then $sgn(\pi)(H_1|a_1 b_{\pi(1)}) \cdots (H_r|a_r b_{\pi(r)}) = sgn(\sigma) \cdot (G_1|a_1 b_{\sigma(1)}) \cdots (G_r|a_r b_{\sigma(r)})$.*

Proof. By setting $\rho := \sigma \cdot \pi^{-1}$ we get $Y_r = \rho(X_r)$. The permutation ρ can be divided into a product $\rho = \tau_n \circ \cdots \circ \tau_1$ of n transpositions $\tau_1, \ldots, \tau_n \in \Sigma_r$, each exchanging consecutive elements. We set $\tau_1(X_r) = T_1$, $\tau_2(T_1) = T_2$, $\ldots\ \tau_n(T_{n-1}) = Y_r = \rho(X_r)$ and thereby obtain

ordered bases $T_1, \ldots, T_{n-1} \in \mathcal{B}^o$. According to Proposition 3.1, for all $i = 1, \ldots, n-1$, there exist $B_{i1}, \ldots, B_{ir} \in \mathcal{B}^o$ such that $A \asymp B_{i1}, \ldots, B_{ir} \asymp T_i$. From proposition 4.2 we conclude $[A, B_{i1}, \ldots, B_{ir}, T_i] = -[A, B_{i+11}, \ldots, B_{i+1r}, T_{i+1}]$ and thus
$[A, B_{i1}, \ldots, B_{ir}, T_i][A, B_{i+11}, \ldots, B_{i+1r}, T_{i+1}] = -1$.
Hence $[A, X_1, \ldots, X_r, X_r][A, B_{11}, \ldots, B_{1r}, T_1][A, B_{11}, \ldots, B_{1r}, T_1][A, B_{21}, \ldots, B_{2r}, T_2] \cdots$
$\cdots [A, B_{n-11}, \ldots, B_{n-1r}, T_{n-1}][A, Y_1, \ldots, Y_r, Y_r] = (-1)^n$ and consequently
$[A, X_1, \ldots, X_r, X_r][A, Y_1, \ldots, Y_r, Y_r] = sgn(\rho)$. Finally we obtain
$(H_1|a_1 b_{\pi(1)}) \cdots (H_r|a_r b_{\pi(r)})(G_1|a_1 b_{\sigma(1)}) \cdots (G_r|a_r b_{\sigma(r)}) = sgn(\sigma) sgn(\pi^{-1})$,
which proves our assertion. □

According to proposition 4.3. the value $[A, X_1, \ldots, X_r, B] = sgn(\pi)(H_1|a_1 b_{\pi(1)}) \cdots (H_r|a_r b_{\pi(r)})$ can be considered as a characteristic invariant of the pair (A, B) of ordered bases. We denote it by $[A, B]$. Obviously the mapping $[.,.] : \mathcal{B}^o \times \mathcal{B}^o \to \{1, -1\}$ is surjective.

For all $A = (a_1, \ldots, a_r), B = (b_1, \ldots, b_r) \in \mathcal{B}^o$ we define:
$A \sim B \;:\Leftrightarrow\; [A, B] = 1$.

Then as an immediate consequence of the previous propositions we conclude:

Proposition 4.4. *$\sim$ is an equivalence relation at $\mathcal{B}^o$ with exactly two nonempty classes.*

Let λ be the canonical homomorphism from $\mathcal{B}^o$ onto $\mathcal{B}^o/_\sim$, which assigns each ordered base to its class. For all $X = (x_1, \ldots, x_r) \in E^r$ a mapping χ_ω is defined by

$$\chi_\omega(X) := \begin{cases} 0 & \text{if } X \notin \mathcal{B}^o \\ \lambda(X) & \text{if } X \in \mathcal{B}^o \end{cases}$$

We observe:

Proposition 4.5. *If H is a hyperplane spanned by $p_1, \ldots, p_{r-1}$ and $a, b \in E \setminus H$, then $\chi_\omega(a, p_1, \ldots, p_{r-1}) = (\overline{p_1, \ldots, p_{r-1}}|ab) \cdot \chi_\omega(b, p_1, \ldots, p_{r-1})$.*

Proposition 4.6. *χ_ω is a chirotope on M.*

Proof. The fact that $(E, |\chi_\omega|)$ is a matroid, is immediate from the definition. If B_0 is an arbitrary but fixed ordered base and B and B' are ordered bases with $B' = \pi(B)$ where π is a permutation, then by proposition 4.2 and 4.3 we conclude $[B_0, B] = sgn(\pi)[B_0, B']$. Hence χ_ω is alternating.

Now, to show that (C3) holds, suppose that $P \in E^{r-2}$ and $a, b, c, d \in E \setminus P$. We put
$\varepsilon_1 := \;\chi_\omega(P, a, b)\chi_\omega(P, c, d)$,
$\varepsilon_2 := -\chi_\omega(P, a, c)\chi_\omega(P, b, d)$ and
$\varepsilon_3 := \;\chi_\omega(P, a, d)\chi_\omega(P, b, c)$.
For proving (C3) we have to show that the set $S := \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ either contains $+1$ and -1 or equals $\{0\}$. Without loss of generality, we may assume that a, b, c, d are pairwise distinct.

(1) If two elements of S are equal to 0, so is the third. Equivalent: If one of the elements of S is unequal 0, so is a second one.

Assume $\varepsilon_3 \neq 0$. Then $B_1 = \{P, a, d\}$ and $B_2 = \{P, b, c\}$ are bases of M. Since $a \in B_1 \setminus B_2$, there exists an element $x \in B_2 \setminus B_1 = \{b, c\}$ such that $(B_1 \setminus a) \cup x \in \mathcal{B}$. Thus
(a) $\{P, d, b\} \in \mathcal{B}$ or $\{P, d, c\} \in \mathcal{B}$

and by similar arguments for d, b and c
(b) $\{P,a,b\} \in \mathcal{B}$ or $\{P,a,c\} \in \mathcal{B}$ and
(c) $\{P,c,a\} \in \mathcal{B}$ or $\{P,c,d\} \in \mathcal{B}$ and
(d) $\{P,b,a\} \in \mathcal{B}$ or $\{P,b,d\} \in \mathcal{B}$.
If ε_1 equals 0, so $\{P,a,b\} \notin \mathcal{B}$ or $\{P,c,d\} \notin \mathcal{B}$. In the first case we apply (b) and (d) and get $\{P,a,c\} \in \mathcal{B}$ and $\{P,b,d\} \in \mathcal{B}$, which implies $\varepsilon_2 \neq 0$. In the second case we apply (a) and (c) and get $\{P,b,d\} \in \mathcal{B}$ and $\{P,a,c\} \in \mathcal{B}$, which also implies $\varepsilon_2 \neq 0$.

If $\varepsilon_2 \neq 0$ or $\varepsilon_1 \neq 0$ an analogous argument shows that at least one of the other elements of S does not equal 0.

(2) If $\varepsilon_{i_1} \neq 0$ and $\varepsilon_{i_2} = 0$, so $\varepsilon_{i_3} = -\varepsilon_{i_1}$ where $\{i_1, i_2, i_3\} = \{1,2,3\}$.
Because of similar arguments in the other cases we can assume $\varepsilon_3 \neq 0$ and $\varepsilon_2 = 0$. Application of (1.) yields $\varepsilon_1 \neq 0$. This implies that $\{P,a,b\}$, $\{P,c,d\}$, $\{P,a,d\}$ and $\{P,b,c\}$ are bases, whereas $\{P,a,c\} \notin \mathcal{B}$ or $\{P,b,d\} \notin \mathcal{B}$. We regard the hyperplanes $A := \overline{P \cup a}$, $B := \overline{P \cup b}$, $C := \overline{P \cup c}$ and $D := \overline{P \cup d}$.

If $\{P,a,c\} \notin \mathcal{B}$, so A and C coincide ($A = C =: H$) and the hyperplanes additionally fulfill $B \cap D \cap H = \overline{P}$, $r(B \cap D \cap H) = r - 2$, $a, c \notin B \cup D$ and $a, c \in H$. Using (H*) we deduce $(B|ac) = (D|ac)$. Now we see:

$$\begin{aligned}
\varepsilon_1 &= \chi_\omega(P,a,b) \cdot \chi_\omega(P,c,d) \\
&= (B|ac) \cdot \chi_\omega(P,c,b) \cdot (D|ac) \cdot \chi_\omega(P,a,d) \\
&\overset{\text{(Prop.4.5)}}{=} -\chi_\omega(P,b,c) \cdot \chi_\omega(P,a,d) \\
&= -\varepsilon_3
\end{aligned}$$

If $\{P,b,d\} \notin \mathcal{B}$, a similar argument works.

(3) Finally we assume $\varepsilon_3, \varepsilon_2, \varepsilon_1 \neq 0$. Then $A := \overline{P \cup a}$, $B := \overline{P \cup b}$, $C := \overline{P \cup c}$ and $D := \overline{P \cup d}$ are hyperplanes and in addition $a, c \notin B, D$ and $b, c \notin A, D$ hold. Thus we get the following equations

$$\varepsilon_1 = \chi_\omega(P,a,b) \cdot \chi_\omega(P,c,d) = (A|bc) \cdot \chi_\omega(P,a,c) \cdot (D|bc) \cdot \chi_\omega(P,b,d) = -(A|bc) \cdot (D|bc) \cdot \varepsilon_2$$
$$\varepsilon_1 = \chi_\omega(P,a,b) \cdot \chi_\omega(P,c,d) = (B|ac) \cdot \chi_\omega(P,c,b) \cdot (D|ac) \cdot \chi_\omega(P,a,d) = -(B|ac) \cdot (D|ac) \cdot \varepsilon_3$$
$$\varepsilon_2 = -\chi_\omega(P,a,c) \cdot \chi_\omega(P,b,d) = -(C|ab) \cdot \chi_\omega(P,b,c) \cdot (D|ab) \cdot \chi_\omega(P,a,d) = -(C|ab) \cdot (D|ab) \cdot \varepsilon_3$$

Suppose that two of the terms are equal, say $\varepsilon_1 = \varepsilon_2$. Consequently $(A|bc) \cdot (D|bc) = -\varepsilon_1 \cdot \varepsilon_2 = -1$. Now we apply (OC2) and get $1 = (B|ac) \cdot (D|ac) = -\varepsilon_1 \cdot \varepsilon_3$ which yields $\varepsilon_1 = -\varepsilon_3$. Similar argumentations in the other cases conclude the proof of the proposition. □

5. Geometrical characterization theorem

In section 3 we saw that in every oriented matroid the chirotope produces a harmonic and strict order function on the underlying combinatorial geometry (proposition 3.4 and 3.5). In section 4 we started vice versa from a harmonic and strict order function and generated a chirotope (proposition 4.6). Summarizing both we get our final result:

Theorem. *Let M be a combinatorial geometry on E. Then M is orientable if and only if there exists an order function on E which is harmonic and strict.*

Remark. There exist combinatorial geometries M which are not orientable, what means that there does not exist any harmonic and strict order function on M. For example, finite affine planes of order $q > 2$ as well as finite projective planes are never orientable (cf. [3]).

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