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On Rings with Zero Total

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Abstract. Rings with zero total and bounded indices are characterized. It is proved that essential left (right) ideals of such rings contain essential two sided ideals generated by central idempotents. In particular, a prime ring with bounded indices has zero total if and only if it is a matrix ring over a skew field. It is shown that the maximal right ring of quotients of a such ring is equal to the maximal left ring of quotients and is isomorphic to finite direct sum of matrix rings over abelian regular selfinjective rings.

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All rings considered will be associative, not necessarily with unity element. Given a ring A, $I \lhd A$ will denote the fact that I is an ideal of A. Given two right ideals $I \subseteq J$ of a ring A, I is said to be *essential* in J, if $I \cap K \neq 0$ for every nonzero right ideal K of A contained in J. A nonzero element $a \in A$ is said to be *partially invertible*, if there exists an element $b \in A$ such that ab is a nonzero idempotent. Following Kasch [12] *the total* $\mathrm{Tot}(A)$ *of a ring* A may be defined as the set

$$\begin{aligned} \mathrm{Tot}(A) &= \{a \in A \mid aA \text{ does not contain a nonzero idempotent}\} \\ &= \{a \in A \mid a \text{ is not partially invertible}\}. \end{aligned}$$

One can check that

(a) $\mathrm{Tot}(A)$ is left and right symmetric,
(b) $\mathrm{Tot}(A)$ is a semi-ideal: $a\mathrm{Tot}(A)b \subseteq \mathrm{Tot}(A)$ for every $a, b \in A$,
(c) $J(A) + \mathrm{Tot}(A) \subseteq \mathrm{Tot}(A)$ where $J(A)$ is the Jacobson radical of A.

Recently [6] it was shown that the class $\mathcal{M}$ of all rings with zero total forms a weakly special class (in particular, $\mathcal{M}$ is closed under taking of ideals and essential extensions) and

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an upper radical $\mathcal{K} = \mathcal{UM}$ is a supernilpotent normal radical. Note that the class $\mathcal{M}$ contains the class of all strongly π-regular rings having no nonzero right nil ideals [7]. In particular, $\mathcal{M}$ contains all von Neumann regular rings and all semisimple rings with dcc for principal right ideals (see also [13, Chapter XIV]). Moreover, $\mathcal{M}$ contains Rickart C^*-algebras [2, Chapter 1, §8, Proposition 4]. The condition $\mathrm{Tot}(A) = 0$ is proved to be useful in study of Baer $*$-rings [9], [10] and [14].

The structure of semiprime rings with bounded indices of nilpotent elements was studied in [3], [1], [8] and [4]. In the present paper we study the structure of rings with zero total and bounded indices of nilpotent elements.

Let A be a ring. A finite subset $\{e_{ij} \mid 1 \leq i, j \leq n\} \subseteq A$ of nonzero elements such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i, j, k, = 1, 2, \ldots, n$ is called a system of n^2 matrix units. Given a nilpotent element $a \in A$, we set $\mathrm{In}(a) = \min\{n \mid a^n = 0\}$. If $a \in A$ is not nilpotent, we set $\mathrm{In}(a) = 1$. Let $S \subseteq A$ be a subset. Set

$$\mathrm{In}(S) = \sup\{\mathrm{In}(a) \mid a \in S\}.$$

Recall that a ring A is called I-ring if every non-nil right ideal of A contains a nonzero idempotent. Noting, that every ring in $\mathcal{M}$ is a Jacobson semisimple I-ring, we can state the following two theorems due to Levitzki as follows.

Theorem 1. [11, Theorem 1, Chapter X, Section 11] *Let A be a ring with zero total and let $a \in A$ be a nilpotent element of index n. Then the ideal (a) of A generated by a contains a system of n^2 matrix units.*

Theorem 2. [11, Theorem 2, Chapter X, Section 11] *Let A be a ring with zero total. Suppose that A contains a system $\{e_{ij}\}$ of n^2 matrix units such that either $e_{11}Ae_{11}$ contains nonzero nilpotent elements or $u = \sum_{i=1}^{n} e_{ii}$ is not in the center of A. Then there exists a system of $\geq (n+1)^2$ matrix units $\{f_{kl}\}$ such that $f_{11} \in e_{11}Ae_{11}$.*

The following result is a direct consequence of Theorem 1.

Corollary 3. *If A is a ring with zero total, then either, for any natural number n, A contains systems of n^2 matrix units or* $\mathrm{In}(A) = n$ *for some natural number n.*

Lemma 4. *Let A be a ring with* $\mathrm{In}(A) = n$ *for some natural number n and $\{e_{ij} \mid 1 \leq i, j \leq n\} \subseteq A$ a system of n^2 matrix units. Then $e_{11}Ae_{11}$ is a reduced ring and $e_{11} + e_{22} + \ldots + e_{nn}$ a central idempotent of A.*

Proof. Suppose that $e_{11}Ae_{11}$ is not a reduced ring. Choose $0 \neq b \in e_{11}Ae_{11}$ such that $b^2 = 0$. Letting $c = b + e_{12} + e_{23} + \ldots + e_{n-1,n}$, we note that $c^n = be_{1n}$ and $c^{n+1} = 0$. Since $\mathrm{In}(A) = n$, we conclude that $be_{1n} = c^n = 0$. Then $b = be_{1n}e_{n1} = 0$ contradicting $b \neq 0$. Thus $e_{11}Ae_{11}$ is a reduced ring.

Next, let $u = e_{11} + e_{22} + \ldots + e_{nn}$. Assume that $(1-u)Au \neq 0$ and choose $0 \neq b \in (1-u)Au$. Clearly $be_{ii} \neq 0$ for some $1 \leq i \leq n$. After a suitable renumbering of e_{ij}'s we may assume that $be_{11} \neq 0$. Set $c = b + e_{12} + e_{23} + \ldots + e_{n-1,n}$. Then $c^n = be_{1n}$ and $c^{n+1} = 0$. It follows that $c^n = 0$ which implies that $b = 0$ contradicting $b \neq 0$. Therefore $(1-u)Au = 0$ and $au = uau$

for all $a \in A$. Analogously one can show that $ua = uau$ for all $a \in A$. Thus $au = ua$ for all $a \in A$ and so u is a central idempotent of A.

Given a subset S of a ring A, we set

$$\begin{aligned} l_A(S) &= \{a \in A \mid aS = 0\} \quad \text{and} \\ r_A(S) &= \{a \in A \mid Sr = 0\}. \end{aligned}$$

Let A be a ring and $E \subseteq A$. The set E is called an orthogonal subset of A if it consists of central idempotents and $uv = 0$ for all $u, v \in E$, $u \neq v$. A subset E of A is said to be a dense orthogonal subset of A if E is an orthogonal subset of A and $l_A(E) = 0$. Given a right ideal L of A and $a \in A$, we set

$$(L : a) = \{x \in A \mid ax \in L\}.$$

Clearly $(L : a)$ is a right ideal of A. Next, if L is an essential right ideal of A, then one can easily show that $(L : a)$ is an essential right ideal of A as well.

Theorem 5. *Let A be a ring with* $\mathrm{In}(A) = n$ *for some natural number n. Then the following two conditions are equivalent:*

1. $\mathrm{Tot}(A) = 0$*;*
2. *A contains an essential ideal I which is a direct sum of ideals $I_k = M_{n_k}(D_k)$, $k = 1, 2, \ldots, t$, where*
   - *(a) $n_1 < n_2 < \ldots < n_t = n$.*
   - *(b) Each $M_{n_k}(D_k)$ is the $n_k \times n_k$ matrix ring over a reduced ring D_k.*
   - *(c) If L is a nonzero right ideal of D_k, then the set of all central idempotents of D_k belonging to L generates an ideal K essential in L.*

If $\mathrm{Tot}(A) = 0$*, then every essential right (left) ideal of A contains a dense orthogonal subset of A.*

Proof. (1) $\Longrightarrow$ (2). We proceed by induction on n. If $n = 1$, the A is a reduced ring and $A = M_1(A)$. It is well-known that each idempotent of a reduced ring is central. Let L be any nonzero right ideal of A and I the ideal generated by all the central idempotents belonging to L. Next, let $0 \neq a \in L$. Since $\mathrm{Tot}(A) = 0$, aA contains a nonzero idempotent and so $I \cap aA \neq 0$.

Now suppose that $n > 1$. Choose $a \in A$ with $\mathrm{In}(a) = n$. Then by Theorem 1 the ideal (a) contains a system $\{e_{ij}\}$ of n^2 matrix units. By Lemma 4, $e_{11}Ae_{11}$ is a reduced ring and $e_{11} + e_{22} + \ldots + e_{nn}$ is a central idempotent of A. According to [11, Proposition 6, Chapter III, Section 8], $Au = M_n(K)$ where

$$K = \{a \in Au \mid ae_{ij} = e_{ij}a \text{ for all } i, j = 1, 2, \ldots, n\}$$

and $K \cong e_{11}Aue_{11} = e_{11}Ae_{11}$. Therefore K is a reduced ring. Let $0 \neq a \in e_{11}Ae_{11}$. Then $(ab)^2 = ab \neq 0$ for some $b \in A$. Set $c = e_{11}be_{11}$. Since $e_{11}a = a = ae_{11}$,

$$(ac)^2 = (ab)^2e_{11} = abe_{11} = ac.$$

Further, $(ac)(ab) = (ab)^2 = ab \neq 0$ and so $ac \neq 0$. It follows that $e_{11}Ae_{11} \in \mathcal{M}$ and hence $K \in \mathcal{M}$. By the case $n = 1$ we have that K satisfies (c).

From the Kuratowski-Zorn Lemma it follows that there exists a maximal family of pairwise orthogonal central idempotents $\{u_j \mid j \in J\}$ such that $Au_j = M_n(K_j)$ where K_j is a reduced subring of A belonging to $\mathcal{M}$. We set

$$M = \sum_{j \in J} Au_j = \oplus_{j\in J} M_n(K_j) = M_n(\oplus_{j\in J} K_j)$$

and note that $D = \oplus_{j\in J} K_j \in \mathcal{M}$. If M is an essential ideal of A, then there is nothing to prove. Suppose that $N = r_A(M) \neq 0$. Clearly N is an ideal of A belonging to $\mathcal{M}$ and $M + N = M \oplus N$ is an essential ideal of A. We claim that $\mathrm{In}(N) < n$. If not, then by the above result N contains a central idempotent u' such that $u'N = M_n(K')$ where K' is a reduced ring belonging to $\mathcal{M}$. Obviously u' is a central idempotent of A and $u'N = u'A$. As $NM = 0$, we have that $u'u_j = 0$ for all $j \in J$, contradicting to the maximality of the family $\{u_j \mid j \in J\}$. Therefore $\mathrm{In}(N) < n$. Applying the induction assumption we complete the proof.

(2) $\Longrightarrow$ (1). It follows from (c) that each $D_k \in \mathcal{M}$. Let $B = M_{n_k}(D_k)$ and

$$R = \left\{ \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n_k} \\ 0 & 0 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_{n_k}(D_k) \mid d_{1i} \in D_k, 1 \leq i \leq n_k \right\},$$

$$S = \left\{ \begin{pmatrix} 0 & d_{12} & \dots & d_{1n_k} \\ 0 & 0 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_{n_k}(D_k) \mid d_{1i} \in D_k, 2 \leq i \leq n_k \right\}.$$

Clearly R is a right ideal of B such that $r_B(R) = 0$, $S = l_R(R)$ and $R/S \cong D_k \in \mathcal{M}$, and hence the conditions of [6, Lemma 2.3] are fulfilled which implies that $B \in \mathcal{M}$. Therefore $I = \oplus_{k=1}^t M_{n_k}(D_k) \in \mathcal{M}$. Since I is an essential ideal of A and $\mathcal{M}$ is closed under taking essential extensions, we conclude that $A \in \mathcal{M}$ and so $\mathrm{Tot}(A) = 0$.

Finally, let L be an essential right ideal of A. Clearly $L \cap I$ is an essential right ideal of the ring I and every dense orthogonal subset of the ring I is a dense orthogonal subset of A. Therefore it is enough to show that $L \cap I$ contains a dense orthogonal subset of I. Since $I = \oplus_{k=1}^t M_{n_k}(D_k)$, we can reduce the problem to the case when $A = M_n(D)$ and D is a reduced ring belonging to $\mathcal{M}$. By Kuratowski-Zorn lemma L contains a maximal orthogonal subset E. Suppose that $P = l_A(E) \neq 0$. Clearly there exists a nonzero ideal J of D such that $M_n(J) \subseteq P$. As $D \in \mathcal{M}$, J contains a nonzero central idempotent u. Therefore P contains a nonzero central idempotent e. Obviously $eA = M_n(eD)$ and $eL = L \cap eA$ is an essential right ideal of the ring eA. Clearly $eE = 0$. It follows from the choice of E that eL contains no nonzero central idempotents of A. We conclude that it is enough to show that L contains a nonzero central idempotent of A provided that $A = M_n(D)$ where $D \in \mathcal{M}$ is a reduced ring with unity 1 and L is an essential right ideal of A. Let $\{e_{ij} \mid 1 \leq i, j \leq n\}$ be a system of n^2 matrix units of A. Clearly $M = \cap_{i=1}^n (L : e_{i1})$ is an essential right ideal of A

and $e_{i1}M \subseteq L$ for all $i = 1, 2, \ldots, n$. As M is an essential right ideal, $e_{11}M \neq 0$. Taking into account that A is semiprime, we obtain that $(e_{11}M)^2 \neq 0$ and so $e_{11}Me_{11} \neq 0$. Choose any $0 \neq x \in e_{11}Me_{11}$. Then $e_{i1}xe_{1i} \in e_{i1}M \subseteq L$ and so $a = \sum_{i=1}^{n} e_{i1}xe_{1i} \in L$. Clearly $0 \neq a \in D$. Therefore there exists $b \in D$ such that ab is a nonzero central idempotent. Since $ab \in L$, we conclude that L contains a nonzero central idempotent. The proof is completed.

Corollary 6. *Let A be a ring with $\mathrm{In}(A) = n < \infty$. Then the following conditions are equivalent:*

1. *A is a prime ring with zero total;*
2. *$A = M_n(D)$ where D is a skew field.*

Proof. The implication (2) $\Longrightarrow$ (1) is obvious. Suppose that $A \in \mathcal{M}$. Since A is a prime ring, it follows from Theorem 5 that A contains an nonzero ideal I of the form $I = M_n(D)$ where D is a reduced ring belonging to $\mathcal{M}$. Clearly I is a prime ring and so D is a prime ring as well. It follows that D is a domain and hence every central idempotent of D is either zero or the unity of D. According to Theorem 5, every nonzero right ideal of D contains a nonzero central idempotent. Therefore D has no proper nonzero right ideals and thus D is a skew field which implies $A = I = M_n(D)$.

Corollary 7. *Let A be a ring with $\mathrm{In}(A) = n$ for some natural number n. Suppose that* $\mathrm{Tot}(A) = 0$. *Then:*

1. *A left ideal L of A is essential if and only if $r_A(L) = 0$.*
2. *A right ideal R of A is essential if and only if $l_A(R) = 0$.*

Proof. By symmetry it is enough to prove (1). If L is an essential left ideal, then L contains a dense orthogonal subset of A and so $r_A(L) = 0$. Next, let L be a left ideal such that $r_A(L) = 0$. Suppose that L is not essential. Choose K to be a left ideal maximal with respect to the property $K \cap L = 0$. It is well-known that $K \oplus L$ is an essential left ideal of A. By Theorem 5, $K \oplus L$ contains a dense orthogonal subset E of A. If $E \subseteq L$, then L contains the ideal $\langle E \rangle$ of A generated by E. Since $r_A(E) = 0$, $\langle E \rangle$ is an essential left ideal of A. Hence L is an essential left ideal of A contradicting to our assumption. Therefore $E \nsubseteq L$. Choose any $e \in E \setminus L$. Then

$$Ae = eAe \subseteq e(K \oplus L) = eK \oplus eL.$$

Clearly Ae is a ring with unity e and eK, eL are left ideals of Ae. Therefore there exist two orthogonal idempotents u and v of Ae such that $e = u + v$, $eK = Aeu$, and $eL = Aev$. As $e \notin L$, $u \neq 0$. Since $eu = u$, we have that

$$Lu = Leu = (eL)u = (Aev)u = Ae(vu) = 0,$$

contradicting to $r_A(L) = 0$. Therefore L is an essential ideal of A.

The reader is referred to [5, Chapter 2] and [15] for basic results of the theory of rings of quotients. Let $\mathrm{Tot}(A) = 0$ and $0 \neq a \in A$. Then there exists $b \in A$ such that ba is a nonzero idempotent. Therefore $r_A(a) \subseteq r_A(ba) = (1 - ba)A$ and so $r_A(a)$ is not an essential right ideal of A. Therefore the condition $\mathrm{Tot}(A) = 0$ implies that A is a right nonsingular ring. Analogously one can show that A is a left nonsingular ring. Recall that a ring A is called *abelian regular* (or *strongly regular*) if it is a von Neumann regular reduced ring.

Corollary 8. *Let A be a ring with* $\mathrm{In}(A) = n$ *for some natural number n. Suppose that* $\mathrm{Tot}(A) = 0$. *Then:*

1. *The left maximal ring of quotients $Q_{ml}(A)$ of A is equal to the right maximal ring of quotients $Q_{mr}(A)$ of A.*
2. *For each $q \in Q_{mr}(A)$ there exists a dense orthogonal subset E_q of A such that $eq \in A$ for all $e \in E_q$.*
3. *$Q_{mr}(A) = \oplus_{i=1}^{t} M_{n_i}(R_i)$ where each D_i is an abelian regular left and right selfinjective ring.*

Proof. According to [16, Theorem 3.3], given a right and left nonsingular ring D, $Q_{mr}(D) = Q_{ml}(D)$ if and only if a right (left) ideal R of D is essential if and only if $l_D(R) = 0$ (respectively, $r_D(R) = 0$). Therefore Corollary 7 implies that $Q_{mr}(A) = Q_{ml}(A)$. The second statement follows from the facts that each dense right ideal of A is an essential right ideal and so contains a dense orthogonal subset of A by Theorem 5. Finally, by Theorem 5 the ring A contains an essential ideal $I = \oplus_{i=1}^{t} M_{n_i}(D_i)$ where each D_i is a reduced ring belonging to $\mathcal{M}$. Clearly $Q_{mr}(A) = \oplus_{i=1}^{t} M_{n_i}(Q_{mr}(D_i))$. Therefore it is enough to show that $Q_{mr}(D_i)$ is an abelian regular ring. Since $D_i \in \mathcal{M}$, we conclude that D_i is a right and left nonsingular ring and $Q_{mr}(D_i) = Q_{ml}(D_i)$. It is well-known that the maximal right ring of quotients of a right nonsingular ring is a right selfinjective von Neumann regular ring (see [5, Theorem 2.1.15]). Therefore $Q_{mr}(D_i)$ is a left and right selfinjective von Neumann regular ring because $Q_{mr}(D_i) = Q_{ml}(D_i)$. Suppose that $q \in Q_{mr}(D_i)$ and $q^2 = 0$. Then $(eq)^2 = eq^2 = 0$ for all $e \in E_q$. Since D_i is reduced, we obtain that $eq = 0$ for all $e \in E_q$. Therefore $q\langle E_q \rangle = 0$ where $\langle E_q \rangle$ is the ideal of D_i generated by E_q. As E_q is a dense orthogonal subset of D_i, we conclude that $\langle E_q \rangle$ is a dense ideal of D_i and so the equality $q\langle E_q \rangle = 0$ implies that $q = 0$. Thus $Q_{mr}(D_i)$ is a reduced ring and the proof is complete.

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