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Finite Sphere Packings and Critical Radii

Dedicated to Edmund Hlawka at the occasion of his 80th birthday

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Abstract. The parametric density permits a joint theory for classical packings and sausage packings of convex bodies, in particular spheres. If the parameter ϱ is small then linear packings ("sausages") are optimal; if ϱ is large then full dimensional packings ("clusters") are optimal and the packings tend to the classical densest infinite packing as the number of bodies tend to infinity. It was conjectured, namely, the Strong Sausage Conjecture, that for sphere packings no intermediate optimal packings exist, i.e. either the minimal or the maximal possible dimension of the packing configuration is attained. The proof of this conjecture would imply a proof of Kepler's conjecture for infinite sphere packings, so even in $\mathbf{E}^3$ only partial results can be expected.
The critical parameter depends on the dimension and on the number of spheres, so if the parameter ϱ is fixed then abrupt changes of the shape of the optimal packings ("sausage catastrophe") occur as the number of spheres grows. In this paper, we give several partial results, in particular in $\mathbf{E}^3$, which support the conjectures.

1. Introduction

Let B^d denote the unit ball in euclidean d-space $\mathbf{E}^d$, $d \geq 2$. For a convex body $K \subset \mathbf{E}^d$ let $V(K)$ denote its volume, e.g. $V(B^d) = \kappa_d$. For a finite set $C_n = \{c_1 \ldots, c_n\} \subset \mathbf{E}^d$ with $||c_i - c_j|| \geq 2$, $i \neq j$ we call $C_n + B^d$ a finite sphere packing or briefly a packing. If $L \subset \mathbf{E}^d$

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is a packing lattice for B^d and $C_n \subset L$, then $C_n + B^d$ is called a lattice packing of spheres. For $\varrho > 0$ the parametric density of $C_n + B^d$ is defined (cf. [8], [28]) by

$$\delta_\varrho(B^d, C_n) = \frac{nV(B^d)}{V(\operatorname{conv} C_n + \varrho B^d)},$$

where '+' denotes the Minkowski sum. Further let

$$\delta_\varrho(B^d, n) = \max\{\delta_\varrho(B^d, C_n) | C_n + B^d \text{ packing}\}.$$

The optimal C_n with $\delta_\varrho(B^d, C_n) = \delta_\varrho(B^d, n)$ are denoted by $C_{d,n,\varrho}$ or briefly by $C_{n,\varrho}$. Linear arrangements, i.e. with $\operatorname{conv} C_n$ being a segment, play an essential role in finite sphere packing. If $\operatorname{conv} C_n$ is a segment of length $2(n-1)$, we write $C_n = S_n$ and call $S_n + B^d$ a sausage. Its density $\delta_\varrho(B^d, S_n)$ can easily be calculated and gives a good lower bound for $\delta_\varrho(B^d, n)$, as the following simple and basic result from [8] (Theorem 1a, slightly different notation) shows: For $d \geq 2$ and $n \geq 3$ there is a $\varrho_{d,n} > 0$ with

$$C_{n,\varrho} = S_n \text{ and } \delta_\varrho(B^d, n) = \delta_\varrho(B^d, S_n) \text{ for } 0 < \varrho \leq \varrho_{d,n}$$

$$C_{n,\varrho} \neq S_n \text{ and } \delta_\varrho(B^d, n) > \delta_\varrho(B^d, S_n) \text{ for } \varrho > \varrho_{d,n}.$$

If one restricts oneself to lattice packings, then one gets analogously $\delta_{L,\varrho}(B^d, n)$ and $\varrho^L_{n,d}$. The densities $\delta_\varrho(B^d, n)$, $\delta_{L,\varrho}(B^d, n)$ and the critical radii $\varrho_{n,d}$ and $\varrho^L_{n,d}$ are the essential functionals in finite sphere packings.

The classical sphere packing density is denoted by $\delta(B^d)$ and in the lattice case by $\delta_L(B^d)$. The density of an 'infinite sausage' is given by

$$\delta_{s,\varrho}(B^d) = \lim_{n\to\infty} \delta_\varrho(B^d, S_n) = (\kappa_d/2\kappa_{d-1})\varrho^{1-d}.$$

Now we can define the relevant functionals for infinite parametric sphere packings:

$$\delta_\varrho(B^d) = \limsup_{n\to\infty} \delta_\varrho(B^d, n)$$
$$\varrho_{d,s} = \sup\{\varrho | \delta_\varrho(B^d) = \delta_{s,\varrho}(B^d)\}$$
$$\varrho_{d,c} = \inf\{\varrho | \delta_\varrho(B^d) = \delta(B^d)\}$$

and the corresponding functionals for lattice packings: $\delta_{L,\varrho}(B^d)$, $\varrho^L_{d,s}$, $\varrho^L_{d,c}$. Clearly

$$\varrho_{d,s} \leq \varrho^L_{d,s} \leq \varrho^L_{d,c} \text{ and } \varrho_{d,s} \leq \varrho_{d,c}.$$

For $d = 2$ Groemer [19] showed, in a different notation:

$$\varrho_{2,n} = \varrho^L_{2,n} = \varrho_{2,s} = \varrho^L_{2,s} = \varrho_{2,c} = \varrho^L_{2,c} = \sqrt{3}/2 \text{ for } n \in \mathbf{N}.$$

The situation for $d \geq 3$ is much more complicated. In this paper some partial results on these critical radii for $d \geq 3$ are given.

The following results from [8] show the close relation between critical radii and classical packing densities:

$$(\kappa_d/2\kappa_{d-1})(\varrho_{d,c})^{1-d} \leq \delta(B^d) \leq (\kappa_d/2\kappa_{d-1})(\varrho_{d,s})^{1-d}$$

$$(\kappa_d/2\kappa_{d-1})(\varrho^L_{d,c})^{1-d} \leq \delta_L(B^d) \leq (\kappa_d/2\kappa_{d-1})(\varrho^L_{d,s})^{1-d}.$$

In [8] it was conjectured

Strong Sausage Conjecture (SSC):

$$\varrho_{d,c} = \varrho_{d,s} \tag{1.1}$$

and

$$\varrho_{d,n} > \varrho_{d,s} \;\; for \;\; d \geq 3,\, n \geq 3; \tag{1.2}$$

and the corresponding

Strong Sausage Conjecture for lattice packings:

$$\varrho^L_{d,c} = \varrho^L_{d,s} \tag{1.3}$$

and

$$\varrho^L_{d,n} > \varrho^L_{d,s} \;\; for \;\; d \geq 3,\, n \geq 3. \tag{1.4}$$

These conjectures are closely related to L. Fejes Tóth's sausage conjecture:

$$\varrho_{d,n} > 1 \text{ for } d \geq 5 \text{ and } n \geq 3$$

and the conjecture on sausage catastrophes [27], [18]:

$$\varrho_{3,n} > 1 \quad \text{for } n < 56$$

$$\varrho_{4,n} > 1 \quad \text{for } n < 375.000.$$

A proof of the SSC in $\mathbf{E}^3$ would also imply a proof of Kepler's conjecture. So at the moment a general proof of the SSC or even the SSC for lattice packings seems to be out of reach. In this paper we give partial results on the SSC for lattice packings. The following three partial approaches are successful:

1. All $n \in \mathbf{N}$ and special lattice C_n. Our Theorems 1,2 and 3 are of this type.
2. Small n and arbitrary lattice C_n. Our Theorems 4, 5 and 6 are of this type. Results of this type for general packings are in [3].
3. $d = 3$, large n and arbitrary lattice C_n. Theorems of this type are in [1].

Final remarks:
1) If $d \geq 3$ then there are even centrally symmetric convex bodies K, for which no analogue of SSC holds. This was first observed by Gritzmann [14] for $d \geq 4$, cf. also [8]. In $\mathbf{E}^3$, the regular octahedron is such an example (cf. [1]).
2) The problem of finding densest sphere packings, in particular for $d = 3$ and small n, occurs also for other packing densities, namely for the Lennard-Jones potential in crystal physics (cf. [22], [23]) and for the second moment density ("penny packings", cf. [17], [26]). Their methods are completely different from ours: They use various effective computer programs, whereas for our parametric density the classical and powerful Brunn-Minkowski theory of convexity is available. Independently of the method chosen, the best packings for $d = 3$ and $n \leq 5$ are the same for all of these densities; whereas they differ for large n.

2. Results

The first three results show that the SSC for lattice packings holds for special classes of lattice polytopes; and for all $n \in \mathbf{N}$, not only for large n.

Theorem 1. *The strong sausage conjecture for lattice packings* (1.3), (1.4) *holds for lattice zonotopes in* $\mathbf{E}^d, d \leq 48$.

Remarks. 1.) The reason for the restriction $d \leq 48$ is only because of limited information on δ_L.
2.) In the plane, any empty lattice triangle is one half of some fundamental parallelogram. It follows that Theorem 1 can be proved for a somewhat wider class of lattice polytopes $Q_n = \operatorname{conv} C_n$, namely if $Q_n = Z + P_1 + \ldots + P_m$, $m \leq d/2$, where Z is a lattice zonotope and $P_1, \ldots, P_m$ are planar lattice polygons so that $\operatorname{aff} P_1, \ldots, \operatorname{aff} P_m$ are independent.

The next classes of polytopes are rather small, but they contain the densest known lattice packings in $\mathbf{E}^3$, besides sausage packings. Some definitions are needed. Let a, b, c be a basis of a lattice $L \subset \mathbf{E}^3$ such that the angles of any face of the tetrahedron $\operatorname{conv}\{o, a, b, c\}$ are $\leq \pi/2$.

Let $T = \operatorname{conv}(O, a, b, c)$ be the lattice tetrahedron generated by a, b, c and the origin O. Any lattice polytope of L with all its facets (at most 8) parallel to one of the 4 facets of T is called a generalized lattice octahedron.

Theorem 2. *The strong sausage conjecture for lattice packings* (1.3),(1.4) *holds for generalized lattice octahedra.*

For the next result we only consider the critical lattice or fcc-lattice $L' \subset \mathbf{E}^3$ (cf. [13], [17]), which generates the densest lattice packing of spheres in $\mathbf{E}^3$. Let Q be the convex hull of the 12 lattice points of L' with distance 2 from the origin. Q is the regular cuboctahedron of edge-length 2. It is centrally symmetric and has 14 facets, namely 6 squares and 8 regular triangles. Any lattice polytope of L' with all its facets (at most 14) parallel to one of the 14 facets of Q is called a generalized L'-cuboctahedron. Observe that each cube, each regular simplex and each regular octahedron of L' is a generalized L'-cuboctahedron.

Theorem 3. *The strong sausage conjecture for lattice packings* (1.3),(1.4) *holds for generalized L'-cuboctahedra.*

Remark. From the proof follows that Theorem 3 also holds for lattices which are close to L' (in the sense of Mahler).

The next 3 results deal with small n. T^d means the regular d-simplex.

Theorem 4. *For $d \geq 3$, $n = 4$ and $\varrho > \varrho^L_{d,4}$ holds* $\operatorname{conv} C_{d,4,\varrho} = T^3$.

Corollary 1. *The equality $\varrho^L_{d,4} = 0.6814\ldots\sqrt{d}$ holds for large d.*

Theorem 5. *For $d = 3$, $n = 5$ and $\varrho > \varrho^L_{3,5}$ holds* $\operatorname{conv} C_{3,5,\varrho} = N$, *where N is the pyramid over the square, with all edges of length* 2.

Denote by D_5 the bipyramid over the regular triangle having edge-length 2. As $V(D) = V(N)$ and $V_i(D_5) < V_i(N)$ for $i = 1, 2$, we deduce a significant difference between lattice and nonlattice packings:

Corollary 2. *For $\varrho > \varrho^L_{3,5}$, the optimal packing is D_5, i.e. not a lattice packing.*

Finally, we present a result supporting the sausage catastrophe. Note that $\varrho^L_{d,4} \sim 0.6814 d^{1/2}$ and (see [21])

$$\liminf_{n\to\infty} \varrho_{d,n} \le \frac{\sqrt{21}}{2} = 2.2912.$$

Theorem 6. 1. *$V(T^d + \varrho B^d) < V(S_n + \varrho B^d)$ holds if $\varrho > 0.6 d^{1/2}$.*
2. *For large d, $V(T^d + \varrho B^d) > V(S_n + \varrho B^d)$ holds if $\varrho < d^{1/2 - o(1)}$.*

3. Proof of Theorem 1

For the proofs, we need some classical definitions which can be found e.g. in [13], [15], or in [25]. We mention them briefly, in order to keep the paper self-contained.

Let K be a convex, compact set in $\mathbf{E}^d$. The *intrinsic i-volume* $V_i(K)$ of K can be defined via Steiner's formula:

$$V(K + \varrho B^d) = \sum_{i=0}^{d} \kappa_i V_{d-i}(K)\, \varrho^i \tag{3.5}$$

where κ_i is the volume of the unit i-ball. Here $V_d(K) = V(K)$, $2V_{d-1}(K)$ is the surface-area $S(K)$ of K and $V_0(K) = 1$. In addition, $\frac{2\kappa_{d-1}}{d\kappa_d} V_1(K)$ is the mean width of K. The intrinsic volumes are continuous, monotone and dimension invariant valuations.

Let P be a convex d-polytope, and denote by $\mathcal{F}_i$ the set of i-faces of it. For an $F \in \mathcal{F}_i$, $\alpha(F, P)$ is the interior angle at F and $\beta(F, P)$ is the exterior angle, normalized so that the total angle is one. We set $\beta(P, P) = 1$. Then $V_i(P)$ can be calculated by

$$V_i(P) = \sum_{F \in \mathcal{F}_i} \beta(F, P)\, |F|, \tag{3.6}$$

where $|F|$ denotes the relative volume in the affine hull.

Let Λ be a packing lattice. The weighted lattice point enumerator of a lattice polytope P (see [6] or [15]) is defined as

$$\widehat{G}_\Lambda(P) = \sum_{x \in \Lambda \cap P} \alpha(x, P).$$

Note that there exists a corresponding simple valuation which is $\widehat{G}_\Lambda(P)$ for d-polytopes and zero otherwise. The number of lattice points in P satisfies the relation (see [6])

$$G_\Lambda(P) = \sum_{i=0}^{d} \sum_{F \in \mathcal{F}_i} \beta(F, P)\, \widehat{G}_\Lambda(F) \tag{3.7}$$

where $\alpha(x,P)=1$ if $x\in \text{int}P$. Comparing (3.6) and (3.7) shows that in order to establish a formula of the type

$$\sum_{i=0}^{d}\lambda_i V_i(P)\geq G_\Lambda(P)$$

for non-negative $\lambda_0,\ldots,\lambda_d$, it is sufficient to prove that

$$\lambda_i|F|\geq \widehat{G}_\Lambda(F) \tag{3.8}$$

holds for any i-face F of P.

Now let $\varrho>0$ and $C_n\subset\Lambda$. We deduce from (3.8) and Steiner's formula that $V(\text{conv}C_n+\varrho B^d)>V(\text{conv}S_n+\varrho B^d)$ holds if

$$\frac{\kappa_{d-i}}{\varrho^{i-1}2\kappa_{d-1}}\,|F|>\widehat{G}_\Lambda(F). \tag{3.9}$$

for any i-face F of $\text{conv}C_n$.

So we assume that $\text{conv}C_n$ is a lattice zonotope which can be dissected into lattice parallelotopes. For any lattice parallelotope P (see [6])

$$|P|=\det(\text{aff}\,P\cap\Lambda)\cdot\widehat{G}_\Lambda(P) \tag{3.10}$$

holds. Let Λ_d be a densest packing lattice for B^d (i.e. a critical lattice) and define

$$\varrho(d,i)=\sqrt[i-1]{\frac{\kappa_{d-i}}{2\kappa_{d-1}}\det\Lambda_i}=\sqrt[i-1]{\frac{\kappa_{d-i}\kappa_i}{2\kappa_{d-1}\delta_L(B^i)}},\quad i=2,\ldots,d.$$

We deduce from (3.10) that (3.9) holds if $\varrho<\varrho(d,i)$.

We claim that

$$\varrho(d,i)>\varrho(d,d) \tag{3.11}$$

for $i=2,\ldots,d-1$ implies Theorem 1. So we assume that (3.11) holds. Then we deduce by (3.9) that $V(\text{conv}C_n+\varrho B^d)>V(\text{conv}S_n+\varrho B^d)$ for $\varrho<\varrho(d,d)$, which in turn yields that

$$\delta(C_n,\varrho(d,d))\leq\delta(S_n,\varrho(d,d))$$

where $\delta_\varrho(\cdot)$ is denoted by $\delta(\cdot,\varrho)$ in order to avoid too complicated indices. Since

$$\lim_{n\to\infty}\delta(S_n,\varrho(d,d))=\delta_L(B^d),$$

one obtains that (3.11) yields Theorem 1. In the following lemmas, we verify (3.11) for various cases.

Lemma 1. 1. *$\varrho(d+1,i)\geq\varrho(d,i)$ holds if $d\geq i\geq 2$.*
2. *$\varrho(d,i)>1$ holds if $d>i$.*

Proof. The first property follows as κ_k/κ_{k+1} is monotonely increasing (see [7]). Therefore it is sufficient to check the second property for $d=i+1$, when

$$\varrho(i+1,i)=\sqrt[i-1]{\frac{1}{\delta_L(B^i)}}>1. \qquad \square$$

Lemma 2. (3.11) *holds if* $i = d-3, d-2, d-1$ *or if* $i = d-4$ *and* $i \geq 7$.

Proof. By Lemma 1, it is sufficient to prove that for $i = d-4, \ldots, d-1$, the inequality

$$\det \Lambda_d < \kappa_{d-i} \cdot \det \Lambda_i \tag{3.12}$$

holds.

According to a result of Blichfeldt (see [24], p.79, Lemma 1), the covering radius of Λ_i is at least $\sqrt{2i/(i+1)}$, and hence constructing suitable $(i+1)$-lattice shows that

$$\det \Lambda_{i+1} \leq \sqrt{\frac{2(i+2)}{i+1}} \cdot \det \Lambda_i.$$

In particular,

$$\det \Lambda_d \leq \sqrt{\frac{d+1}{i+1}} \cdot \sqrt{2}^{\,d-i} \cdot \det \Lambda_i.$$

Finally, for $k = 1, 2, 3$ and for $k = 4$ and $d \geq 11$, the inequality

$$\kappa_k > \sqrt{\frac{d+1}{d+1-k}} \cdot \sqrt{2}^{\,k}$$

yields (3.12). □

Table 1 collects upper bounds $\varrho^*(d,d)$ for $\varrho(d,d)$. For these upper bounds, we use the list of good packing lattices in [10].

d	$\varrho^*(d,d)$	d	$\varrho^*(d,d)$	d	$\varrho^*(d,d)$	d	$\varrho^*(d,d)$
2	0.8660	14	1.2329	26	1.3129	38	1.4703
3	0.9488	15	1.2335	27	1.3431	39	1.4768
4	0.9847	16	1.2249	28	1.3653	40	1.4706
5	1.0347	17	1.2465	29	1.3872	41	1.4780
6	1.0564	18	1.2574	30	1.3954	42	1.4803
7	1.0755	19	1.2650	31	1.3884	43	1.4808
8	1.0781	20	1.2640	32	1.3761	44	1.4767
9	1.1367	21	1.2647	33	1.4397	45	1.4732
10	1.1626	22	1.2580	34	1.4541	46	1.4655
11	1.1231	23	1.2500	35	1.4659	47	1.4567
12	1.1354	24	1.2361	36	1.4716	48	1.4443
13	1.2179	25	1.2788				

Table 1

On the other hand, the known values of $\det \Lambda_i$, $i = 2, \ldots, 6$ yield that $\varrho(6,2) = 1.6237$, $\varrho(7,3) = 1.6434$, $\varrho(8,4) = 1.6105$, $\varrho(9,5) = 1.6194$, $\varrho(10,6) = 1.5963$. In addition, we deduce by Blichfeldt's bound (see [11] or [13])

$$\det \Lambda_d \geq \frac{2\,\kappa_d}{d+2} \sqrt{2}^{\,d}$$

and by $\kappa_5 = 8\pi^2/15$ that

$$\varrho(d, d-5) > \sqrt[d-6]{\frac{\sqrt{2}^{\,d-5}\kappa_5\kappa_{d-5}}{(d-3)\kappa_{d-1}}} = \sqrt{2}\,\sqrt[d-6]{\frac{2\sqrt{2}(d-1)}{15}}.$$

Since the function $(t+5)^{1/t}$ is monotonely decreasing for positive t, it follows that

$$\varrho(d, d-5) > \sqrt[42]{\frac{2\sqrt{2}\cdot 47}{15}} = 1.4896\ldots$$

for $d \leq 48$.

Each entry of Table 1 is smaller than $\varrho(i, i{+}4)$ for $i = 2, \ldots, 6$ or $\varrho(i, i{+}5)$ for $i = 7, \ldots, 43$. Hence (3.11) follows by Lemma 1 and Lemma 2, which in turn yields Theorem 1.

Remark. Using Stirling's estimate for the Gamma function and Blichfeldt's lower and Minkowski's upper bound (see [11] or [13]) for $\det \Lambda_d$, it is not hard to establish that $\varrho(d, i) > \varrho(d, d)$ if $i \leq d/2$. On the other hand, these bounds are not good enough to yield that Theorem 1 holds in general.

Actually, (3.11) is a necessary condition for the SSC for lattice packings, so these relations for $\varrho(d, i)$ support the general statement as well.

4. Proof of Theorem 2

A dissection or tiling of a set $S \subset \mathbf{E}^d$ by sets S_i, $i = 1, \ldots, m$ means:

$$S = \bigcup_{i=1}^{m} S_i \quad \text{and} \quad \operatorname{int}(S_i \cap S_j) = \emptyset \quad \text{for} \quad i \neq j.$$

Let $C_n + B^d$ and $\varrho > 0$ be given. Further let $P_1, \ldots, P_m$ be a tiling of $\mathbf{E}^d$. This tiling induces a tiling $P'_1, \ldots, P'_m$ on $C_n + B^d$ and a tiling $P''_1, \ldots, P''_m$ on $\operatorname{conv} C_n + \varrho B^d$. For simplicity we always assume that the P_i and hence the P'_i, P''_i are in the convex ring, i.e. unions of finitely many convex bodies.

Let $V(P'_i) = v'_i$, $V(P''_i) = v''_i$ and $\Theta_i = v'_i/v''_i$, $i = 1, \ldots, m$. Then $\delta_\varrho(B^d, C_n) = (\sum_{i=1}^m v'_i) / (\sum_{i=1}^m v''_i) = (\sum_{i=1}^m \Theta_i v''_i) / (\sum_{i=1}^m v''_i)$. We further assume that the tilings always contain the outer normal cones of the vertices of $\operatorname{conv} C_n$ as tiles. We call these tiles vertex tiles. For vertex tiles holds $\Theta_i = \varrho^{-d}$. These tilings just described we call *appropriate tilings*, and we get the simple but useful

Lemma 3. *Let $C_n + B^d$ a packing, $\varrho > 0$ and $P'_1, \ldots, P'_m$ and $P''_1, \ldots, P''_m$ appropriate tilings of $C_n + B^d$ and $\operatorname{conv} C_n + \varrho B^d$.*
Further let the inequality $\Theta_i \leq (\kappa_d/2\kappa_{d-1})\varrho^{1-d}$ hold for all tiles except the vertex tiles, with strict inequality for at least one i. Then

$$\delta_\varrho(B^d, C_n) < \delta_\varrho(B^d, S_n).$$

Proof. We have $V(\mathrm{conv}\, C_n + \varrho\, B^d) = \sum_i^1\, v_i'' + \sum_i^2\, v_i''$, where $\sum_i^2$ contains exactly the vertex tiles. Hence $\sum_i^2\, v_i'' = V(\varrho\, B^d) = \kappa_d\, \varrho^d$ and $V(\mathrm{conv}\, C_n + \varrho B^d) - \kappa_d\, \varrho^d = \sum_i^1\, v_i'' = \sum_i^1\, v_i'/\Theta_i$ and

$$\textstyle\sum_i^1\, v_i' = (n-1)\kappa_d. \tag{4.13}$$

We now consider the sausage packing $S_n + B^d$ and the appropriate tiling induced by the n hyperplanes orthogonal to $\mathrm{conv}\, S_n$ and containing one of the sphere centers c_i. If we denote the tiles and densities by $\widehat{v}_j$, $\widehat{\Theta}_j$, then $\widehat{\Theta}_j = \Theta_j = (\kappa_d/2\kappa_{d-1})\, \varrho^{1-d}$, and

$$V(\mathrm{conv}\, S_n + \varrho\, B^d) - \kappa_d\, \varrho^d = \textstyle\sum_j^1\, \widehat{v}_j'/\widehat{\Theta}_j = (2\kappa_{d-1}/\kappa_d)\varrho^{d-1}\, \sum_j^1\, \widehat{v}_j' = 2(n-1)\kappa_{d-1}\, \varrho^{d-1}.$$

So $V(\mathrm{conv}\, S_n + \varrho\, B^d) - V(\mathrm{conv}\, C_n + \varrho\, B^d) = 2(n-1)\kappa_{d-1}\, \varrho^{d-1} - \sum_i\, v_i'/\Theta_i$. By assumption $\Theta_i \le (\kappa_d/2\kappa_{d-1})\, \varrho^{1-d}$, and for at least one i with $<$, so we get with (4.13)

$$V(\mathrm{conv}\, S_n + \varrho\, B^d) - V(\mathrm{conv}\, C_n + \varrho\, B^d) < 2(n-1)\kappa_{d-1}\, \varrho^{d-1}\left(1 - ((n-1)\kappa_d)^{-1}\textstyle\sum_i^1\, v_i'\right) = 0. \ \square$$

With Lemma 3 we are ready for the

Proof of Theorem 2. Let $L \subset \mathbf{E}^3$ be a packing lattice and let a, b, c be a basis of L such that the four heights h_0, h_a, h_b, h_c of the simplex $T = \mathrm{conv}\,(0, a, b, c)$ are all ≥ 1. This is the case if e.g. all angles at the facets of T are $\le \pi/2$, and hence for each packing lattice such bases exist.

Without restriction we assume $1 \le h_0 \le h_a \le h_b \le h_c$, because otherwise we rearrange $0, a, b, c$ without changing L and T.

The corresponding facets of T are

$$f_0 = \mathrm{conv}\,(a, b, c),\ f_a = \mathrm{conv}\,(0, b, c),\ f_b = \mathrm{conv}\,(0, a, c),\ f_c = \mathrm{conv}\,(0, a, b).$$

For simplicity we use the same letters also for the 2-dimensional volumes and so we have

$$f_0 \ge f_a \ge f_b \ge f_c.$$

Now let $Q \subset \mathbf{E}^3$ be a generalized lattice-octahedron of L with respect to T. The facets parallel to f_0 are denoted by F_0, F_0'; similarly for $F_a, F_a', F_b, F_b', F_c, F_c'$. Some of these facets may be empty. Then the tiling is constructed as follows:

From T one gets the corresponding fundamental cell Z of L with $V(Z) = \det\, L = 6V(T)$. The parallelotope Z can be dissected into 3 tiles as follows:

T, a translate of $T' = -T$ and the remaining octahedron O' with $V(O') = 4V(T)$. The translates Z_i of Z and their dissections into translates T_i of T, T_i' of T' and O_i' of O' induce a tiling of $\mathbf{E}^3$.

Easy considerations show that this tiling of $\mathbf{E}^3$ induces also a tiling of Q, and this tiling has the nice property that the Z_i can be dissected at most by F_0, F_0', but not by $F_a, F_a', F_b, F_b', F_c, F_c'$, because these facets are parallel to those of the Z_i.

In the following, it turns out that the proof is simple if the facets F_0, F_0' do not exist.

The tiling of Q induces a tiling on the facets of Q into triangles congruent to f_0, f_a, f_b, f_c and hence a tiling on $Q + \varrho_c\, B^d$ into the tiles in Q and triangular prisms of height ϱ_c over the facets, denoted by $P_{0,i}, P_{a,i}, P_{b,i}, P_{c,i}$, further cylindrical tiles at the edges and vertex tiles at the vertices.

From Lemma 3 follows that the vertices need not be considered. The density of the cylindrical tiles is δ_L^3, the same holds for any cell Z_i; hence these tiles need not be considered.

All triangular prisms over the facets of Q have height $\varrho_c = 2^{2/3}\pi^{-1/2}$ and density at most $(\pi/(3\sqrt{2}))^3 = (\delta_L^3)^3 < \delta_L^3$. These prisms will be needed to compensate the critical tiles of possibly high density, namely

tetrahedra T_i or T_i' or octahedra O_i' having one triangle in a facet F_0 or F_0' of Q. These triangles are either translates of f_0 or of $-f_0 = f_0'$, and we denote them by $f_{0,i}$ and $f_{0,i}'$. If we join these triangles in pairs along their common edge parallel to the edge $\overline{b_c}$, then we get a tiling of aff F_0 and aff F_0' into parallelograms.

In the following we consider only F_0, because everything holds also for F_0'. Each parallelogram in F_0 is incident with one tetrahedron and one octahedron. Here it suffices to consider one half of the octahedron, because the other half either coincides with another tetrahedron and hence is a tile with density $\leq \delta_L^3$ or has one triangular face in F_0' and is considered in F_0'.

So each parallelogram in F_0 incident with one tetrahedron and one half of an octahedron is a tile of density $\leq \delta_L^3$.

The remaining triangles of F_0, which cannot be completed to a parallelogram, have the property that their edge parallel to $\overline{b_c}$ is contained in one of the two edges E_1, E_2 of F_0 parallel to $\overline{b_c}$, and belonging to F_a or F_a' (E_1, E_2 may be empty).

Let m_1 be the number of these triangles in E_1, m_2 in E_2, and without restriction let $m_1 \geq m_2$. Then m_2 pairs of triangles belong again to pairs of tetrahedra and half octahedra and give new tiles of density $\leq \delta_L^3$.

The remaining $m = m_1 - m_2$ cannot be completed to parallelograms. They are incident to m tetrahedra or m half octahedra and we associate to them 4 m prisms as follows:

a) m prisms over these m triangles in F_0 with 2-volume f_0.

b) $m-1$ prisms over triangles in F_0, which are adjacent to just 2 of the triangles in a). They also have 2-volume f_0.

c) m prisms over triangles in F_a (or F_a') having a common edge with one of the m triangles in a). They have 2-volume f_a.

d) At least $m+1$ prisms over triangles in F_a, F_b, F_c which have one edge in F_0. At least one of these triangles has 2-volume f_b and at least one: f_c.

Now the volume of these at least $4m$ prisms is at least:

$$((2m-1)f_0 + mf_a + f_b + mf_c)\varrho_c \geq m(f_0 + f_a + f_b + f_c)\varrho_c$$

by (4.13). So each of the m tetrahedra or half octahedra can be associated to 4 prisms of volume $f_0\varrho_c$, $f_a\varrho_c$, $f_b\varrho_c$ and $f_c\varrho_c$ and for these new tiles T_j it is to show $\delta(T_j) < \delta_L^3$.

This is equivalent to showing that in the tetrahedral case

$$\delta_{\varrho_c}(B^3, T) < \delta_{\varrho_c}(B^3, \mathrm{conv} S_4) \tag{4.14}$$

or, for the octahedron

$$\delta_{\varrho_c}(B^3, O') < \delta_{\varrho_c}(B^3, \mathrm{conv} S_6). \tag{4.15}$$

But we know from Theorem 5 that $\delta_1(B^3, T) < \delta_1(B^3, \mathrm{conv}S_4)$ and from [4] that $\delta_1(B^3, O') < \delta_1(B^3, \mathrm{conv}S_6)$.

So (4.14) and (4.15) hold a fortiori and we have for these tiles that their density is $< \delta_L^3$. Finally, we conclude that we have a tiling of $Q + \varrho_c B^3$, such that for all tiles except the vertex tiles the density is $\leq \delta_L^3$ and in at least one case we have $< \delta_L^3$. This proves Theorem 2.

5. Proof of Theorem 3

This proof is simpler than that of Theorem 2, because we consider only the critical lattice. Again we use Lemma 3 and construct an appropriate tiling.

Let $L \subset \mathbf{E}^3$ be the critical lattice and let a, b, c be a reduced basis of L such that a, b, c have length 2 and mutual angles $\pi/3$. The basis a, b, c generates a basic cell Z of L with $V(Z) = \det L$ and a tiling of $\mathbf{E}^3$ by translates Z_i of Z. Each Z_i can be dissected in the obvious way into two regular simplices $S_{i,1}, S_{i,2}$ of volume $\frac{1}{6}\det L$ and a regular octahedron O_i of volume $\frac{2}{3}\det\ L$.

Now let Q be a generalized octahedron. The facets of Q induce a dissection of Z_i and hence a tiling of Q. We classify those parts of Z_i which tile Q.

Easy considerations show that no simplex S_{i1}, S_{i2} can be dissected. An octahedron O_i can be dissected by any of the planes which meet 4 of its 6 vertices. But at most 2 of them can dissect one octahedron simultaneously, because otherwise its center c is a vertex of the lattice polytope Q, which contradicts the fact that $\mathrm{int}\, O_i$ contains no lattice point. So Q can be dissected into two types of tiles, both tetrahedra of volume $\frac{1}{6}\det L = \frac{2}{3}\sqrt{2}$:

a) T_1: regular tetrahedra of edge-length 2;

b) T_2: distorted tetrahedra, a quarter of an O_i.
This dissection induces a dissection of the boundary region $(Q + \varrho_c B^d)\backslash Q$ into two types of orthogonal prisms of height ϱ_c:

c) P_1 prisms with a regular triangle as base and volume $\sqrt{3}\varrho_c$,

d) P_2 prisms with an orthogonal triangle as base and volume $2\varrho_c$,

e) C_i: cylindrical pieces, generated by the outer normals of Q at its edges and of radius ϱ_c,

f) vertex tiles.

We now have an appropriate tiling and we are ready to calculate the densities Θ_i as for Theorem 2: For the vertex tiles, which need not be considered by Lemma 1, again $\Theta_i = \left(\varrho_{3,c}^L\right)^{-3} > \delta_L^3$. Further, for C_i

$$\Theta_i = (\kappa_3/2\kappa_2)\left(\varrho_{3,c}^L\right)^{-2} = \delta_L^3$$

holds. For the other Θ_i, we have to investigate $\Theta(T_1), \Theta(T_2), \Theta(P_1), \Theta(P_2)$. For T_1 we have the well-known Rogers bound $\Theta_i = \sigma_3 > \delta_L^3$. For T_2 one has $\Theta(T_2) = \Theta(O_i)$ and $\frac{1}{3}\Theta(T_1) + \frac{2}{3}\Theta(T_2) = \delta_L^3$, so $\Theta(T_2) = \Theta(O_i) = \tau_3 = 0,7208 < \delta_L^3$.

Further

$$\Theta(P_1) = \frac{1}{4}\kappa_3(V(P_1)\varrho_c)^{-1} = \pi(3\sqrt{3}\varrho_c)^{-1} < \delta_L^3$$

and

$$\Theta(P_2) = \frac{1}{4}\kappa_3(V(P_2)\varrho_c)^{-1} = \pi(6\varrho_c)^{-1} < \delta_L^3.$$

So the only critical case is T_1. The crucial idea to overcome this difficulty is to put each T_1 together with some other tiles such that the new tile has density $\Theta \leq \delta_L^3$.

For each T_1 at least one of the following combinations as a new tile T' are possible:

1. T_1 and one half of an octahedron of the same Z_i.
   Then $\Theta(T') = \frac{1}{3}\Theta(T_1) + \frac{2}{3}\Theta(T_2) = \delta_L^3$
2. T_1 and nothing of an octahedron of the same Z_i. Then at least one facet of T_1 belongs to bd(conv C_n) and we can associate a prism P_1. Then with $\varrho_{3,c}^L = 8^{\frac{1}{4}}\pi^{-\frac{1}{2}}$:

$$\begin{aligned}\Theta(T'') &= (V(T_1)\Theta(T_1) + V(P_1)\Theta(P_1))(V(T_1) + V(P_1))^{-1} = \\ &= \left(\frac{2}{3}\sqrt{2}\cdot\sigma_3 + \frac{\pi}{3}\right)\left(\frac{2}{3}\sqrt{2} + \sqrt{3}\,\varrho_{3,c}^L\right) = (1,782\ldots)(2,586\ldots)^{-1} = \\ &= 0,689\ldots < \delta_L^3.\end{aligned}$$

3. T_1 and exactly one quarter of an octahedron of the same Z_i, i.e. exactly one T_2. Then at least one of the nonregular facets of T_2 belongs to bd(conv C_n) and we can associate a prism P_2. Then

$$\begin{aligned}\Theta(T'') &= (V(T_1)\Theta(T_1) + V(T_2)\Theta(T_2) + V(P_2)\Theta(P_2))(V(T_1) + V(T_2) + V(P_2))^{-1} = \\ &= \left(\frac{2}{3}\sqrt{2}\,\sigma_3 + \frac{2}{3}\sqrt{2}\cdot\tau_3 + \frac{\pi}{3}\right)\left(\frac{4}{3}\sqrt{2} + 2\varrho_c\right)^{-1} < \Theta(T'') < \delta_L^3.\end{aligned}$$

So in all cases, except for the vertex tiles, we have $\Theta_i \leq \delta_L^3$ and in at least one case $<$; so with Lemma 3 the theorem is proved.

6. Some isoperimetric inequalities

For some hyperplane H, denote by π the reflection in H. Let $0 < \varepsilon < 1$. We define the *ε-Steiner symmetrization* K' of a convex body K so that for any line l orthogonal to H, the following holds:

$$l \cap K' = (1-\varepsilon)(l \cap K) + \varepsilon(l \cap \pi(K)).$$

It is well-known (see [20], p.256) that $V(K') = V(K)$ and for $i = 1, \ldots, d-1$, $V_i(K') \leq V_i(K)$ with equality if and only if K has a symmetry hyperplane parallel to H. If the choice of ε is irrelevant, we speak simply about the Steiner symmetrization. The case $\varepsilon = \frac{1}{2}$ is the classical Steiner-symmetrization and then K' is symmetric in H.

If K is a simplex, Hadwiger proved (cf. [20], p.273)

Lemma 4. *Among all simplices of a given volume, the regular simplex has the minimal intrinsic i-volume for $i = 1, \ldots, d-1$.*

Note that the result as it is stated in [20], concerns only the surface area, but the same proof using the Steiner symmetrization yields Lemma 4 also for the other intrinsic volumes.

For 3-polytopes having five vertices, the isoperimetric problem is solved in [4]. We remind the reader that $S(K) = 2V_{d-1}(K)$ is the surface-area of K.

Lemma 5. *For a 3-polytope P with five vertices, we have*

$$\frac{V_1(P)}{V(P)^{1/3}} \geq 3.0925\ldots \quad \textit{and} \quad \frac{S(P)}{V(P)^{2/3}} \geq 6.2402\ldots.$$

The extremal bodies are certain bipyramids over triangles.

For a generalization of the next results of Steiner, see [4].

Lemma 6. *In $\mathbf{E}^3$, among all quadrilateral-based bipyramids of a given volume, the regular octahedron has the minimal intrinsic i-volume for $i = 1, 2$.*

We consider a very specific inequality, which is needed for the proof of Lemma 9. Let N be the pyramid over the square having edge-length 2 and set $\phi(K) = V_1(K) + 0.3S(K)$. If B is the base of N and O is the regular octahedron with edge-length 2 then $V_i(N) = \frac{1}{2}(V_i(O) + V_i(B))$ for $i = 1, 2$.

Lemma 7. *Let K be a pyramid in $\mathbf{E}^3$ with $V(K) \geq V(N)$ and assume that the base P of K is a parallelogram whose edges have length at least 2. Then the function $\phi(K)$ is minimal if and only if $K = N$.*

Remark. It is not hard to see that $V_1(K) \geq V_1(N)$. The reason for such a complicated calculation is that $S(K)$ can be actually smaller than $S(N)$.

Proof. If $A(P) \geq 4$ then define O' as the union of K and its reflected image through the center of P. Lemma 6 and $V(K) \geq V(N)$ yield that $V_i(K) = \frac{1}{2}(V_i(O') + 2V_i(P)) \geq V_i(N)$, with equality if and only if $K = N$.

So assume that $A(P) \leq 4$. Using ε-Steiner symmetrization with respect to the planes orthogonal to the sides of P for small $\varepsilon > 0$ shows that we may assume that P is a rhombus having side-length 2, K is a right pyramid and $V(K) = V(N)$.

Denote by α the smaller angle of P and set $t = \cos\alpha$, $0 \leq t < 1$. Some simple calculations show that $\psi_1(t) = V_1(K)$ can be written as

$$\begin{aligned}
\psi_1(t) &= 2\left[1 - \frac{1}{\pi}\arctan\left(\frac{\sqrt{2}}{1-t^2}\right)\right] \\
&+ \frac{1}{\sqrt{2}}\sqrt{\frac{1}{1-t^2} + 1 + t}\left[1 - \frac{2}{\pi}\arctan\left(\sqrt{\frac{2}{1+t} - t - t^2 + t^3}\right)\right] \\
&+ \frac{1}{\sqrt{2}}\sqrt{\frac{1}{1-t^2} + 1 - t}\left[1 - \frac{2}{\pi}\arctan\left(\sqrt{\frac{2}{1-t} + t - t^2 - t^3}\right)\right],
\end{aligned}$$

and for the surface area,

$$\psi_2(t) = S(K) = \sqrt{1-t^2} + \sqrt{\frac{2}{1-t^2} + 1 - t^2}.$$

We write the derivative of $\psi_1(t)$ as $\psi_1'(t) = \psi_{11}(t) + \psi_{12}(t)$ where

$$\begin{aligned}\psi_{11}(t) &= \frac{1}{2\sqrt{2}}\sqrt{\frac{1-t^2}{1+(1+t)(1-t^2)}}\left(\frac{2t}{(1-t^2)^2}+1\right)\times \\ & \left[1-\frac{2}{\pi}\arctan\left(\sqrt{\frac{2}{1+t}}-t-t^2+t^3\right)\right]\end{aligned}$$

and

$$\begin{aligned}\psi_{12}(t) &= \frac{1}{2\sqrt{2}}\sqrt{\frac{1-t^2}{1+(1-t)(1-t^2)}}\left(\frac{2t}{(1-t^2)^2}-1\right)\times \\ & \left[1-\frac{2}{\pi}\arctan\left(\sqrt{\frac{2}{1-t}}+t-t^2-t^3\right)\right].\end{aligned}$$

If $2t/(1-t^2)^2 \geq 1$ then $\psi_1'(t) \geq \psi_{11}(t)$. Using the monotonicity of the argument of arctan, $t \geq 0$ and $1+t < 2$ for $\psi_{11}(t)$ show that

$$\psi_1'(t) \geq \frac{1}{\sqrt{6}}\left(1-\frac{2}{\pi}\arctan\sqrt{2}\right)t = 0.1599\,t.$$

Now assume that $2t/(1-t^2)^2 < 1$, and hence $t < 0.5$. Set $\omega = \frac{1}{2\sqrt{2}}(1-\frac{2}{\pi}\arctan\sqrt{2})$. We deduce that

$$\psi_1'(t) \geq 2\omega\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)t - \omega\left(\frac{1}{\sqrt{1+(1-t)(1-t^2)}}-\frac{1}{\sqrt{1+(1+t)(1-t^2)}}\right).$$

The derivative of the expression in the parentheses is at most 1.5 because $0 \leq t < 0.5$, which in turn yields that

$$\psi_1'(t) \geq 2\omega\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{3}{4}\right)t = 0.1480\,t.$$

On the other hand, differentiating $\psi_2(t)$ *via* a substitution $s = 1-t^2$ results in

$$\phi_2'(t) = \left(\frac{2}{s^{3/2}\sqrt{2+s^2}}-\frac{1}{\sqrt{s}}-\frac{\sqrt{s}}{\sqrt{2+s^2}}\right)t.$$

Here the expression in the parentheses is monotonely decreasing for $0 < s \leq 1$ as one can verify by differentiation, and hence

$$0.3\psi_2'(t) \geq 0.3\left(\frac{1}{\sqrt{3}}-1\right)t = -0.1267\,t.$$

It follows that $\psi(t) = \psi_1(t) + 0.3\psi_2(t)$ is strictly monotonely increasing for $0 \leq t < 1$, which in turn yields that $\phi(K) \geq \phi(N)$ with equality if and only if $K = N$. $\square$

7. Four-ball and five-ball packings

With respect to four-ball packings in $\mathbf{E}^d$, $d \geq 3$, and five-ball packings in $\mathbf{E}^3$, we give a complete characterization of the optimal lattice packing for $\varrho > 0$ (see Theorem 7 and Theorem 8). The result about five-ball packings points to an interesting phenomenon: the optimal packing is not a lattice packing for $\varrho \geq 1.0483\ldots$.

First we consider four-ball packings. We need the following lemma proved in [4]:

Lemma 8. *Assume that $x_0 \in C$ for $C = \mathrm{conv}\{x_1, x_2, x_3\}$ and $d(x_i, x_j) \geq 2$ for $0 \leq i < j \leq 3$. Then $V_1(C) \geq 3 + \sqrt{3} = 4.7320\ldots$. In addition, $V_1(C) \geq 3 + \sqrt{3}$ for a planar $C \in \mathcal{F}_5^3$.*

Note that $V_1(\mathrm{conv}S_4) = 6$ and using Steiner's formula, the inequality $V(\mathrm{conv}C_4 + \varrho B^d) \geq V(\mathrm{conv}S_4 + \varrho B^d)$ can be written in the form

$$\frac{\kappa_{d-3}}{\kappa_{d-1}}\,\frac{1}{\varrho^2}\,V_3(\mathrm{conv}C_4) + \frac{\kappa_{d-2}}{2\kappa_{d-1}}\,\frac{1}{\varrho}\,S(\mathrm{conv}C_4) \geq 6 - V_1(\mathrm{conv}C_4). \tag{7.16}$$

Let $\varrho_{d,4}^L$ be the unique positive ϱ such that equality holds in (7.16) for $\mathrm{conv}C_4 = T^3$.

Theorem 7. *For a given $\varrho > 0$, assume that $V(\mathrm{conv}C_4 + \varrho B^d)$ is minimal. Then*
i) $C_4 = S_4$ *if* $\varrho < \varrho_{d,4}^L$,
ii) $C_4 = S_4$ *or* $\mathrm{conv}C_4 = T^3$ *if* $\varrho = \varrho_{d,4}^L$,
iii) $\mathrm{conv}C_4 = T^3$ *if* $\varrho > \varrho_{d,4}^L$.

Remark. Let $\omega = 0.2927\ldots$ be the positive root of the equation

$$4V_3(T^3)\,\omega^2 + S(T^3)\,\omega = 6 - V_1(T^3).$$

Then for large d, $\varrho_{d,4}^L \sim (2\sqrt{2\pi}\omega)^{-1}\sqrt{d} = 0.6814\ldots\sqrt{d}$. If $d = 3$ then $\varrho_4 = 1.1332$.

Proof of Theorem 7. If $\dim(\mathrm{conv}C_4) = 3$ then $V(\mathrm{conv}C_4) \geq V(T^3)$ because the edges of T^3 determine the densest lattice packing of B^3, which in turn yields $V_i(\mathrm{conv}C_4) \geq V_i(T^3)$ by Lemma 4, with equality only if $\mathrm{conv}C_4 = T^3$.

So assume $\dim(\mathrm{conv}C_4) = 2$ and $\varrho \geq \varrho_{d,4}^L$. Then we deduce the inequality

$$\varrho^2\pi(V_1(\mathrm{conv}C_4) - V_1(T^3)) \geq 1.4158 > V(T^3)$$

as $V_1(\mathrm{conv}C_4) \geq 4$ by Lemma 8, and hence $V(\mathrm{conv}C_4 + \varrho B^d) > V(T^3 + \varrho B^d)$ by $S(\mathrm{conv}C_4) \geq S(T^3)$ and by Steiner's formula. Finally, (7.16) yields the case $\varrho < \varrho_{d,4}^L$ since strict inequality holds for $\varrho = \varrho_{d,4}^L$. □

For five-ball packings, the following lemma is the core of the proof. Again, let $\phi(K) = V_1(K) + 0.3S(K)$.

Lemma 9. *If* $\mathrm{int}(\mathrm{conv}C_5)$ *is non-empty then* $\phi(\mathrm{conv}C_5) \geq \phi(N)$ *with equality iff* $\mathrm{conv}C_5 = N$.

Proof. Let $\mathrm{conv}C_5 = \mathrm{conv}\{x_0, \dots, x_4\}$ where $x_0, \dots, x_4$ are points of some packing lattice. If $\mathrm{conv}C_5$ can be dissected into at least three simplices then $V(\mathrm{conv}C_5) \geq 2\sqrt{2}$. We deduce $S(\mathrm{conv}C_5) > S(N)$ and $V_1(\mathrm{conv}C_5) > V_1(N)$ by Lemma 5, which in turn yields $\phi(\mathrm{conv}C_5) > \phi(N)$.

So assume that $\mathrm{conv}C_5$ is a pyramid over the base P where $x_1, \dots, x_4$ are the only lattice points in P. If P is a triangle then $V_1(P) \geq 3 + \sqrt{3}$ by Lemma 8. It follows by $S(\mathrm{conv}C_5) \geq 6\sqrt{3}$ and $V_1(N) = 4.3509\dots$ that

$$\phi(\mathrm{conv}C_5) - \phi(N) \geq (3 + \sqrt{3} - V_1(N)) + 0.3(2\sqrt{3} - 4) = 0.2203 > 0.$$

The only remaining case, if P is a parallelogram, is settled in Lemma 7. □

Define $\varrho^L_{3,5} = 1.1025\dots$ by the equality $V(N + \varrho^L_{3,5}B^3) = V(\mathrm{conv}S_5 + \varrho^L_{3,5}B^3)$.

Theorem 8. *For a given $\varrho > 0$, assume that $V(\mathrm{conv}C_5 + \varrho B^3)$ is minimal. Then*
i) $C_5 = S_5$ *if* $\varrho < \varrho^L_{3,5}$,
ii) $C_5 = S_5$ *or* $\mathrm{conv}C_5 = N$ *if* $\varrho = \varrho^L_{3,5}$,
iii) $\mathrm{conv}C_5 = N$ *if* $\varrho > \varrho^L_{3,5}$.

Proof. By Steiner's formula, the inequality $V(\mathrm{conv}C_5 + \varrho B) \geq V(\mathrm{conv}S_5 + \varrho B)$ can be written in the form

$$\frac{1}{\varrho^2\pi}V(\mathrm{conv}C_5) + \frac{1}{\varrho\pi}S(\mathrm{conv}C_5) + V_1(\mathrm{conv}C_5) \geq 8. \tag{7.17}$$

First let $\varrho \leq \varrho^L_{3,5}$ and let $\dim(\mathrm{conv}C_5) = 2$. According to Oler's inequality (cf. [15]),

$$\frac{1}{2\sqrt{3}}S(\mathrm{conv}C_5) + V_1(\mathrm{conv}C_5) \geq 8,$$

which in turn yields (7.17) by $\varrho\pi \leq 3.4635 < 2\sqrt{3}$.

If $\varrho \geq \varrho^L_{3,5}$ and $\dim(\mathrm{conv}C_5) = 3$ then $V(\mathrm{conv}C_5 + \varrho B) \geq V(N + \varrho B)$ is equivalent with

$$\frac{1}{\varrho^2\pi}V(\mathrm{conv}C_5) + \frac{1}{\varrho\pi}S(\mathrm{conv}C_5) + V_1(\mathrm{conv}C_5) \geq \frac{1}{\varrho^2\pi}V(N) + \frac{1}{\varrho\pi}S(N) + V_1(N). \tag{7.18}$$

Here $V(\mathrm{conv}C_5) \geq V(N)$ as $\mathrm{conv}C_5$ can be dissected into at least two packing-lattice simplices, and hence it is sufficient to verify

$$V_1(\mathrm{conv}C_5) - V_1(N) \geq \frac{1}{\varrho\pi}(S(N) - S(\mathrm{conv}C_5)).$$

This inequality follows by $1/(\varrho\pi) = 0.2887\dots < 0.3$ and by Lemma 9.

So far we have proved ii), which in turn yields i) by (7.17). If $\varrho \geq \varrho^L_{3,5}$ then the only remaining case to consider is $\dim(\mathrm{conv}C_5) = 2$. Set $t = 1/\varrho$ and $t_5 = 1/\varrho^L_{3,5}$. The inequality $V(\mathrm{conv}C_5 + \varrho B) \geq V(N + \varrho B)$ can be written in the form

$$\frac{V(N)}{\pi}t^2 + \frac{S(N) - S(\mathrm{conv}C_5)}{\pi}t \leq V_1(\mathrm{conv}C_5) - V_1(N).$$

As 0 is a root of the quadric polynomial on the left-hand side and the inequality holds for $t = t_5$ by the case ii), strict inequality also holds for $t < t_5$. $\square$

Finally, we compare general packings and lattice packings. Let D_5 be the bipyramid with all edges of length 1, and define $\varrho_5 = 1.0483 < \varrho^L_{3,5}$ by the formula $V(D_5 + \varrho_5 B^3) = V(\mathrm{conv} S_5 + \varrho_5 B^3)$. Most probably $\varrho_5 = \varrho_{3,5}$. Observe that $V(D_5) = V(N)$ and $V_i(D_5) < V_i(N)$ for $i = 1, 2$. Therefore Lemma 8 yields that a lattice packing can be the optimal general packing only for small ϱ; namely, when it is the sausage.

Phenomenon 1. *For $\varrho > \varrho^*_5$, the general packing of five balls minimizing $V(\mathrm{conv} C_5 + \varrho B^3)$ is not induced by a lattice packing.*

8. Proof of Theorem 6

We compare the sausage arrangement of $d+1$ balls with the arrangement determined by T^d. The edges of T^d determine a packing lattice Λ such that T^d is a lattice simplex.

Note that if F is an i-face of T^d then

$$\frac{\widehat{G}_\Lambda(F) \cdot \kappa_i}{|F|} = \sigma_i,$$

namely, the classical Rogers bound. It follows by (3.9) (actually with the opposite inequality sign), that $V(T^d + \varrho B^d) < V(\mathrm{conv} S_n + \varrho B^d)$ holds if

$$\varrho > \sqrt[i-1]{\frac{\kappa_{d-i}}{2\kappa_{d-1}} \cdot \frac{1}{\sigma_i}}$$

for $i = 2, \ldots, d$. The inequalities (see [7])

$$\frac{\kappa_{m-2}}{\kappa_m} = \frac{m}{2\pi} \quad \text{and} \quad \frac{\kappa_{m-1}}{\kappa_m} < \sqrt{\frac{m+1}{2\pi}}$$

yield that in order to prove 1), it is sufficient to verify that

$$0.6 \cdot \sqrt{d} > \sqrt{\frac{d}{2\pi}} \cdot \sqrt[i-1]{\frac{1}{2\sigma_i}}. \tag{8.19}$$

If $i \leq 8$ then we deduce (8.19) from the known values of the Rogers constant (see [11]). On the other hand, Blichfeldt's method yields that $\sigma_i > \frac{i+2}{i+1}\sqrt{2}^{\,-i}$ (see [13]), and if $i > 8$ then (8.19) follows using this estimate.

Now we verify 2). Let $\varepsilon > 0$, and choose k so that $2/(k-1) < \varepsilon$. Observe that $V(T^d + \varrho B^d) > V(\mathrm{conv} S_n + \varrho B^d)$ is equivalent to

$$\sum_{i=1}^{d} \frac{\kappa_{d-i}}{2\kappa_{d-1}} \frac{1}{2d} V_i(T^d) \cdot \frac{1}{\varrho^{i-1}} > 1. \tag{8.20}$$

For large d, the asymptotic value of $V_k(T^d)$ is calculated in [2]. It follows that for large d,

$$\frac{\kappa_{d-k}}{2\kappa_{d-1}} \frac{1}{2d} V_k(T^d) \sim c(k) \cdot \frac{\sqrt{d}^{k-1}\sqrt{\ln d}^k}{d} > \sqrt{d}^{k-3}.$$

Now the choice of k and (8.20) yield that if $\varrho < \sqrt{d}^{1-\varepsilon}$ then

$$V(T^d + \varrho B^d) > V(\mathrm{conv} S_n + \varrho B^d).$$

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