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On the Centre of Gravity and Width of Lattice-constrained Convex Sets in the Plane

M. A. Hernández Cifre[1] **P. R. Scott**[2] **S. Segura Gomis**[3*]

[1,3] *Departamento de Matemáticas, Universidad de Murcia*
30100-Murcia, Spain
[2] *Department of Pure Mathematics, University of Adelaide*
South Australia 5005
e-mail: mhcifre@fcu.um.es pscott@maths.adelaide.edu.au salsegom@fcu.um.es

Abstract. In 1982, Scott [5] proposed the following conjecture: Let $\mathbf{Z}^2$ denote the lattice of integer points in the plane, and let K be a convex set having the centroid as the only interior lattice point; then, $\omega(K) \leq \frac{3\sqrt{2}}{2}$. In this paper we give a counterexample to that conjecture and propose another bound. Besides, we prove that $\omega(K) \leq \frac{3\sqrt{2}}{2}$ for the family of all triangles.

MSC 1991: 52C05, 52A40.

1. Introduction

Let $\mathbf{Z}^2$ denote the integer lattice in the Euclidean plane E^2. Let K be a convex set which contains the origin O, but no nonzero points of $\mathbf{Z}^2$ in its interior. Let $\omega(K)$ denote the minimal width of K, that is, the smallest distance between two parallel support lines of K.

In 1982 [5], Scott conjectured that if the centroid of K lies at O, then the inequality

$$\omega(K) \leq \frac{3\sqrt{2}}{2}$$

holds, with equality if and only if K is the triangle whose vertices are $(1,-1)$, $(1,2)$ and $(-2,-1)$, shown in Figure 1.

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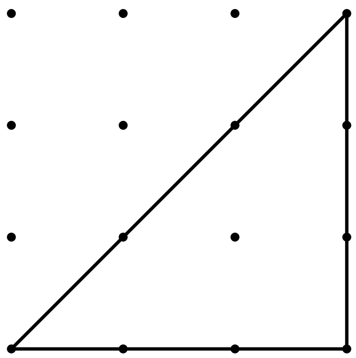

Fig. 1: Scott's triangle

In this paper we obtain a counterexample to this conjecture, and we prove that if the question is restricted only to the family of all triangles, then the answer is "yes"; but the equality sign is not only attained by the triangle shown in Figure 1, but also for an infinite family of triangles.

Several related results estimating $\omega(K)$, according to the different number of points of $\mathbf{Z}^2$ contained in the interior of K, can be found in [4], [6] and [7]. The 3-dimensional analogs promise to be awkward. Good general references for this kind of problems are [1], [3].

2. The counterexample and the new conjecture

The quadrilateral K_0 shown in Figure 2, with vertices

$$P = (-1,-1) \quad R = \left(\frac{2}{3}, \frac{2}{3}\right)$$

$$Q = \left(\frac{4}{3}, -\frac{2}{3}\right) \quad S = \left(-\frac{2}{3}, \frac{4}{3}\right)$$

does not contain any nonzero point of $\mathbf{Z}^2$ in its interior.

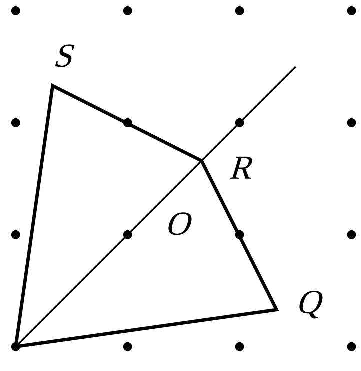

Fig. 2: Counterexample

Besides, its center of gravity G is

$$G = \frac{1}{2}\left[\frac{1}{3}(P+Q+R) + \frac{1}{3}(P+S+R)\right] = (0,0)$$

and its minimal width

$$\omega(K_0) = \min\{d(P,QR), d(Q,PS)\} =$$

$$= \min\left\{\frac{8\sqrt{2}}{5}, \sqrt{5}\right\} = \sqrt{5} > \frac{3\sqrt{2}}{2}.$$

So, the conjecture is false.

It can be proved that K_0 has maximum minimal width among the family of all quadrilaterals satisfying the assumptions which are symmetric with respect to the straight line $y = x$.

As the solutions to analogous problems concerning the integer lattice and looking for sets with maximum minimal width use to be polygons with a small number of vertices, it seems reasonable to state the following conjecture:

Conjecture 1. *Let $K \subset E^2$ be a convex set, which does not contain any other point of the integer lattice $\mathbf{Z}^2$ in its interior besides the origin O; if the centroid of K lies at O, then*

$$\omega(K) \leq \sqrt{5}$$

when the equality sign is attained if and only if K is the quadrilateral K_0 shown in Figure 2.

3. Scott's conjecture for the family of triangles

The next proposition proves that if the conjecture is restricted to the family of triangles, the minimal width is indeed smaller or equal than $\dfrac{3\sqrt{2}}{2}$.

There are infinite triangles for which the maximum minimal width is attained: it can be proved that the maximum minimal width is attained (up to integral unimodular transformations) only for the triangles $T_{\varepsilon,\delta}$ whose vertices are:

$$A = (1-\varepsilon, -1-\varepsilon), \quad B = (2\varepsilon+\delta-2, 2\varepsilon+\delta-1), \quad C = (1-\varepsilon-\delta, 2-\varepsilon-\delta),$$

where

$$\varepsilon \in \left[0, 1 - \frac{\sqrt{2}}{2}\right]$$

and

$$\delta \in \left[0, \frac{3 - 4\varepsilon - \sqrt{4\varepsilon^2+3}}{2}\right].$$

For instance, in the case $\varepsilon = 0$, we obtain the family (see Figure 3)

$$T_{0,\delta} = \left\{ conv\,\{(1,-1), (-2+\delta, -1+\delta), (1-\delta, 2-\delta)\}, \delta \in \left[0, \frac{3-\sqrt{3}}{2}\right]\right\}.$$

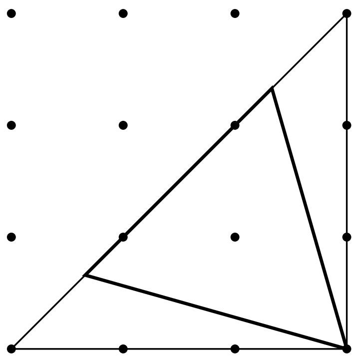

Fig. 3

Note that when $\delta = 0$, the triangle shown in Figure 1 is obtained.

Proposition 1. *Let T be a triangle whose centroid lies at the origin O and such that T does not contain any other point of $\mathbf{Z}^2$ in its interior besides O. Then,*

$$\omega(T) \leq \frac{3\sqrt{2}}{2}.$$

Proof. Since T contains no non-zero lattice-points, one side of the triangle separates O from an adjacent pair of lattice-points $(\pm 1, 0)$, $(0, \pm 1)$. It follows that the distance of this side from O is at most $\frac{\sqrt{2}}{2}$. Since it is well known [2] that the centre of gravity of a convex figure divides a chord through it in a ratio between $\frac{1}{2}$ and 2, it follows that

$$\omega(T) \leq \frac{3\sqrt{2}}{2}. \qquad \square$$

The following corollary can be easily established:

Corollary 1. *If K is any convex set, and a support line of K separates O from an adjacent pair of the lattice-points $(\pm 1, 0)$, $(0, \pm 1)$, then*

$$\omega(K) \leq \frac{3\sqrt{2}}{2}.$$

References

[1] Croft, T.; Falconer, J.; Guy, K.: *Unsolved Problems in Geometry*, Springer-Verlag, New York 1991.

[2] Ehrhart, E.: *Sur les ovales et les ovoides.* Comptes Rendus **240** (1955), 583–385.

[3] Erdös, P.: Gruber, P. M.; Hammer, J.: *Lattice Points.* Longman Scientific and Technical, New York 1989.

[4] Scott, P. R.: *A lattice problem in the plane*. Mathematika **20** (1973), 247-252.

[5] Scott, P. R.: *Two problems in the plane*. Amer. Math. Monthly **89** (1982), 460-461.

[6] Scott, P. R.: *On planar convex sets containing one lattice point*. Quart. J. Math. Oxford Ser. 2 **36** (1985), 105-111.

[7] Scott, P. R.: *On the width of a planar convex set containing zero or one lattice points.* Bull. Austral. Math. Soc. **48** (1993), 47–53.

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