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Minimal and Maximal Ideals in Rings with Involution

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Abstract. We investigate the relationships between the ideal structure and the $*$-ideal structure of a ring with involution $(*)$. Descriptions of $*$-minimal and $*$-maximal ideals are obtained in terms of minimal and maximal ideals, respectively. Furthermore conditions are provided allowing us to associate with each minimal or maximal ideal a $*$-minimal or $*$-maximal ideal, respectively. These connections are strong enough to permit the transfer of various properties from the $*$-ideal structure to the ideal structure or vice-versa.

An *involution* on a ring R is a unary operation $*$ on R such that: $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, and $(ab)^* = b^*a^*$, for each a, $b \in R$. A ring with such an operation is called a "ring with involution", or a "$*$-ring". Rings and linear algebras with involutions have been an object of study since von Neumann observed the role played by the classical adjoint in the algebra of linear operators on a Hilbert space. An abstract theory of such rings developed with involution playing the role of an additional property for objects in the category of rings. This subject has been studied intensely, especially in the context of applications to Lie algebras, Jordan algebras, or rings of operators. More recently, the point-of-view of considering the category of rings with involution has been taken under investigation. (Exemplary of this is the section devoted to rings with involution in Rowen's treatise on ring theory, [5, 2.13], and the survey article by Wiegandt [6])

We consider the category whose objects are rings with involution and whose morphisms must respect this structure. The view point here is much like that given by Rowen [5, 299–307], with the important exception that herein the rings need not have unity and morphisms need not preserve a unity when acting on a ring which does have one. Thus an ideal of an object in this category is the kernel of a morphism in this category, and we call such an ideal a "$*$-ideal". In contrast an ideal in the category of rings will be called an "ideal". This terminology is carried through with $*$-minimal ideal, $*$-maximal ideal, and so on. (This follows the convention adopted by Rowen [5, p.299].)

For clarity it is worthwhile to elucidate some of the terminology to be used. Let R be a $*$-ring and let I be an ideal of the ring R. Then I is a $*$-ideal if $I^* \subseteq I$. (Here for any nonempty set S, we define $S^* = \{s^* : s \in S\}$.) We say R is $*$-simple if R has no nonzero proper $*$-ideals. (Similarly, we say the ring R is simple if R has no nonzero proper ideals. This allows the pathological case, where $R^2 = 0$ and $(R, +)$ is C_p, the cyclic group of prime order p, to be a simple ring.) If X is an ideal of R, then X is said to be *essential* (*$*$-essential)* in R if $X \neq 0$ and if $X \cap I \neq 0$ for each nonzero ideal ($*$-ideal) of R. Observe that if K is an ideal of R, then KK^*, $K \cap K^*$, and $K + K^*$ are $*$-ideals of R.

In this paper we investigate relationships between the $*$-ideal structure and the ring ideal structure for several types of ideals: minimal, maximal, and prime. This leads to results concerning the $*$-socle, $*$-simple rings, and the $*$-heart in relation to their respective counterparts: the socle, simple rings, and the heart. Examples are provided to illustrate and to delimit the theory developed.

We end the paper with an application of the previous results: let R be a $*$-ring; then every prime ideal of R is maximal if and only if every $*$-prime ideal of R is $*$-maximal.

1. $*$-Minimal ideals

In this section we develop close connections between minimal ideals and $*$-minimal ideals. These are used in establishing results concerning the $*$-socle and $*$-essentiality. Examples are given illustrating some of the results obtained.

Proposition 1.1 *Let R be a $*$-ring. If I is a $*$-minimal ideal of R, then either:*

(i) I is a minimal ideal of R; or

(ii) for any nonzero ideal K of R, with $K \subset I$, then $I = K \bigoplus K^*$, and K and K^* are minimal ideals of R; if furthermore $I^2 \neq 0$, then there are exactly two nonzero ideals of R properly contained in I.

If $R^2 = 0$, then only case (i) can occur.

Proof. Assume I is not a minimal ideal of R and let K be any nonzero ideal of R with $K \subset I$. Since $K + K^*$ and $K \cap K^*$ are $*$-ideals of R and are contained in I, we have $K + K^* = I$ and $K \cap K^* = 0$. So $I = K \bigoplus K^*$. Let Y be a nonzero ideal of R such that $Y \subseteq K$. Then, by the above argument, $I = Y \bigoplus Y^*$. For any $k \in K$, we have $k = y_1 + y_2$, where $y_1 \in Y$, $y_2 \in Y^*$, and hence $k - y_1 = y_2$. Thus $y_2 \in K \cap K^* = 0$, yielding $k \in Y$ and hence $K = Y$. So K and K^* are minimal ideals of R. Next, let B be nonzero ideal of R such that $B \subset I$ and B is neither K nor K^*. As with the argument for K, B is a minimal ideal of R. So $BK = 0$. Similarly, BK^*, B^*K, and B^*K^* are zero, yielding $I^2 = (B \bigoplus B^*)\,(K \bigoplus K^*) = 0$. When $R^2 = 0$, the set $\{k + k^* : k \in K\}$ is a $*$-ideal of R and is contained in I. So case (i) must hold when $R^2 = 0$.

In part (ii) of Proposition 1.1, when $I^2 \neq 0$ we have $K^2 \neq 0$. Recall that a minimal ideal which is not square zero is a simple ring, [2, p.135]. So in the case at hand we have K and K^* are simple rings. From $I = K \bigoplus K^*$ and $I/K^* \approx K$, we see that K^*, and hence K, are maximal ideals of the ring I. This, and similar situations in the sequel, occur frequently enough to warrant looking more closely at the setting where a ring is a direct sum of two simple rings. We give two lemmas, the proofs of which are routine and hence are omitted.

Lemma 1.2 *Let R be a ring with ideals S_1 and S_2 such that $R = S_1 \bigoplus S_2$, S_1 and S_2 are simple rings, and S_1 and S_2 are either both square zero or neither is. Then:*

(i) S_1 and S_2 are maximal ideals and minimal ideals of R;

(ii) $R^2 = 0$ or $R^2 = R$;

(iii) if $R^2 = R$, then the only proper nonzero ideals of R are S_1 and S_2;

(iv) if $R^2 = 0$, then $R^+ \approx C_p \bigoplus C_p$, p a prime, and each subgroup is an ideal; all nonzero proper ideals have p elements.

Lemma 1.3 *Let R be a $*$-ring and let S_1 and S_2 be ideals of R such that $R = S_1 \bigoplus S_2$ and $S_1^* = S_2$. If S_1 and S_2 are simple rings, then either:*

(i) $R^2 = R$ and R is $*$-simple; or

(ii) $R^2 = 0$, $R^* = C_p \bigoplus C_p$, where p is prime, and R has at least one nonzero proper $*$-ideal.

An example illustrating part (ii) of Lemma 1.3 is afforded by using $(a,b)^* = (\alpha^{-1}(b), \alpha(a))$, where $\alpha \in \mathrm{Aut}(C_p)$. Then a nonzero proper $*$-ideal is given by $D = \{(a,a) : a \in C_p\}$, where $S_1 = (C_p, 0)$ and $S_2 = (0, C_p)$. Another is given by $A = \{(a,-a) : a \in C_p\}$. Of course $A = D$ if (and only if) $p = 2$. In the case where $\alpha = 1$, the identity mapping on C_p, then one gets a special case of the exchange involution. (Recall, [5, Example 2.13.11], that for any ring T, then $E = T \bigoplus T^{opp}$ is a ring and the mapping $(t_1, t_2) \to (t_2, t_1)$ yields an involution on E, the *exchange involution*.) Note that there are $p-1$ such automorphisms α. It is routine to show that given any α_1, $\alpha_2 \in \mathrm{Aut}(C_p)$, the resulting two rings with involution on $R = C_p \bigoplus C_p$, with $R^2 = 0$, are isomorphic in the category of rings with involution. (To see this use $f(a,b) = (\beta(a), \gamma(b))$, where $\gamma = \alpha\beta$, to get that the exchange involution ring, so given, is isomorphic to the one given by α.)

Immediately from Lemma 1.3 and the comment following Proposition 1.1 we have:

Corollary 1.4 *Let R be a $*$-ring and let I be a $*$-minimal ideal of R. If $I^2 \neq 0$, then $I^2 = I$ and I is a $*$-simple ring.*

Proposition 1.5 *Let R be a $*$-ring and let I be a minimal ideal of R such that $I^2 \neq 0$. Then either (exclusively):*

(i) I is a $*$-ideal of R; or

(ii) $I \bigoplus I^*$ is a $*$-simple ring and hence is a $*$-minimal ideal of R.

Proof. Assume $I \neq I^*$. Then minimality of I yields $I \cap I^* = 0$ and hence $I + I^* = I \bigoplus I^*$. Since $I^2 \neq 0$, the minimal ideals I and I^* are simple rings and the ring $I \bigoplus I^*$ has only I and I^* as proper nonzero ideals. Let Y be a nonzero proper $*$-ideal of the $*$-ring $I \bigoplus I^*$. So Y is I or I^*. Since Y is invariant under $*$, either case leads to the contradictory $I = I^*$.

So $I \bigoplus I^*$ is $*$-simple and consequently the only nonzero $*$-ideal of R contained in $I \bigoplus I^*$ is itself.

Note that in Proposition 1.1 we started with a $*$-minimal ideal and connected it to minimal ideals, while in Proposition 1.5 we started with a minimal ideal and connected it with $*$-minimal ideals. Observe that Lemma 1.3 shows that $I^2 \neq 0$ is necessary in Proposition 1.5(ii).

The following example illustrates Proposition 1.1(ii) as well as showing that the condition $I^2 \neq 0$ in Proposition 1.5(ii) is necessary.

Example 1.6 Let S be a ring and let X be an ideal of S with $X^2 = 0$. Let $R = \begin{bmatrix} S & X \\ X & 0 \end{bmatrix}$. Then with the standard matrix operations R is a ring and matrix transposition is an involution, denoted here by $*$. If S has unity, then $R^2 = R$. If X is a minimal ideal of S, then $\bar{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ is a minimal ideal of R. Observe that $\bar{X}^2 = 0$, $\bar{X} \bigoplus \bar{X}^*$ is a $*$-minimal ideal of R, but $\bar{X} \bigoplus \bar{X}^*$ is not a $*$-simple ring.

For a $*$-ring R the sum of all minimal $*$-ideals of R is denoted by $\mathrm{Soc}_*(R)$, the $*$-socle of R. (If R has no minimal $*$-ideals then define $\mathrm{Soc}_*(R)$ to be zero.) In light of Proposition 1.1 we have $\mathrm{Soc}_*(R) \subseteq \mathrm{Soc}(R)$. As we shall see in Example 1.9, it is possible for $\mathrm{Soc}_*(R) \neq \mathrm{Soc}(R)$.

Proposition 1.7 *Let R be a $*$-ring. Then $\mathrm{Soc}(R) = T \bigoplus \mathrm{Soc}_*(R) = T^* \bigoplus \mathrm{Soc}_*(R)$, where T is a direct sum of minimal ideals K of R such that $K^2 = 0$ and $K \bigoplus K^*$ is not a $*$-minimal ideal of R.*

Proof. Using a standard argument it can be shown that $\mathrm{Soc}(R)$ is a direct sum of minimal ideals. (A model for this argument is found in [4, 59–60].) Label the summands with ordinals so that $\mathrm{Soc}(R) = \sum \bigoplus M_\lambda$, $\lambda \in \Lambda$, and $\mathrm{Soc}_*(R) = \sum_{\lambda<\alpha} \bigoplus M_\lambda$, $T = \sum_{\alpha \leq \lambda} \bigoplus M_\lambda$. Then $\mathrm{Soc}(R) = (\mathrm{Soc}_*(R)) \bigoplus T$. From Proposition 1.5 we see that if I is a minimal ideal of R and $I^2 \neq 0$, then $I \subseteq \mathrm{Soc}_*(R)$. Then for $\alpha \leq \lambda$ we have $M_\lambda^2 = 0$ and $M_\lambda + M_\lambda^*$ is not a $*$-minimal ideal of R. Since $T \cap \mathrm{Soc}_*(R) = 0$, we have $T^* \cap \mathrm{Soc}_*(R) = 0$, and consequently $\mathrm{Soc}(R) = (\mathrm{Soc}_*(R)) \bigoplus T^*$.

Corollary 1.8 *Let R be a $*$-ring and let T be as in Proposition 1.7.*

(i) If R is a semiprime ring, then $\mathrm{Soc}(R) = \mathrm{Soc}_*(R)$.

(ii) If R has D.C.C. on $*$-ideals (or on $*$-ideals contained in T), then T contains no nonzero $*$-ideals. Consequently under such a chain condition if B is an ideal of R and $B \subseteq T$, then $B^* \nsubseteq T$.

(iii) If K is a minimal ideal of R and $K \subseteq T$ with $KR = RK = 0$, then $K^* \cap T = 0$.

Proof. The proofs of (i) and (ii) are immediate. To see (iii) observe that $(K, +) \approx C_p$, p prime, and that $K + K^* = K \bigoplus K^*$ contains a $*$-minimal ideal of R, namely $\{k + k^* : k \in K\}$. So $K \bigoplus K^* \subseteq \mathrm{Soc}_*(R)$ and hence $K^* \nsubseteq T$, which yields $K^* \cap T = 0$. (Note that R need not be semiprime for $\mathrm{Soc}(R)$ and $\mathrm{Soc}_*(R)$ to coincide, as Example 1.6 shows.)

The following example illustrates several of the preceding results.

Example 1.9 Let p be a prime and let A be the ring $\mathbb{Z}_p$ and B be the zero multiplication ring on the group C_p. Let $R = A \bigoplus A \bigoplus B \bigoplus B$. Define an involution $*$ on R via $(a_1, a_2, b_1, b_2)^* = (a_2, a_1, b_2, b_1)$. We list some properties of R that are relevant to the preceding results of this section.

(i) $R = \mathrm{Soc}(R)$ and $\mathrm{Soc}_*(R) = \{(a_1, a_2, b, b) : a_1, a_2 \in A,\ b \in B\}$.

(ii) $\mathrm{Soc}(R) = (0, 0, 0, B) \bigoplus \mathrm{Soc}_*(R) = (0, 0, B, 0) \bigoplus \mathrm{Soc}_*(R)$. Thus $(0, 0, B, 0)$ can be identified with T in Proposition 1.7.

(iii) Let $I = \{(0, 0, b, b) : b \in B\}$. Then $I^2 = 0$ and I is a $*$-ideal of R which is also a minimal ideal of R. (While Proposition 1.5 guarantees that all nonnilpotent minimal ideals of R are contained in $\mathrm{Soc}_*(R)$, this example shows that $\mathrm{Soc}_*(R)$ can also contain nilpotent ones.)

(iv) In general, $\mathrm{Soc}_*(R)$ does not contain all $*$-ideals of R which are contained in $\mathrm{Soc}(R)$. To illustrate this, observe that $(0, 0, B, B)$ is a $*$-ideal of R which is contained in $\mathrm{Soc}(R)$, but $(0, 0, B, B)$ is not contained in $\mathrm{Soc}_*(R)$.

(v) $\mathrm{Soc}_*(R)$ is a maximal ideal of R which is essential in R.

Comment (v) leads us to inquire into the relation between $\mathrm{Soc}_*(R)$ and $*$-essential ideals of R in general.

Proposition 1.10 *Let R be a $*$-ring. Then* $\mathrm{Soc}_*(R)$ *is equal to the intersection of all the $*$-essential ideals of R.*

Proof. The proof is similar to that given in Rowen [5, 176–178] where it is shown that the socle of a module is equal to the intersection of the essential submodules of the module.

2. $*$-Simple rings and $*$-maximal ideals

In this section we establish connections between $*$-maximal ideals and maximal ideals. Of course $*$-simple rings and simple rings play a key role in this.

Our first result of this section, together with Lemma 1.3 and the associated remarks following it, give an extension of the known characterization of $*$-simple in the category of $*$-rings with unity to the category of $*$-rings. (See [5, Proposition 2.13.24]) The situation in the larger category is somewhat more involved.

Proposition 2.1 *Let R be a $*$-simple ring. Then either:*

(i) R is simple; or

(ii) R contains a maximal ideal K such that $R = K \bigoplus K^*$, K and K^* are simple rings, $R^2 \neq 0$, and the only proper nonzero ideals of R are K and K^*.

Proof. Observe that if R is not simple, then R is a $*$-minimal ideal of itself which is not a minimal ideal. So $R = K \bigoplus K^*$, where K and K^* are minimal ideals of R. If T is an ideal of the ring K, then T is an ideal of R, so minimality of K yields $T = 0$ or $T = K$. Thus K is a simple ring. Since $R/K^* \approx K$, we have K^* is a maximal ideal of R. Similarly we obtain K^* is a simple ring and K is a maximal ideal of R.

Since by Lemma 1.3, $R^2 = 0$ implies the existence of a nonzero proper $*$-ideal of R, we must have $R^2 \neq 0$. Then Lemma 1.2 yields that K and K^* are the only nonzero proper ideals of R.

Corollary 2.2 *Let R be a $*$-simple ring. Then R is a subdirectly irreducible ring if and only if R is a simple ring.*

Corollary 2.3 *Let R be a $*$-simple ring. Then $(R,+)$ is either isomorphic to a direct sum of copies of the additive group of rational numbers or to a direct sum of copies of $(C_p,+)$, where p is a fixed prime.*

Proof. First take $R^2 = 0$. By Proposition 2.1, R is simple. So $(R,+)$ is isomorphic to some $(C_p,+)$. Next, consider $R^2 \neq 0$. Recall that a simple ring which is not square zero is an algebra over its centroid and the centroid is a field. So for R simple the desired result for $(R,+)$ follows immediately. For R not simple, observe that $R = K \bigoplus K^*$, where K is a simple ring with $K^2 \neq 0$ and the centroids for K and K^* are isomorphic.

In the next result we show that if a $*$-maximal ideal is not a maximal ideal, then it fits "tightly" in a maximal ideal.

Theorem 2.4 *Let R be a $*$-ring and let M be a $*$-maximal ideal of R. If M is not a maximal ideal of R, then there exists a maximal ideal K of R such that:*

(i) $K + K^* = R$ *and* $K \cap K^* = M$; *so* R/K^* *is simple and* $R/K^* \approx K/M$;

(ii) $R^2 + M = R$ *and the only proper ideals of R which properly contain M are K and K^*;*

(iii) K *and* K^* *are $*$-essential in R;*

(iv) *either M is essential as an ideal in the ring R, or there exists a minimal ideal I of R such that $K^* = M \bigoplus I$ and $R = I \bigoplus I^* \bigoplus M$.*

Proof. Since $\bar{R} = R/M$ is a $*$-simple ring which is not simple, we have that there exists a proper ideal K of R with $M \subset K$ such that $\bar{R} = K/M \bigoplus (K/M)^*$, with K/M and $(K/M)^* = K^*/M$ being both maximal as ideals in $\bar{R}$ and simple as rings. Thus K and K^* are maximal ideals of R, $K + K^* = R$, and $K \cap K^* = M$. Since $\bar{R}^2 = \bar{R}$, the only proper nonzero ideals of $\bar{R}$ are K/M and K^*/M. The Correspondence Theorem then yields that there are no proper ideals of R that properly contain M except K and K^*.

For purposes of contradiction assume there exists a nonzero $*$-ideal T of R such that $T \cap K = 0$. Also $T \cap K^* = 0$. Then $TK = 0 = TK^*$ and hence $TR = T(K + K^*) = TK + TK^* = 0$. Since $T \cap M = 0$ also, we have $R = M + T = M \bigoplus T$. Hence $R/M \approx T$, and T is a $*$-simple ring which is not a simple ring. By Proposition 2.1 we have $T^2 \neq 0$, a contradiction to $TR = 0$. Thus K is $*$-essential in R, which immediately yields that K^* is also.

To establish (iv), consider M to be not essential in the set of ideals of the ring R. Observe that if K is essential in the set of ideals of the ring R, then so is K^*, and consequently the nonzero intersection $M = K \cap K^*$ would also be essential. Thus there exists a nonzero ideal I of R such that $I \cap K = 0$. The only ideals of R which contain M are M, K, K^*, and R. So either $I \bigoplus M = K^*$ or $I \bigoplus M = R$. In the former case, from $I \approx M/K^*$ we have that I is a simple ring and hence I is a minimal ideal of R. Finally, consider $I \bigoplus M = R$. For each $k \in K$ there exists $i \in I$, $m \in M$ such that $k = i + m$. But $i = k - m$ is in K, forcing $i = 0$.

So $k = m$. Since k is arbitrary, this yields the contradictory $M = K$. Thus $I \bigoplus M \neq R$ and the proof is completed.

We next examine maximal ideals in $*$-rings.

Theorem 2.5 *Let R be a $*$-ring. If M is a maximal ideal of the ring R, then either (exclusively):*

(i) $M \cap M^*$ is a $*$-maximal ideal of R; or

(ii) for any $*$-ideal K of R such that $M \cap M^* \subset K \subset R$, then K is a maximal ideal of R and there is no ideal of R properly contained between $M \cap M^*$ and K, and between $M \cap M^*$ and M.

Proof. Assume $M \cap M^*$ is not a $*$-maximal ideal of R. Then $M \neq M^*$. Let K be any $*$-ideal of R such that $M \cap M^* \subset K \subset R$. From $R/M^* = (M + M^*)/M^* \approx M/(M \cap M^*)$, we have that $M/(M \cap M^*)$ is a simple ring. Thus $M \cap M^*$ is a maximal ideal of the ring M and there are no ideals of the ring R properly contained between $M \cap M^*$ and M. From $M \cap M^* \subseteq M \cap K \subseteq M$, we have either $M \cap K = M$ or $M \cap K = M \cap M^*$. The former yields $M \subseteq K$ and hence $M = K$, a contradiction to M not a $*$-ideal. Consider $M \cap K = M \cap M^*$. Then $K \nsubseteq M$ and $R/K = (M + K)/K \approx M/(M \cap K) = M/(M \cap M^*)$. Thus R/K is a simple ring and consequently K is a maximal ideal of R. From $R/M = (K + M)/M \approx K/(K \cap M) = K/(M \cap M^*)$, we see that $K/(M \cap M^*)$ is a simple ring and hence there are no ideals of R properly contained in between $M \cap M^*$ and K.

Corollary 2.6 *Let R be a $*$-ring with $R^2 = R$. If M is a maximal ideal of the ring R, then $M \cap M^*$ is a $*$-maximal ideal of R.*

Proof. Assume for purposes of contradiction that M is not a $*$-maximal ideal. Then $\bar{R} = R/(M \cap M^*)$ is not $*$-simple. However, $M \neq M^*$ and $\bar{R} = \bar{M} \bigoplus \bar{M}^*$, where $\bar{M} = M/(M \cap M^*)$, and $\bar{M}$ and $\bar{M}^*$ are simple rings. Since $\bar{R}^2 = \bar{R}$, Lemma 1.3(i) yields $\bar{R} = \bar{M} \bigoplus \bar{M}^*$ is $*$-simple.

Observe that if R is a $*$-ring and M a maximal ideal of the ring R such that $M \cap M^* = 0$, then either $M = 0$ and R is simple, or $R = M \bigoplus M^*$. Since M and M^* are simple rings, the situation $R = M \bigoplus M^*$ is the one addressed in detail in Lemma 1.3. This observation, together with that at the beginning of this section and Theorem 2.5 yield a thorough extension to the category of $*$-rings of the known result for the category of $*$-rings with unity, [5, Lemma 2.13.23]. Again, the larger category yields a somewhat richer situation.

Observe that Example 1.9 can be used to illustrate Theorems 2.4 and 2.5. For Theorem 2.4, identify M with $(0, 0, B, B)$ of Example 1.9. Then K will be either $(A, 0, B, B)$ or $(0, A, B, B)$. In Theorem 2.5(i), take M to be $(A, 0, B, B)$; then $M \cap M^* = (0, 0, B, B)$. For Theorem 2.5(ii), identify M with $(A, A, B, 0)$; then $K = \mathrm{Soc}_*(R)$.

3. Applications

In this section results from the previous sections are used to investigate $*$-subdirectly irreducible rings, $*$-prime rings, and $*$-prime ideals. First, some terminology and basic results. Let R be a $*$-ring. The intersection of all the nonzero ideals of R, called the *heart* of R, is denoted by $H(R)$. So R is subdirectly irreducible if and only if $H(R) \neq 0$. The intersection of all nonzero $*$-ideals of R is denoted by $H_*(R)$. Then R is $*$-subdirectly irreducible if and only if $H_*(R) \neq 0$. If $H_*(R) = 0$, then a routine application of Birkhoff's Theorem yields that R is a subdirect product of $*$-subdirectly irreducible rings. (The $*$-heart and $*$-subdirectly irreducible algebras have been investigated in detail in [3].) It is immediate that $H(R) \subseteq H_*(R)$. Following Rowen [5, 2.13], a $*$-ideal P is said to be $*$-prime if whenever A, B are $*$-ideals such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. (An investigation in depth concerning $*$-prime ideals has been carried out in [1].) So R is a $*$-prime ring if and only if (0) is a $*$-prime ideal.

Proposition 3.1 *Let R be a $*$-ring. The following are equivalent:*

(i) $(H_*(R))^2 \neq 0$*;*

(ii) R is a $*$-prime ring with at least one minimal ring ideal.

Proof. Let $H_* = H_*(R)$. Assume $H_*^2 \neq 0$. Then H_* is a minimal $*$-ideal of R. So from Proposition 1.1, either H_* is a minimal ideal of the ring R, or $H_* = K \bigoplus K^*$, where K and K^* are minimal ideals of R. In either case, if A and B are nonzero $*$-ideals of R, then $H_* \subseteq A$, $H_* \subseteq B$, and hence $0 \neq H_*^2 \subseteq AB$. So R is a $*$-prime ring.

Conversely, assume (ii). Let I be a fixed minimal $*$-ideal of R and let A be any nonzero $*$-ideal of R. Then $AI \neq 0$. So $A \cap I = I$. Thus $I \subseteq A$ for each $*$-ideal A of R, yielding $H_* = I$. Since R is $*$-prime we have $H_*^2 \neq 0$.

For many important classes of rings every prime ideal is maximal. Our final result connects this condition with the analogous one in the category of $*$-rings.

Theorem 3.2 *Let R be a $*$-ring. The following are equivalent:*

(i) every proper $*$-prime ideal of R is a $*$-maximal ideal of R;

(ii) every proper prime ideal of R is a maximal ideal of R.

Proof. Assume (i). Let P be a proper prime ideal of R. If P is $*$-prime, then P is $*$-maximal and hence R/P is $*$-simple. However, R/P is also a prime ring. Consequently R/P cannot be the direct sum of two distinct nonzero ideals. Using this and Proposition 2.1 yields that R/P is a simple ring, and hence P is a maximal ideal.

Next, consider P not $*$-prime. Then $P \neq P^*$ and $P \cap P^*$ is a $*$-prime ideal which is not a maximal ideal of the ring R. Since $P \cap P^*$ is $*$-maximal, we have that $R/(P \cap P^*)$ is a $*$-simple ring which is not a simple ring. Use Proposition 2.1 to obtain that $R/(P \cap P^*) = (P/(P \cap P^*)) \bigoplus (P^*/(P \cap P^*))$, and $P^*/(P \cap P^*)$ is a simple ring. Then $R/P \approx (R/(P \cap P^*)) / (P/(P \cap P^*)) \approx P^*/(P \cap P^*)$. So R/P is a simple ring and P is a maximal ideal.

Assume (ii). Let X be a $*$-prime ideal of R. We make use of the following result: a $*$-ring is a $*$-prime if and only if it contains a prime ideal P such that $P \cap P^* = 0$. (This is given in [5, Proposition 2.13.35], in the context of rings with unity. However, the proof there does not make use of a unity element in the ring.) So there is an ideal M of the ring R such that $X \subseteq M$, M/X is a prime ideal of the ring R/X, and $(M/X) \cap (M/X)^* = 0$. Since $(M/X)^* = M^*/X$ is also a prime ideal, we have that M and M^* are prime ideals of the ring R and $X = M \cap M^*$. By assumption, M and M^* are maximal ideals of R; so M/X and M^*/X are maximal ideals of R/X. Since $(R/X)^2 \neq 0$, Corollary 2.6 yields that R/X is $*$-simple. Consequently X is $*$-maximal.

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