Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 38 (1997), No. 2, 213-215.

A simple, (almost) non-inductive Proof for the Existence of Composite Fields and Algebraic Closures

Andreas W. M. Dress

FSMP - Strukturbildungsprozesse, Universität Bielefeld
Postfach 10 01 31, 33501 Bielefeld
e-mail: jordan@mathematik.uni-bielefeld.de

In the theory of fields, it is often convenient to know that for any two extension fields L_1 and L_2 of a common ground field K, there exists a field extension L of K together with two embeddings $\varphi_1 : L_1 \hookrightarrow L$ and $\varphi_2 : L_2 \hookrightarrow L$ with $\varphi_1\big|_K = \varphi_2\big|_K = Id_K$. In addition, L may be assumed to be generated by its subfields $\varphi_1(L_1)$ and $\varphi_2(L_2)$ in which case L is often called a composite field extension of K – that is, a field extension composed from L_1 and L_2 (over K). Standard proofs of this fact either use tensor products (a rather time consuming concept for an undergraduate lecture) or construct L by inductively adjoining to L_2 elements $x_1, x_2, \ldots$ from L_1 with $L_1 = K(x_1, x_2, \ldots)$ (a rather unpleasant procedure, in particular if L_1 needs an infinite number of generators). Hence, it might be worthwhile to observe that – provided one is willing to use the existence of maximal ideals M in any ring R (with $1_R \in R$ and $1_R \neq 0_R$!) – a much simpler, non-inductive (except for the use of the existence of maximal ideals), and much less time consuming way is to consider the K-algebra $R' := End_K(V)$ of K-endomorphisms of the K-vectorspace $V := Hom_K(L_1, L_2)$ into which L_1 can be embedded K-linearly by

$$\Phi_1 : L_1 \to R' : x_1 \mapsto (V \to V : \psi \mapsto (\psi^{x_1} : L_1 \to L_2 : y_1 \mapsto \psi(y_1 \cdot x_1)))$$

while L_2 can be embedded K-linearly into R' via

$$\Phi_2 : L_2 \to R : x_2 \mapsto (V \to V : \psi \mapsto ({}^{x_2}\psi : L_1 \to L_2 : y_1 \mapsto x_2 \cdot \psi(y_1))).$$

As $\Phi_1(x_1) \circ \Phi_2(x_2) = \Phi_2(x_2) \circ \Phi_1(x_1)$ holds for all $x_1 \in L_1$ and $x_2 \in L_2$, the subring R of R' generated by $\Phi_1(L_1)$ and $\Phi_2(L_2)$ is commutative. In addition, $\Phi_1\big|_K = \Phi_2\big|_K = Id_K$

clearly holds by the definition of K-linearity (relative to the canonical identification of the elements of K with the associated scalar multiplication in V considered as an element in R). And $V \neq \{0\}$ implies $1_R \neq 0_R$. So, choosing a maximal ideal M in R, putting $L := R/M$, $\varphi_1 : L_1 \to L : x_1 \mapsto \Phi_1(x_1) + M$, and $\varphi_2 : L_2 \to L : x_2 \mapsto \Phi_2(x_2) + M$, the existence of a composite field extension L of K for given field extensions $K \subseteq L_1$ and $K \subseteq L_2$ is easily established.

To get a more symmetric version of this proof, we may replace $V = Hom_K(L_1, L_2)$ by the K-vectorspace $B(L_1 \times L_2, K)$ of all K-bilinear maps from L_1 and L_2 into K and R' by the ring of its K-endomorphisms into which L_1 and L_2 embed in a similar fashion. The advantage of this approach is that it generalizes immediately to an arbitrary number of field extensions: let I be an arbitrary set and, for each $i \in I$, let L_i denote some field extension of a common ground field K. Let

$$\begin{aligned} V := Mult(\prod_{i\in I} L_i, K) := \{f : \prod_{i\in I} L_i \to K \mid \ & \text{for each } i_0 \in I \\ & \text{and any choice of elements } x_j \in L_j \ (j \in I - \{i_0\}), \\ & \text{the induced map } f : L_{i_0} \to K : x_{i_0} \mapsto f((x_i)_{i\in I}) \text{ is } K\text{-linear}\} \end{aligned}$$

denote the K-vectorspace of all multilinear maps from $\prod_{i\in I} L_i$ into K and let $R' := End_k(V)$ denote the K-algebra of its K-endomorphisms. Clearly, we have $1_{R'} \neq 0_{R'}$ in view of $V \neq \{0\}$ which in turn is an immediate consequence of the fact that for any choice of *surjective* K-linear maps $\lambda_i : L_i \twoheadrightarrow K (i \in I)$, the map

$$f_{(\lambda_i)_{i\in I}} : \prod_{i\in I} L_i \to K : (x_i)_{i\in I} \mapsto \begin{cases} \prod_{i\in I} \lambda_i(x_i) & \text{if } \lambda_i(x_i) = 1 \text{ for almost all } i \in I, \\ 0 & \text{else} \end{cases}$$

is a non-zero element in V. Hence, as above, there exist canonical embeddings

$$\Phi_{i_0} : L_{i_0} \to R' : x_{i_0} \mapsto \left(V \to V : f \mapsto \left(\prod_{i\in I} L_i \to K : (x_i)_{i\in I} \mapsto f(y_i) \right.\right.$$
$$\left.\left.\text{with } y_i := \begin{cases} x_i & \text{if } i \neq i_0 \\ x_{i_0} x_i & \text{else} \end{cases} \right) \right),$$

the subring R of R' generated by all the images $\Phi_i(L_i)$ $(i \in I)$ of the L_i is commutative and satisfies the condition $1_R \neq 0_R$ and, for all $i, j \in I$, one has $\Phi_i\big|_K = \Phi_j\big|_K = Id_K$. Hence, choosing a maximal ideal M in R as above, the existence of a composite field extension L of K for any arbitrarily large family of field extensions L_i $(i \in I)$ of K is established.

Choosing for I the set of all irreducible (or just all non-constant) polynomials in $K[x]$ and for $i = i(x) \in I$ the splitting field of $i(x)$ as the associated field extension L_i of K, it is easily seen that this way an algebraically closed field extension L of K will result, thus establishing in a rather simple and straightforward fashion the basic fact that any field K is a subfield of an algebraically closed field.

It is also obvious that the same approach yields, for any family of simple associative K-algebras A_i $(i \in I)$, the existence of a simple associative K-algebra A together with K-algebra embeddings $\varphi_i : A_i \to A$ $(i \in I)$ such that A is generated by $\bigcup_{i \in I} \varphi_i(A_i)$ and the identity $\varphi_i(a_i) \cdot \varphi_j(a_j) = \varphi_j(a_j)\varphi_i(a_i)$ holds for all $i, j \in I$ with $i \neq j$ and all elements $a_i \in A_i$ and $a_j \in A_j$.

It is also easy to derive, either by the same method or simply as a consequence of the above result, the following slightly more general result needed in the analysis of normal field extensions:

Given an arbitrary index set I, a family $K_i \hookrightarrow L_i$ $(i \in I)$ of field extensions and embeddings $\alpha_i : K_i \hookrightarrow K$ $(i \in I)$ of the various ground fields K_i into a common "super" ground field K, there exists a field extension $\psi : K \hookrightarrow L$ of K and embeddings $\varphi_i : L_i \hookrightarrow L$ $(i \in I)$ with $\varphi_i \big|_{K_i} = \psi \circ \alpha_i$ for all $i \in I$:

Indeed, first choosing composite field extensions $K_i \hookrightarrow L_i'$ of $K_i \hookrightarrow L_i$ and $\alpha_i : K_i \hookrightarrow K$ we get field extensions $\psi_i : K \hookrightarrow L_i'$ with embeddings $\beta_i : L_i \hookrightarrow L_i'$ satisfying the relation $\beta_i \big|_{K_i} = \psi_i \circ \alpha_i$ for each $i \in I$ individually, which then can be composed to one big field extension $\psi : K \hookrightarrow L$ with embeddings $\varphi_i' : L_i' \hookrightarrow L$ satisfying the relation $\varphi_i' \circ \psi_i = \psi$ and, hence, with embedding $\varphi_i := \varphi_i' \circ \beta_i : L_i \hookrightarrow L$ satisfying the relation

$$\varphi_i \big|_{K_i} = \varphi_i' \circ \beta_i \big|_{K_i} = \varphi_i' \circ \psi_i \circ \alpha_i = \psi \circ \alpha_i$$

for each $i \in I$, as claimed.

Similarly, one can go directly for this more general result by first considering the K-vectorspace V of all maps $f : \prod_{i \in I} L_i \to K$ which for each $i_0 \in I$ and all $x_j \in L_j$ $(j \in I \setminus \{i_0\})$ induce a K_{i_0}-linear map $L_{i_0} \to K : x_{i_0} \mapsto f((x_i)_{i \in I})$ relative to the K_{i_0}-vectorspace structure of L induced by $\alpha_{i_0} : K_{i_0} \to L$ and observing that, as above, one has $V \neq \{0\}$, and then embedding the L_i into the K-endomorphism ring R of V and further arguing as above.

And finally, also as above, there is a non-commutative version of this result, relating to simple K_i-algebras A_i giving rise to a simple K-algebra A relative to field extensions $\alpha_i : K_i \hookrightarrow K$ or – only seemingly more generally – embeddings $\alpha_i : K_i \hookrightarrow K$ of the various ground fields K_i into the centre of a simple algebra K.

Received May 6, 1996