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Four-dimensional Ball-homogeneous and C-spaces

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Abstract. We prove that four-dimensional *ball-homogeneous* Einstein manifolds which are 2-stein or Hermitian are locally symmetric. Most of the results are extended to the class of four-dimensional *C-spaces.*

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1. Introduction

Locally homogeneous Riemannian manifolds have the property that the volume of "small" geodesic spheres or balls only depends on the radius. Riemannian manifolds having this property are called *ball-homogeneous spaces* [14]. Conversely, two-dimensional, connected ball-homogeneous spaces are clearly locally homogeneous but for all other dimensions such a possible converse property is to our knowledge and in general still an intriguing open problem.

Harmonic spaces or, more generally, Riemannian manifolds all of whose local geodesic symmetries are volume-preserving, that is, *D'Atri spaces* (see [13] for a survey and further references), are ball-homogeneous [10]. But also for these subclasses no examples which are not locally homogeneous are known. Three-dimensional D'Atri spaces are locally homogeneous [12] and a similar result has been proved for some special classes of four-dimensional

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Einstein D'Atri spaces. More specifically, taking into account Jensen's result about locally homogeneous Einstein spaces of dimension four [11], one may state the problem whether such spaces are locally symmetric or not. A positive answer has been given for four-dimensional Kähler manifolds (without the Einstein condition) [17], Hermitian Einstein spaces [7] and 2-stein spaces [18]. In the first part of this paper (Section 3), we shall extend these results to the broader class of ball-homogeneous spaces.

In the second part (Section 4), we shall treat similar problems and prove analogous results for the class of *C-spaces*, that is, the Riemannian manifolds such that the eigenvalues of the Jacobi operator are constant along each geodesic. These spaces have been introduced in [3] (see also [4], [5] for a survey and more references) where it is proved that two- and three-dimensional C-spaces are D'Atri spaces, and conversely. For higher dimensions both classes seem to be related but it is yet unknown if a C-space is necessarily ball-homogeneous. Moreover, all known examples are again locally homogeneous but it remains an open problem whether this holds in general or not.

In Section 2 we start the paper with a collection of results about four-dimensional Einstein spaces which will be useful in proving our main results.

2. Four-dimensional Einstein manifolds

Let (M, g) be an Einstein manifold of dimension four which we suppose to be connected. Let R be the Riemannian curvature tensor associated to the Levi Civita connection and taken with the sign convention

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \ ,$$

for all smooth vector fields X, Y on M. ϱ denotes the Ricci tensor of type $(0, 2)$, τ the scalar curvature and Δ the Laplacian.

As is well-known, at each point $m \in M$ there exists a *Singer-Thorpe basis* $\{e_1, e_2, e_3, e_4\}$ for the tangent space T_mM, that is, an orthonormal basis such that the components of R are given by

$$\begin{cases} R_{1212} = R_{3434} = a, \ R_{1313} = R_{2424} = b, \ R_{1414} = R_{2323} = c \,, \\ R_{1234} = \alpha, \ R_{1342} = \beta, \ R_{1423} = \gamma \,, \\ R_{ijkl} = 0 \text{ whenever three of the indices } i, j, k, l \text{ are distinct} \end{cases} \tag{2.1}$$

[20] (see also [2]). Here $R_{XYZW} = g(R_{XY}Z, W)$.

In what follows we shall often make use of a series of scalar curvature invariants. In particular, besides τ, we shall consider the quadratic invariant $||R||^2 = \sum_{i,j,k,l} R^2_{ijkl}$ where $\{e_i \ , i = 1, 2, 3, 4\}$ is an arbitrary orthonormal basis, and the cubic invariants

$$\check{R} = \sum_{i,j,k,l,p,q} R_{ijkl}R_{klpq}R_{pqij} \ ,$$

$$\check{\check{R}} = \sum_{i,j,k,l,p,q} R_{ikjl}R_{kplq}R_{piqj} \ .$$

Further, we note that

$$(1/2)\Delta||R||^2 = g(\Delta R, R) + ||\nabla R||^2 \tag{2.2}$$

and on a four-dimensional Einstein space we always have [9]

$$\begin{cases} \check{R} = -\tau^3/12 + (2\tau/3)||R||^2 - (1/3)g(\Delta R, R)\,, \\ \check{\check{R}} = \tau^3/48 - (\tau/24)||R||^2 - (1/6)g(\Delta R, R)\,. \end{cases} \tag{2.3}$$

With respect to a Singer-Thorpe basis, we then have

$$\begin{aligned} &\tau = 4(a+b+c)\,, \\ &||R||^2 = 8(a^2+b^2+c^2+\alpha^2+\beta^2+\gamma^2)\,, \\ &\check{R} = 16(a^3+b^3+c^3+3a\alpha^2+3b\beta^2+3c\gamma^2)\,, \\ &\check{\check{R}} = 24(abc+a\beta\gamma+b\alpha\gamma+c\alpha\beta)\,. \end{aligned} \tag{2.4}$$

We will also make use of the following yet unpublished result of A. Derdziński:

Proposition 2.1. *Let (M,g) be a four-dimensional Einstein manifold such that its Weyl conformal tensor $W \in C^\infty(\mathrm{End}\Lambda^2 M)$ has constant eigenvalues. Then (M,g) is locally symmetric.*

Since W has constant eigenvalues if and only if $a\,,b\,,c\,,\alpha\,,\beta$ and γ are constant, and hence, if and only if (M,g) is *curvature homogeneous* [19], we can rewrite Proposition 2.1 in the following way:

Proposition 2.2. *A four-dimensional curvature homogeneous Einstein manifold is locally symmetric.*

An Einstein manifold (M,g) is called a *2-stein space* if $F(x) = \Sigma_{a,b} R^2_{xaxb}$ is independent of the unit vector $x \in T_m M$, for all $m \in M$. For a four-dimensional Einstein manifold we have [16]

Lemma 2.3. *Let (M,g) be a four-dimensional Einstein manifold. Then (M,g) is a 2-stein space if and only if*

$$\pm\alpha = a - \tau/12 \;\;,\;\; \pm\beta = b - \tau/12 \;\;,\;\; \pm\gamma = c - \tau/12 \tag{2.5}$$

for each $m \in M$.

Note that if we have the + signs for one orientation of $T_m M$, then we have the – signs for the other orientation.

Now, let (M,g) be an oriented four-dimensional Riemannian manifold, m a point of M and fix an orientation on $T_m M$. The space $\Lambda^2(T_m M)$ of skew-symmetric (0,2)-tensors over $T_m M$ splits as $\Lambda^2(T_m M) = \Lambda^+ \oplus \Lambda^-$, $\Lambda^\pm$ being the eigenspaces of the *Hodge star operator* $* \in \mathrm{End}\Lambda^2(T_m M)$. The Weyl conformal tensor $W \in \mathrm{End}\Lambda^2(T_m M)$ at m leaves $\Lambda^\pm$ invariant. We denote by $W^\pm = \frac{1}{2}(W \pm *W)$ the restriction of W to $\Lambda^\pm$. (M,g) is called *self-dual* (respectively, *anti-self-dual*) if $W^- = 0$ (respectively, $W^+ = 0$) [1]. Since $W^\pm = 0$ if and only if (2.5) is verified with respect to a "positive" Singer-Thorpe basis (see [17]), it follows from Lemma 2.3 that *a four-dimensional Einstein space is 2-stein if and only if* $W^+ = 0$ *or* $W^- = 0$.

A Riemannian manifold is called a *globally Osserman space* if the eigenvalues of the Jacobi operator $R_x = R(x,\cdot)x$ are independent of $m \in M$ and of the unit vector $x \in T_mM$. It is called a *pointwise Osserman space* if the eigenvalues of R_x only depend on m and not on x. For four-dimensional manifolds we have the following result (see [8] for more details):

Proposition 2.4. *Let (M,g) be a four-dimensional manifold. Then the following statements are equivalent:*

(a) (M,g) is a pointwise Osserman space;

(b) (M,g) is 2-stein;

(c) locally there is a choice of the orientation of M for which g is self-dual and Einstein.

Moreover, we have

Proposition 2.5. [6] *A four-dimensional globally Osserman space is flat or locally isometric to a rank-one symmetric space.*

Now we recall some facts about *four-dimensional Kähler-Einstein spaces.* Let (M,g,J) be such a space and $m \in M$. An orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_mM is called an *adapted basis* if $e_2 = Je_1$ and $e_4 = Je_3$. We have

Lemma 2.6. [17] *Let (M,g) be a four-dimensional Kähler-Einstein space. Then there exists an adapted Singer-Thorpe basis of T_mM for each $m \in M$.*

With respect to an adapted Singer-Thorpe basis we obtain easily, by using the Kähler identity for the curvature tensor R,

$$\alpha = b + c = \tau/4 - a\,, \quad \beta = -b\,, \quad \gamma = -c\ . \tag{2.6}$$

Hence and since a Ricci-flat Riemannian manifold is obviously Einsteinian, from (2.6) and Lemma 2.3 we obtain at once the following result:

Proposition 2.7. *A four-dimensional Ricci-flat Kähler manifold is a 2-stein space.*

Finally, we recall the following useful formula for $||R||^2$ when (M,g,J) is a *Hermitian Einstein* manifold [7, (2.50)]:

$$\begin{aligned} ||R||^2 g(x,x)g(x,x) \quad &= \quad 16\,F(x) + 2\,(\tau - 3\tau^*)\,H(x) + \\ &\quad + (1/4)(3{\tau^*}^2 - \tau^2)g(x,x)g(x,x)\,, \end{aligned} \tag{2.7}$$

for all $m \in M$ and $x \in T_mM$. Here, τ^* is the $*$-*scalar curvature* of (M,g,J), that is, $\tau^* = \mathrm{tr}Q^*$ where $g(Q^*x,y) = \varrho^*(x,y) = -(1/2)\mathrm{tr}(z \mapsto R(x,Jy)Jz$, and $H(x)$ is the *holomorphic sectional curvature* of the holomorphic plane determined by x. In particular, when (M,g,J) is Kählerian, we have $\tau = \tau^*$ and hence (2.7) becomes

$$||R||^2 g(x,x)g(x,x) = 16F(x) + (\tau^2/2)g(x,x)g(x,x) - 4\tau H(x)\,. \tag{2.8}$$

This last formula implies at once that a four-dimensional 2-stein Kähler manifold with non-vanishing scalar curvature has constant holomorphic sectional curvature (see also [17]).

3. Four-dimensional ball-homogeneous spaces

Following [14], a Riemannian manifold (M,g) is called *ball-homogeneous* if the volume $V_m(r)$ of any geodesic ball $B_m(r)$ of sufficiently small radius r and center m does not depend on m.

For the volume of geodesic balls we have the following power series expansion:

Proposition 3.1. [9] *For any point $m \in M$ and any small radius $r>0$ we have*

$$V_m(r) = V_0^n(r)(1 + A(m)r^2 + B(m)r^4 + C(m)r^6 + O(r^8)) \tag{3.1}$$

where $V_0^n(r)$ denotes the volume of a Euclidean ball of dimension n=dim$M$ and radius r. The coefficients A, B and C are given, in function of linear, quadratic and cubic scalar curvature invariants, respectively by:

$$\begin{aligned}
A(m) &= -\frac{\tau(m)}{6(n+2)}\,, && (3.2)\\
B(m) &= \frac{(-3||R||^2 + 8||\varrho||^2 + 5\tau^2 - 18\tau\Delta\tau)(m)}{360(n+2)(n+4)}\,, && (3.3)\\
C(m) &= \frac{1}{720(n+2)(n+4)(n+6)}\{-\frac{5}{9}\tau^3 - \frac{8}{3}\tau||\varrho||^2 + && (3.4)\\
&\quad +\tau||R||^2 + \frac{64}{63}\check{\varrho} - \frac{64}{21}g(\varrho\otimes\varrho,\bar{R}) + \frac{32}{7}g(\varrho,\dot{R}) - \\
&\quad - \frac{110}{63}\check{R} - \frac{200}{63}\check{\check{R}} + \frac{45}{7}||\nabla\tau||^2 + \frac{45}{14}||\nabla\varrho||^2 + \\
&\quad +\frac{45}{7}\alpha(\varrho) - \frac{45}{14}||\nabla R||^2 + 6\tau\,\Delta\tau + \frac{48}{7}g(\Delta\varrho,\,\varrho) + \\
&\quad +\frac{54}{7}g(\nabla^2\tau,\,\varrho) - \frac{30}{7}g(\Delta R,\,R) - \frac{45}{7}\Delta^2\tau\}\ (m)
\end{aligned}$$

where

$$\begin{aligned}
\check{\varrho} &= \sum_{i,j,k} \varrho_{ij}\varrho_{jk}\varrho_{ki}\,,\\
g(\varrho\otimes\varrho,\bar{R}) &= \sum_{i,j,k,l} \varrho_{ij}\varrho_{kl}R_{ikjl}\,,\\
g(\varrho,\dot{R}) &= \sum_{i,j,p,q,r} \varrho_{ij}R_{ipqr}R_{jpqr}\,,\\
\alpha(\varrho) &= \sum_{i,j,k} \nabla_i\varrho_{jk}\nabla_k\varrho_{ij}\,.
\end{aligned}$$

So, if (M,g) is ball-homogeneous, then A, B and C must be constant functions on M.

Now, we first prove

Lemma 3.2. *Let (M,g) be a four-dimensional ball-homogeneous Einstein space. Then* $||R||^2$ *and* $||\nabla R||^2$ *are constant on M.*

Proof. Let (M,g) be a ball-homogeneous Einstein space. Since τ and B are constant, from (3.3) it follows that $8||\varrho||^2 - 3||R||^2$ is constant. Moreover, the Einstein condition $\varrho = \frac{\tau}{n} g$ is equivalent to $||\varrho||^2 = \frac{\tau^2}{n}$ and so $||\varrho||^2$ is constant. Hence $||R||^2$ is also constant on M. From (3.4) we then obtain, with $\check{\varrho} = \frac{\tau^3}{n^2}$ and $\nabla\varrho = 0$,

$$\begin{aligned} &- \frac{64}{21} g(\varrho \otimes \varrho, \bar{R}) + \frac{32}{7} g(\varrho, \dot{R}) - \frac{110}{63}\check{R} - \frac{200}{63}\check{\check{R}} - \\ &- \frac{45}{14}||\nabla R||^2 - \frac{30}{7} g(\Delta R, R) = \text{const.} \end{aligned} \tag{3.5}$$

Further, on an Einstein manifold we also have (see [9, p.191])

$$\begin{aligned} &g(\varrho, \dot{R}) = (\tau/n)||R||^2\,, \\ &g(\varrho \otimes \varrho, \bar{R}) = \tau^3/n^2\,, \\ &g(\Delta R, R) = (2\tau/n)||R||^2 - \check{R} - 4\check{\check{R}} \end{aligned}$$

and so, (3.5) yields

$$\frac{160}{63}\check{R} + \frac{880}{63}\check{\check{R}} - \frac{45}{14}||\nabla R||^2 = \text{const.} \tag{3.6}$$

Next, we suppose that (M,g) is four-dimensional. Then, since $||R||^2$ is constant, from (2.2) and (2.3) we have

$$\begin{cases} \check{R} = -\tau^3/12 + (2\tau/3)||R||^2 + (1/3)||\nabla R||^2\,, \\ \check{\check{R}} = \tau^3/48 - (\tau/24)||R||^2 + (1/6)||\nabla R||^2\,, \end{cases} \tag{3.7}$$

and hence, $||\nabla R||^2$ is constant. □

Next, we prove our main results.

Theorem 3.3. *Let (M,g) be a four-dimensional 2-stein space. If (M,g) is ball-homogeneous, then (M,g) is flat or locally isometric to a rank-one symmetric space.*

Proof. First we observe that (M,g) is a pointwise Osserman space (Proposition 2.4). Further, from (2.2), (2.3), (2.4) and (2.5) we obtain, with respect to a Singer-Thorpe basis,

$$\begin{cases} a + b + c = \tau/4\,, \\ ab + bc + ca = -(1/32)||R||^2 + (5/192)\tau^2\,, \\ 96abc = \tau^3/12 - (1/6)\tau||R||^2 + (1/6)||\nabla R||^2\,. \end{cases} \tag{3.8}$$

Since τ, $||R||^2$ and $||\nabla R||^2$ are constant (Lemma 3.2), it follows that a, b and c are constant on M. Hence, (M,g) is a globally Osserman space and this completes the proof because of Proposition 3.5. □

We note that the hypothesis of ball-homogeneity can be replaced by the condition $||\nabla R||^2$ *is constant.*

Lemma 3.4. *Let (M,g) be a four-dimensional Kähler-Einstein space. If (M,g) is ball-homogeneous, then (M,g) is locally symmetric.*

Proof. Since (M,g,J) is a Kähler-Einstein space, we may consider for T_mM an adapted Singer-Thorpe basis as in Lemma 2.6. From (2.4) and (2.6) we then obtain

$$\begin{cases} \tau = 4(a+b+c)\,, \\ ||R||^2 = 8(a^2+3b^2+3c^2+2bc)\,, \\ \check{R} = 48bc(a-b-c)\,. \end{cases} \tag{3.9}$$

Further, formulas (3.7) are still valid and hence, from (3.9) we have

$$\begin{cases} \tau = 4(a+b+c)\,, \\ ||R||^2 = 8(a^2+3b^2+3c^2+2bc)\,, \\ \tau^3/48 - (\tau/24)||R||^2 + (1/6)||\nabla R||^2 = 48bc(a-b-c) \end{cases} \tag{3.10}$$

where τ, $||R||^2$ and $||\nabla R||^2$ are constant on M. Now, we substitute a from the first equation in (3.10) into the other equations. We then have

$$\begin{cases} (b+c)^2 - (\tau/8)(b+c) - bc = \text{const.}\,, \\ bc(\tau/8 - b - c) = \text{const.} \end{cases} \tag{3.11}$$

It follows that bc and $b+c$ are constant, that is, b and c, and hence also a, are constant on M. Then, (M,g) is curvature homogeneous and so Proposition 2.2 yields that (M,g) is locally symmetric. □

Now, we follow the same method as in [7] to prove

Theorem 3.5. *Let (M,g,J) be a connected four-dimensional Hermitian Einstein space. If (M,g) is ball-homogeneous, then (M,g) is locally symmetric.*

Proof. For any four-dimensional Hermitian manifold we have (see [7])

$$\begin{aligned} 2g((\nabla_X J)Y,Z) &= \omega(JX)g(X,Z) + \omega(Y)g(X,JZ) - \\ &\quad - \omega(JZ)g(X,Y) - \omega(Z)g(X,JY) \end{aligned} \tag{3.12}$$

for all smooth vector fields X, Y, Z on M and where $\omega = \delta\Omega \circ J$, Ω being the $Kählerform$ of (M,g,J). The form ω is related to τ and τ^* by the formula

$$\tau - \tau^* = 2\delta\omega + ||\omega||^2\,. \tag{3.13}$$

Now, put

$$\begin{cases} K(x) &= x(\tau^*)H(x)\,, \\ N(x) &= x(\tau^{*^2})g(x,x)g(x,x)\,, \\ P(x) &= (\nabla_x R)(x,Jx,x,Jx) - \omega(x)H(x) - \\ & \quad - \sum_{i=1}^4 \omega(Je_i)\, R(x,e_i,x,Jx)\, g(x,x) \end{cases} \tag{3.14}$$

for $x \in T_mM$, $m \in M$. Using (2.7), (3.12) and (3.14) we obtain by differentiation (see also [7])

$$64xF(x) - 24K(x) + 8(\tau - 3\tau^*)P(x) + 3N(x) = 0\,. \tag{3.15}$$

Next, let D denote the Euclidean Laplacian of T_mM. We first observe that

$$xF(x) = 2\sum_{a,b} R_{xaxb}\nabla R_{xaxb} = 2G(x)\,. \tag{3.16}$$

Then, since (M,g) is Einsteinian and hence Ricci-parallel, we have [17]

$$D^2G(x) = 12x||R||^2\,. \tag{3.17}$$

In our case, since $||R||^2$ is constant, from (3.17) we get $D^2(xF(x)) = 24x||R||^2 = 0$. Calculating $D^2K(x)$, $D^2P(x)$ and $D^2N(x)$ and making use of (3.16) we then obtain

$$(\tau - 3\tau^*)(\tau + 3\tau^*)\omega(x) = 0\,. \tag{3.18}$$

Now, put

$$\begin{aligned} M_0 &= \{m \in M \ / \ (\tau - 3\tau^*)(\tau + 3\tau^*) \neq 0 \ at \ m\}\,, \\ M_1 &= \{m \in M \ / \ \tau + 3\tau^* = 0 \ at \ m\}\,, \\ M_2 &= \{m \in M \ / \ \tau - 3\tau^* = 0 \ at \ m\}\,. \end{aligned}$$

If $M_0 \neq \emptyset$, then, since $\omega(x) = 0$ on M_0, (M_0, g, J) is Kählerian and hence locally symmetric (Lemma 3.4). Then (3.13) implies $\tau = \tau^* =$ const. . So, $M_0 = \bar{M}_0$ and hence $M_0 = M$. Next, if $M_0 = \emptyset$, we have $M = M_1 \cup M_2$.

We consider first the case $\tau \neq 0$. Then, $M_1 \cap M_2 = \emptyset$ and hence $M = M_1$ or $M = M_2$ since M is connected. First, assume $M = M_2$. Then from (2.7) we obtain that $F(x)$ does not depend on $x \in T_mM$, that is, (M,g) is a 2-stein space and hence it is locally symmetric (Theorem 3.3). Next, let $M = M_1$. Following [7] we can construct an adapted Singer-Thorpe basis $\{\ e_1, e_2, e_3, e_4\ \}$ of T_mM and with respect to such a basis we have

$$\alpha = -\tau/12 - a\,, \quad \beta = \tau/6 - b\,, \quad \gamma = \tau/6 - c\,. \tag{3.19}$$

From (2.2), (2.3), (2.4) and (3.19) we then obtain

$$\begin{cases} (a+\tau/8)+b+c=3\tau/8\,, \\ 16\{(a+\tau/8)^2+b^2+c^2\}=5\tau^2/12+||R||^2\,, \\ 16\{4[(a+\tau/8)^3+b^3+c^3]-\tau[(a+\tau/8)^2+b^2+c^2]\}= \\ =(2\tau/3)||R||^2-(13/14)\tau^3+(1/3)||\nabla R||^2\,. \end{cases} \tag{3.20}$$

Since τ, $||R||^2$ and $||\nabla R||^2$ are constant, it follows from (3.20) that a, b and c are constant. So, (M,g) is curvature homogeneous and hence locally symmetric.

Finally, if $\tau=0$, from (3.18) we set $\tau^*=0$ since $M_0=\emptyset$ and hence $\omega\neq 0$. Then (2.7) implies that (M,g) is 2-stein and thus locally symmetric. □

Remark 3.6. Since a D'Atri space is always ball-homogeneous [10], Theorems 3.3 and 3.5 extend the analogous results for D'Atri spaces ([18] and [7], respectively). In that case Lemma 3.4 holds without the Einstein condition [17].

Remark 3.7. It is clear that Lemma 3.4 may be replaced by the following: *Let (M,g,J) be a four-dimensional Kähler manifold. Then it is locally symmetric if and only if it is Ricci-parallel and ball-homogeneous.*

4. Four-dimensional Einstein C-spaces

Let (M,g) be a connected Riemannian manifold, m a point of M and $x\in T_mM$ a unit vector. Denote by $\gamma: r\mapsto \exp(rx)$ the geodesic with arc length r tangent to x at m. Further, let $R_\gamma=R(\dot\gamma,\cdot)\dot\gamma$ be the *Jacobi operator* along γ. (M,g) is called a *C-space* if the eigenvalues of R_γ are constant along γ, for each geodesic γ of (M,g). From this definition we obtain easily the following

Lemma 4.1. *Let (M,g) be a C-space. Then*

$$\nabla_x\varrho_{xx}=0\,, \tag{4.1}$$

$$G(x)=\sum_{a,b}R_{xaxb}\nabla_xR_{xaxb}=0\,, \tag{4.2}$$

$$L(x)=\sum_{a,b,c}R_{xbxc}R_{xcxa}\nabla_xR_{xaxb}=0\,, \tag{4.3}$$

for each $m\in M$ and $x\in T_mM$ unit vector.

Proof. Since $\mathrm{tr}R_x=\varrho_{xx}$ is constant along the geodesic tangent to x at m, we have at once $\nabla_x\varrho_{xx}=0$. Further, we observe that $\mathrm{tr}R_x^2=\sum R_{xaxb}^2$ and $\mathrm{tr}R_x^3=\sum R_{xaxb}R_{xbxc}R_{xcxa}$. Then (4.2) and (4.3) follow in the same way as (4.1). □

Lemma 4.2. *Let (M,g) be a Ricci-parallel C-space. Then $||R||^2$ is constant on M.*

Proof. From Lemma 4.1 it follows that G $=0$. Since (M,g) is Ricci-parallel, (3.18) holds and hence $||R||^2$ is constant on M. □

Remark 4.3. If (M, g, J) is a Kählerian C-space, then $||R||^2$ is constant on M. In fact, (M, g, J) is a Kähler manifold with $\nabla_x \varrho_{xx} = 0$. Then, (M, g) is Ricci-parallel (see [16]) and so the result follows from Lemma 4.2.

Now, we can prove our main results about four-dimensional C-spaces.

Theorem 4.4. *Let (M,g) be a four-dimensional 2-stein space. If (M,g) is a C-space, then (M,g) is locally flat or locally isometric to a rank-one symmetric space.*

Proof. We proceed as in the proof of Theorem 3.3. Since (M, g) is Einsteinian, it is Ricci-parallel and hence $||R||^2$ is constant (Lemma 4.2). Moreover, since (M, g) is 2-stein, we have (see [18], p.221)

$$D^3 L(x) = 40x||\nabla R||^2 \,. \tag{4.4}$$

But $L(x) = 0$ (Lemma 4.1) and hence $||\nabla R||^2$ is constant. So, a, b and c are constant on M and we can conclude, as in the proof of Theorem 3.3, that (M, g) is flat or locally isometric to a rank-one symmetric space. □

Theorem 4.5. *A four-dimensional Kählerian C-space is locally symmetric.*

Proof. First, we observe that (M, g, J) is Ricci-parallel (Remark 4.3). So, if (M, g) is locally reducible, it is locally symmetric.

Now, let (M, g) be locally irreducible. Then it is Einsteinian. We consider first the case $\tau \neq 0$. Let m be a point of M, $x \in T_m M$ a unit vector and γ the geodesic tangent to x at m. Since (M, g) is a C-space, we have $G(\dot\gamma) = 0$ (formula (4.2)). But $G(\dot\gamma) = (1/2)\nabla_{\dot\gamma(r)} F(\dot\gamma(r))$. Then, $F(\dot\gamma(r))$ is constant along γ. Since $||R||^2$ is constant, from (2.8) it follows that $H(\dot\gamma(r))$ is constant along γ. Thus, by deriving along γ, we obtain

$$\nabla_{\dot\gamma(r)} R_{\dot\gamma(r) J\dot\gamma(r)\dot\gamma(r) J\dot\gamma(r)} = \nabla_{\dot\gamma(r)} H(\dot\gamma(r)) = 0\,, \tag{4.5}$$

that is, $\nabla_x R_{xJxxJx} = 0$ for each $x \in T_m M$ and hence, (M, g) is locally symmetric [16, Lemma 10].

If $\tau = 0$, (2.8) implies that (M, g, J) is 2-stein and hence locally symmetric (Theorem 4.4). □

Theorem 4.6. *Let (M,g,J) be a four-dimensional Hermitian Einstein space. If (M,g) is a C-space and $||\nabla R||^2$ is constant on M, then (M,g) is locally symmetric.*

Proof. We may follow the same argument as for Theorem 3.5. In this case, formula (3.15) is simpler, since $xF(x) = 2G(x) = 0$. □

If we suppose that M is compact, we can delete the hypothesis "$||\nabla R||^2$ *is constant on* M" in Theorem 4.6. In fact, we have

Theorem 4.7. *A compact, four-dimensional, Hermitian Einstein C-space is locally symmetric.*

Proof. We can follow again the proof of Theorem 3.5 up to the case $M = M_1$. For this case we consider the following two possibilities:
a) $\tau = 0$. Then, $\tau^* = -\tau/3 = 0$ and (2.7) implies that (M, g) is 2-stein and hence locally symmetric.
b) $\tau \neq 0$. We shall prove that this case does not occur. Indeed, because of the compactness of M we can integrate (3.13) on M. We have

$$\int_M (\tau - \tau^*)\, dM = \int_M ||\omega||^2 dM \geq 0 \ . \tag{4.6}$$

Since $\tau + 3\tau^* = 0$, (4.6) yields

$$\int_M \tau \geq 0 \ . \tag{4.7}$$

But since τ is a nonvanishing constant, (4.7) implies $\tau > 0$. Hence, $\tau^* = -\tau/3 < 0$. Then (M, g) is Kählerian [15]. So, $\omega = 0$ and because of (3.13), $\tau = \tau^*$. Since $\tau + 3\tau^* = 0$, we have $\tau = 0$ which contradicts the hypothesis. □

Remark 4.8. It follows from Proposition 2.7 and Theorems 3.3 and 4.4 that any four-dimensional Ricci-flat Kähler manifold which is ball-homogeneous or a C-space is locally flat. This implies that a K_3-space (see [2]) cannot be ball-homogeneous nor a C-space.

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