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Entropy, Capacity and Arrangements on the Cube

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Abstract. Capacity numbers of convex sets are characterized as a kind of packing numbers. After that entropy numbers and capacity numbers of the unit ball of l_∞^d are computed.

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1. Introduction

Let $(E, \|\cdot\|)$ be a (real) normed space and $X \subseteq E$ a fixed bounded subset of E. The closed balls of radius $r > 0$ centered in $x_0 \in E$ are denoted by $B(x_0, r)$. We need the following basic geometric concepts for explaining two kinds of entropy quantities (cf. [3], [2]) of X: A system $\mathcal{C} \subseteq 2^E$ of subsets of X is called a *covering of* X if $X \subseteq \bigcup \mathcal{C}$. A *packing in* X is a system $\mathcal{P} \subseteq 2^X$ of subsets of X which are pairwise disjoint. By a *ρ-distant subset in* X, $\rho > 0$, we mean a subset $D \subseteq X$ with $\|x - y\| > 2\rho$ for all $x, y \in D$, $x \neq y$.

The *n-th entropy number of* X, $n \in \{1, 2, 3, \ldots\}$, is defined by

$$\begin{aligned} \varepsilon_n(X) \;\; = \;\; & \inf\{\, \varepsilon > 0 : \text{ there exists a covering of } X \\ & \qquad \text{consisting of at most } n \text{ balls of radius } \varepsilon \,\} \,. \end{aligned} \tag{1}$$

The *n-th capacity number* (or inner entropy number) *of* X is given by

$$\begin{aligned} \varphi_n(X) \;\; = \;\; & \sup\{\, \rho > 0 : \text{ there exists a } \rho\text{-distant} \\ & \qquad \text{subset of } X \text{ of cardinality } n+1 \,\} \end{aligned} \tag{2}$$

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if $\operatorname{card}(X) \geq n+1$, and by $\varphi_n(X) = 0$ if $\operatorname{card}(X) < n+1$. Of course, both concepts can be defined in the same way for bounded subsets of general metric spaces instead of $(E, \|\cdot\|)$.

If the set X is convex we can give a characterization of $\varphi_n(X)$ by special packings which is based on the close relation between ρ-distant subsets of X and packings in the so-called *parallel set* $X + B(0,\rho) = \{\, x+y : x \in X,\ y \in B(0,\rho) \,\}$ of X.

Proposition 1. *Let X be a convex subset of a normed space E, n a positive integer, and $\rho > 0$. Then the following are equivalent:*
(i) There exists a ρ-distant subset D of X of cardinality $n+1$.
(ii) There exists a packing $\mathcal{P}$ in $X + B(0,\rho)$ consisting of $n+1$ balls of radius ρ.

Proof. The implication (i) $\Rightarrow$ (ii) is simple to verify and does not make use of the convexity of X: Assume $D = \{x_0, x_1, \dots, x_n\}$ to be a ρ-distant subset in X. We obtain the asserted packing by setting $\mathcal{P} = \{B(x_0,\rho), B(x_1,\rho), \dots, B(x_n,\rho)\}$.

To prove (ii) $\Rightarrow$ (i) we consider a packing $\mathcal{P} = \{B(x_0,\rho), B(x_1,\rho), \dots, B(x_n,\rho)\}$ in $X + B(0,\rho)$. First we show that the centres x_i, $i = 0, 1, \dots, n$, belong to the closure $\overline{X}$ of X. Assume the contrary $x_0 \notin \overline{X}$. Then x_0 can be seperated from the closed convex set $\overline{X}$ by a suitable functional $f \in E'$. Thus we obtain

$$
\begin{aligned}
\langle x_0, f\rangle &> \sup\{\, \langle x, f\rangle : x \in \overline{X} \,\}, \\
\langle x_0, f\rangle + \sup\{\, \langle y, f\rangle : y \in B(0,\rho) \,\} &> \sup\{\, \langle x, f\rangle : x \in \overline{X} \,\} + \\
&\qquad \sup\{\, \langle y, f\rangle : y \in B(0,\rho) \,\}, \\
\sup\{\, \langle x_0 + y, f\rangle : y \in B(0,\rho) \,\} &> \sup\{\, \langle x+y, f\rangle : x \in \overline{X}, y \in B(0,\rho) \,\}, \\
\sup\{\, \langle z, f\rangle : z \in B(x_0,\rho) \,\} &> \sup\{\, \langle z, f\rangle : z \in \overline{X} + B(0,\rho) \,\}
\end{aligned}
$$

in contradiction to $B(x_0,\rho) \subseteq \overline{X} + B(0,\rho)$ since $B(x_0,\rho)$ belongs to the packing $\mathcal{P}$ in $X + B(0,\rho)$. Consequently, $x_0, x_1, \dots, x_n \in \overline{X}$.

We have $\|x_i - x_j\| > 2\rho$ for all $i, j \in \{0, 1, \dots, n\}$, $i \neq j$, since the balls $B(x_i,\rho)$ are pairwise disjoint. So we can find an $\varepsilon > 0$ such that $\|x_i - x_j\| > 2\,(\rho+\varepsilon)$ for $i, j \in \{0, 1, \dots, n\}$, $i \neq j$. Finally we choose points y_i from the non-empty sets $X \cap B(x_i,\varepsilon)$, $i = 0, 1, \dots, n$, and thus obtain

$$\|y_i - y_j\| \geq \|x_i - x_j\| - \|y_i - x_i\| - \|y_j - x_j\| > 2\,(\rho+\varepsilon) - \varepsilon - \varepsilon = 2\rho$$

for $i \neq j$ which shows that $D = \{y_0, y_1, \dots, y_n\}$ is a ρ-distant subset in X. □

Combining formula (2) and Proposition 1 we get the following characterization of the capacity numbers of X:

Theorem 1. *Let X be a convex and bounded infinite subset of a normed space E and n a positive integer. Then*

$$
\begin{aligned}
\varphi_n(X) = \sup\{\, \rho > 0 :\ & \textit{there exists a packing in } X + B(0,\rho) \\
& \textit{consisting of } n+1 \textit{ balls of radius } \rho \,\}\,.
\end{aligned}
$$

□

It can be shown by an example in the euclidean plane l_2^2 that the supposition of convexity of X is not superfluous in Proposition 1 and Theorem 1.

The capacity numbers of a ball in E can be given not only as a kind of packing numbers in a parallel set as above but also as packing numbers in the ball itself:

Theorem 2. *Let $B(x_0, r)$ be a (closed) ball in a normed space E and n a positive integer. Then*

$$\begin{aligned}\varphi_n(B(x_0,r)) &= \frac{r\,\mu_n(B(x_0,r))}{r-\mu_n(B(x_0,r))} \qquad \textit{with} \\ \mu_n(B(x_0,r)) &= \sup\{\,\mu\in(0,r): \textit{ there exists a packing in } B(x_0,r) \\ &\qquad\qquad \textit{consisting of } n+1 \textit{ balls of radius } \mu\,\}\,.\end{aligned}$$

Proof. We use Theorem 1. Considering $B(x_0,r)+B(0,\rho)=B(x_0,r+\rho)$ we obtain

$$\begin{aligned}\varphi_n(B(x_0,r)) &= \sup\{\,\rho>0: \text{ there exists a packing in } B(x_0,r+\rho) \\ &\qquad\qquad \text{consisting of } n+1 \text{ balls of radius } \rho\,\} \\ &= \sup\{\,\rho>0: \text{ there exists a packing in } B(x_0,r) \\ &\qquad\qquad \text{consisting of } n+1 \text{ balls of radius } \frac{r\,\rho}{r+\rho}\,\} \\ &= \sup\{\,\frac{r\,\mu}{r-\mu}: \text{ there exists a packing in } B(x_0,r) \\ &\qquad\qquad \text{consisting of } n+1 \text{ balls of radius } \mu\in(0,r)\,\}\end{aligned}$$

which shows the asserted equation since the function $\frac{r\,\mu}{r-\mu}$ is monotonically increasing for $\mu\in(0,r)$. □

The simplest set $X\subseteq E$ whose sequences $(\varepsilon_n(X))_{n=1}^\infty$ and $(\varphi_n(X))_{n=1}^\infty$ are of interest is the unit ball $X=B(0,1)$ of E. In [2] it is shown that

$$n^{-\frac{1}{d}} \;\le\; \varepsilon_n(B(0,1)) \;\le\; 2\,\varphi_n(B(0,1)) \;\le\; 4\,n^{-\frac{1}{d}} \qquad \text{if} \qquad \dim(E)=d\,,$$

but the exact calculation of entropy and capacity numbers of the unit ball is a nontrivial open problem. The goal of this paper is to give the solution for the space l_∞^d.

2. Entropy and capacity numbers of the cube

We consider the d-dimensional Banach space l_∞^d the unit ball $X=B(0,1)$ of which is the d-dimensional cube $[-1,1]^d$. The following result on the sequences $(\varepsilon_n(X))_{n=1}^\infty$ and $(\varphi_(X))_{n=1}^\infty$ seems to be obvious because one expects intuitively that lattice-like arrangements are optimal on the cube. Nevertheless, an exact proof requires detailed considerations.

Theorem 3. *Let X be the unit ball of the space l_∞^d and m a positive integer. Then*

$$\varepsilon_{m^d}(X)=\varepsilon_{m^d+1}(X)=\ldots=\varepsilon_{(m+1)^d-1}(X)=\frac{1}{m}$$

and

$$\varphi_{m^d}(X)=\varphi_{m^d+1}(X)=\ldots=\varphi_{(m+1)^d-1}(X)=\frac{1}{m}\,.$$

Proof. For a fixed $n \in \{m^d, m^d+1, \dots, (m+1)^d-1\}$ we have

$$\begin{aligned}\varepsilon_n(X) &= \inf\{\,\varepsilon \in (0,1] : \text{ there exists a covering of } X \\ &\qquad \text{consisting of at most } n \text{ balls of radius } \varepsilon\,\}\end{aligned} \tag{3}$$

in accordance with formula (1) and

$$\varphi_n(X) = \frac{\mu_n(X)}{1-\mu_n(X)} \qquad \text{with} \tag{4}$$

$$\begin{aligned}\mu_n(X) &= \sup\{\,\mu \in (0,1) : \text{ there exists a packing in } X \\ &\qquad \text{consisting of at least } n+1 \text{ balls of radius } \mu\,\}\end{aligned} \tag{5}$$

according to Theorem 2. In the following $H = [0,2)^d$ denotes the half open ball of radius 1 centered in $(1,1,\dots,1)$. The characterizations of $\varepsilon_n(X)$ and $\varphi_n(X)$ as an infimum or supremum, respectively, allow to replace all the balls in (3) and (5) by half open balls, i.e.

$$\begin{aligned}\varepsilon_n(X) &= \inf\{\,\varepsilon \in (0,1] : \text{ there exists a covering of } H \text{ consisting} \\ &\qquad \text{of at most } n \text{ balls from } \{\,x+\varepsilon H : x \in \mathbb{R}^d\,\}\,\}\,,\end{aligned} \tag{6}$$

$$\begin{aligned}\mu_n(X) &= \sup\{\,\mu \in (0,1) : \text{ there exists a packing in } H \text{ consisting} \\ &\qquad \text{of at least } n+1 \text{ balls from } \{\,x+\mu H : x \in \mathbb{R}^d\,\}\,\}\,.\end{aligned} \tag{7}$$

The next step is to show that we can restrict our considerations in formulas (6) and (7) to lattice like arrangements on H. Therefore we make use of the arrangements $\mathcal{L}_\delta = \{\,x+\delta H : x \in 2\delta\mathbb{Z}^d\,\}$ which form partitions of l_∞^d into half open balls of radius $\delta \in (0,1]$. We obtain coverings of H and packings in H by setting

$$\mathcal{C}_\delta = \{\,M \in \mathcal{L}_\delta : M \cap H \neq \emptyset\,\} = \{\,x+\delta H : x \in 2\delta\mathbb{Z}^d \cap H\,\}\,, \tag{8}$$

$$\mathcal{P}_\delta = \{\,M \in \mathcal{L}_\delta : M \subseteq H\,\} = \{\,x+\delta H : x \in 2\delta\mathbb{Z}^d \cap [0,2(1-\delta)]^d\,\}\,. \tag{9}$$

The cardinalities of these arrangements are

$$\begin{aligned}\operatorname{card}(\mathcal{C}_\delta) &= \operatorname{card}(2\delta\mathbb{Z}^d \cap H) = \operatorname{card}(\mathbb{Z}^d \cap \frac{1}{2\delta}[0,2)^d) \\ &= \operatorname{card}((\mathbb{Z} \cap [0,\frac{1}{\delta})\,)^d) = (\operatorname{card}(\mathbb{Z} \cap [1,\frac{1}{\delta}+1)\,))^d\end{aligned}$$

which means that

$$\operatorname{card}(\mathcal{C}_\delta) = c_\delta^d \qquad \text{with} \qquad c_\delta \in \mathbb{Z} \cap [\frac{1}{\delta},\frac{1}{\delta}+1) \tag{10}$$

and

$$\begin{aligned}\operatorname{card}(\mathcal{P}_\delta) &= \operatorname{card}(2\delta\mathbb{Z}^d \cap [0,2(1-\delta)]^d) = \operatorname{card}(\mathbb{Z}^d \cap \frac{1}{2\delta}[0,2(1-\delta)]^d) \\ &= \operatorname{card}((\mathbb{Z} \cap [0,\frac{1-\delta}{\delta}]\,)^d) = (\operatorname{card}(\mathbb{Z} \cap [1,\frac{1}{\delta}]\,))^d\end{aligned}$$

showing that

$$\operatorname{card}(\mathcal{P}_\delta) = p_\delta^d \qquad \text{with} \qquad p_\delta \in \mathbb{Z} \cap (\frac{1}{\delta} - 1, \frac{1}{\delta}] \,. \tag{11}$$

Now let us show that formula (6) can be rewritten as

$$\varepsilon_n(X) = \inf\{\, \varepsilon \in (0,1] : \operatorname{card}(\mathcal{C}_\varepsilon) \le n \,\} \tag{12}$$

by verifying the following: Each covering $\mathcal{C}$ of H consisting of half open balls from $\{\, x + \varepsilon H : x \in \mathbb{R}^d \,\}$, $\varepsilon \in (0,1]$, does not contain less balls than $\mathcal{C}_\varepsilon$.

For showing this we map each ball $x_0 + \varepsilon H$ from $\mathcal{C}_\varepsilon$ (i.e. $x_0 \in 2\varepsilon\mathbb{Z}^d \cap H$ due to (8)) onto *one* ball B_0 from $\mathcal{C}$ which covers x_0. This mapping from $\mathcal{C}_\varepsilon$ into $\mathcal{C}$ is injective since two different points $x_0, x_1 \in 2\varepsilon\mathbb{Z}^d \cap H$ have a distance $\|x_0 - x_1\| \ge 2\varepsilon$ and cannot be included in the same half open ball $B_0 \in \mathcal{C}$ of radius ε. Consequently, $\operatorname{card}(\mathcal{C}) \ge \operatorname{card}(\mathcal{C}_\varepsilon)$.

Formula (7) can be sharpened to

$$\mu_n(X) = \sup\{\, \mu \in (0,1) : \operatorname{card}(\mathcal{P}_\mu) \ge n+1 \,\} \tag{13}$$

which is based on the fact that each packing $\mathcal{P}$ in H formed with half open balls from $\{\, x + \mu H : x \in \mathbb{R}^d \,\}$, $\mu \in (0,1)$, does not contain more balls than $\mathcal{P}_\mu$:

$\mathcal{P}_\mu$ covers $[0, 2(1-\mu)]^d$ in accordance with formula (9). On the other hand for each ball $x_0 + \mu H \in \mathcal{P}$ we have $x_0 \in [0, 2(1-\mu)]^d$ since $x_0 + \mu H \subseteq H$. Therefore we can map each $x_0 + \mu H \in \mathcal{P}$ onto that ball $B_0 \in \mathcal{P}_\mu$ which contains x_0. For two different $x_0 + \mu H, x_1 + \mu H \in \mathcal{P}$ we have $\|x_0 - x_1\| \ge 2\mu$ since $x_0 + \mu H$ and $x_1 + \mu H$ are disjoint. Thus x_0 and x_1 can not belong to the same half open ball $B_0 \in \mathcal{P}_\mu$. Hence the mapping from $\mathcal{P}$ into $\mathcal{P}_\mu$ defined above is injective which implies that $\operatorname{card}(\mathcal{P}) \le \operatorname{card}(\mathcal{P}_\mu)$.

Combining formulas (12) and (10) and remembering that n has been chosen from $\{m^d, m^d + 1, \ldots, (m+1)^d - 1\}$ we obtain

$$\begin{aligned}
\varepsilon_n(X) &= \inf\{\, \varepsilon \in (0,1] : c_\varepsilon^d \le n \text{ with } c_\varepsilon \in \mathbb{Z} \cap [\frac{1}{\varepsilon}, \frac{1}{\varepsilon} + 1) \,\} \\
&= \inf\{\, \varepsilon \in (0,1] : c_\varepsilon \le m \text{ with } c_\varepsilon \in \mathbb{Z} \cap [\frac{1}{\varepsilon}, \frac{1}{\varepsilon} + 1) \,\} \\
&= \inf\{\, \varepsilon \in (0,1] : \frac{1}{\varepsilon} \le m \,\} \\
&= \frac{1}{m}
\end{aligned}$$

which proves the first part of Theorem 3. Similarly we make use of (13) and (11) and get

$$\begin{aligned}
\mu_n(X) &= \sup\{\, \mu \in (0,1) : p_\mu^d > n \text{ with } p_\mu \in \mathbb{Z} \cap (\frac{1}{\mu} - 1, \frac{1}{\mu}] \,\} \\
&= \sup\{\, \mu \in (0,1) : p_\mu \ge m+1 \text{ with } p_\mu \in \mathbb{Z} \cap (\frac{1}{\mu} - 1, \frac{1}{\mu}] \,\} \\
&= \sup\{\, \mu \in (0,1) : \frac{1}{\mu} \ge m+1 \,\} \\
&= \frac{1}{m+1}
\end{aligned}$$

which implies

$$\varphi_n(X) = \frac{1}{m}$$

according to (4). This completes the proof of Theorem 3. □

Let us remark that, of course, entropy numbers and capacity numbers of a set do not coincide in general. We see that the sequences $(\varepsilon_n(X))_{n=1}^{\infty}$ and $(\varphi_n(X))_{n=1}^{\infty}$ are strictly decreasing only in the one-dimensional case $d = 1$. In [1] it is shown that there exist convex polytopes $P \subseteq l_\infty^d$ which have a strictly decreasing sequence $(\varepsilon_n(P))_{n=1}^{\infty}$.

The question for the sequence $\varepsilon_n(S)$ of a given set S in a normed or, more general, in a metric space arises for instance in branches of approximation theory. Especially the jumps in the sequence are of interest (cf. [5], [4]). The proof of Theorem 3 is closely related to the geometry of the space l_∞^d. We encourage the reader to search for new results. Even the question for the exact values of entropy numbers and capacity numbers of balls in the euklidean space l_2^d is unsolved.

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