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On the Stability of the Restricted Bundle of Space Curves Contained in a Quartic Surface

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Abstract. Here we show the existence of a smooth complex projective curve $Z \subset \mathbf{P}^3$ of degree d and genus g such that $T\mathbf{P}^3|Z$ is stable for all integers d, g such that (roughly) $d^2/9 < g < (d-1)^2/8$. Following a construction of Gruson and Peskine we will find such a curve Z on an integral rational quartic surface with a double line. We will use in an essential way also the main properties of stable vector bundles on reduced (but reducible) curves. In this way we will obtain examples of space curves with restricted tangent bundle with high degree of stability. Then for $n = 3, 4$ and 5 we will construct curves in $\mathbf{P}^n$ with simple restricted tangent bundle for all invariants (d, g) with (roughly) $2d \leq g \leq d^2/(4n-4)$ and contained in a surface of degree $2n-2$.

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Let $X \subset \mathbf{P}^n$ be a smooth projective curve. The restricted tangent bundle $T\mathbf{P}^n|X$ is an interesting object to study, since its properties are often related to the geometry of the embedding of X (at least in the most "extremal" cases). Furthermore, $T\mathbf{P}^n(-1)$ is an interesting bundle (the universal rank n quotient bundle of $\mathbf{P}^n$). Furthermore (see e.g. [12]) the space $H^1(X, T\mathbf{P}^n|X)$ is related to the germ at X of the map from the neighborhood of X in the Hilbert scheme $\mathrm{Hilb}(\mathbf{P}^n)$ to the moduli scheme of genus g curves M_g, $g := p_a(X)$. Several papers (see e.g. [10], [7], [2] and references there) studied the stability or the semistability of the restricted tangent bundle. Our aim here is to find a huge set of pairs of integers (d, g) such that there is a smooth curve $X \subset \mathbf{P}^3$ of degree d and genus g with $T\mathbf{P}^3|X$ stable . More precisely, we want to find real numbers α, β with $0 < \alpha < \beta$ and such that if $\alpha d^2 \leq g \leq \beta d^2$, there is a smooth curve $X \subset \mathbf{P}^3$ of degree d and genus g with $T\mathbf{P}^3|X$ stable. Note that by the upper bound on the genus of smooth space curves, all the space curves have $g \leq d^2/4 - d + 1$ ([6], Th. IV.6.4), all space curves

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not contained in a quadric surface have $g \leq d^2/6 - d/2 + 1$ (see e.g. the introduction of [5]) and all space curves with $d \geq 10$ not contained in a cubic surface have $g \leq 1 + d^2/8$ (see e.g. [5]). For smooth space curves contained in a quadric surface the picture is quite clear (see Remark 1.5). Here we will succeed (in characteristic 0) with $\alpha = 1/9$ and $\beta = 1/8$ (see Theorem 1.10). More precisely, in this range we will show the existence of a curve X with these invariants d, g and with restricted tangent bundle which has high degree of stability (see Theorem 1.10). Our curves will be contained in a rational surface Y of degree 4 with a double line. The use of this surface for the construction of smooth space curves with invariants d, g was an essential part of the proof of a "conjecture" of Halphen obtained by L. Gruson and Ch. Peskine ([5]). We will use in an essential way their study of curves on Y ([5], §1). Another key tool will be a semicontinuity result for the order of stability of vector bundles on flat families of reduced curves ([7]). Indeed it will be essential to have a good setup for our problem when X is a reduced but reducible curve. The definitions given in [7] (see section 1) seem to be such a setup.

In the last section we consider the much easier problem of the construction of projective curves with simple restricted tangent bundle. We will consider curves in $\mathbf{P}^n$ with $n = 3, 4$ and 5. Again we will consider curves contained in suitable surface of degree $2n-2$. We will obtain smooth nondegenerate curves in $\mathbf{P}^n, n = 3, 4$ and 5, with simple restricted tangent bundle for all invariants (d, g) with (roughly) $2d \leq g < d^2/(4n-4)$.

1. Stable restricted tangent bundle for space curves

We will always work over an algebraically closed base field with characteristic 0. Following [7] we recall a useful definition of stability for vector bundles on reduced (but not necessarily irreducible) projective curves. Let Z be a reduced proper curve and E a vector bundle on Z with constant rank r. A subsheaf A of E is said to be of constant rank k if for all irreducible components T of Z the sheaf $A|T$ has rank k at a general point of T. The degree $\deg(A)$ of a sheaf with constant rank k is defined by the formula $\deg(A) = \chi(A) - k\chi(\boldsymbol{O}_Z)$ which makes Riemann-Roch just a tautology. E is called stable (resp. semistable) if for every integer k with $1 \leq k < r$ and every subsheaf A of E with constant rank k we have $\deg(A)/k < \deg(E)/r$ (resp. $\deg(A)/k \leq \deg(E)/r$).

We recall the following key result ([7], Lemma 2.3).

Lemma 1.1. ([7], Lemma 2.3). *Let $\pi : X \to S$ be a flat projective family of reduced curves and E be a rank r vector bundle on X. Fix an integer i with $1 \leq i < r$. For every $s \in S$, let $f(i)(s)$ be the maximal degree of a subsheaf of $E|\pi^1(s)$ with constant rank i. Then the function $f(i) : S \to Z$ is upper semicontinuous.*

Lemma 1.2. *Let E be a rank r vector bundle on the reduced curve T. Assume $T = U \cup V$ with T with only nodal singularities along $U \cap V$. Fix an integer i with $1 \leq i < r$. Let a (resp. b) be the maximal degree of a subsheaf of $E|U$ (resp. $E|V$) with constant rank i. Then every subsheaf of E with constant rank i has degree $\leq a + b$.*

Proof. Let F be a subsheaf of E of constant rank i with $\deg(F)$ maximal. This implies that E/F has no torsion outside $\mathrm{Sing}(T)$. We may also assume that E/F has no torsion outside $U \cap V$. Let P be a nodal point of T and $R := \boldsymbol{O}_{T,P}$ the local ring of T at P. Assume by contradiction the existence of a nontrivial subsheaf G of E/F with support at P. Taking the kernel of the composition $E \to E/F \to (E/F)/G$ we would obtain a subsheaf H of E with constant rank i and $\deg(H) > \deg(F)$, contradiction. Therefore $\mathrm{Tor}_2^R(E/F, k(P)) = 0$. Since E is locally trivial we obtain $\mathrm{Tor}_i^R(F, k(P)) = 0$ for every $i > 0$. Hence F is locally free at P (but in general not a subbundle of E). Hence $\deg(F) = \deg(F|U) + \deg(F|V)$. Let F'_U (resp. F'_V) be the image of $F|U$ (resp. $F|V$) in $E|U$ (resp. $E|V$). From the Mayer-Vietoris exact sequence

$$0 \to \boldsymbol{O}_{U \cup V} \to \boldsymbol{O}_U \oplus \boldsymbol{O}_V \to \boldsymbol{O}_{U \cap V} \to 0. \tag{1}$$

we obtain the exact sequence

$$0 \to \cup_{P \in (U \cap V)} \mathrm{Tor}_1^{\boldsymbol{O}_{T,P}}(E/F, k(P)) \to F|U \oplus F|V \to F'_U \oplus F'_V \to 0. \tag{2}$$

Hence $\deg(F'_U) + \deg(F'_V) = \deg(F|U) + \deg(F|V) - m$ where $m := \sum_{P \in (U \cap V)} \mathrm{length}(\mathrm{Tor}_1^{\boldsymbol{O}_{T,P}}(E/F, k(P)))$. The subsheaf F'_U (resp. F'_V) of $E|U$ (resp. $E|V$) defines a saturated rank i subsheaf F''_U (resp. F''_V) of $E|U$ (resp. $E|V$) with $\deg(F''_U) = \deg(F'_U) + m$ (resp. $\deg(F''_V) = \deg(F'_V) + m$). Since $\deg(F''_U) \le a$ and $\deg(F''_V) \le b$ by assumption, we have $\deg(F) = \deg(F|U) + \deg(F|V) \le a + b - m$, proving the lemma. Note that equality holds if and only if $m = 0$, i.e. if and only F is a subbundle of E near $U \cap V$. □

Corollary 1.3. ([2], Lemma 1.1) *Let E be a rank r vector bundle on the reduced curve T. Assume $T = U \cup V$ with T with only nodal singularities along $U \cap V$.*
(i) *If $E|U$ and $E|V$ are semistable, then E is semistable.*
(ii) *If $E|U$ is stable and $E|V$ is semistable, then E is stable.*

Remark 1.4. Under the assumptions of 1.2, let F''_U (resp. F''_V) be a saturated subsheaf of $E|U$ (resp. $E|V$) with constant rank i. The proof of 1.2 shows how to obtain from F''_U and F''_V a saturated subsheaf F of E with constant rank i and $\deg(F''_U) + \deg(F''_V) - \mathrm{rank}(E) \cdot \mathrm{card}(U \cap V) \le \deg(F) \le \deg(''_U) + \deg(F''_V)$.

Now we consider the restricted tangent bundle of smooth space curves.

Remark 1.5. Let Z be a smooth space curve contained in a smooth quadric surface Q, say Z of type (x, y) with $1 \le x \le y$. Note ([2], Remark 2.5) that the two rulings of Q define two rank 1 subbundles of $T\mathbf{P}^3(-1)|Z$, one of degree $x - y$ and one of degree $y - x$. Hence ([2], Remark 2.5) if $y \ge 2x$ then $T\mathbf{P}^3(-1)|Z$ is not stable and if $y > 2x$ then $T\mathbf{P}^3(-1)|Z$ is not semistable. It was proved in [2], Lemma 2.4, that if Z is a general curve of bidegree (x, y) and $y < 2x$ (resp. $y \le 2x$) and $(x, y) \ne (1, 1)$, then $T\mathbf{P}^3(-1)|Z$ is stable (resp. semistable). Now assume that C is a smooth curve contained in a quadric cone Q'. By [6], V. Ex. 2.9, either C is a complete intersection $Q' \cap S$ for some surface S or C is linked to a line by a complete intersection $Q' \cap S$. The ruling of Q' does not induce a destabilizing rank 1 subbundle of $T\mathbf{P}^3(-1)|C$ but the possible invariants $\deg(C)$, $p_a(C)$

are very few and they arise as invariants of curves on a smooth quadric with type (x, y) and $x \le y \le x+1$.

Remark 1.6. Let $D \subset \mathbf{P}^n$ be a smooth linearly normal elliptic curve (hence $\deg(D) = n+1$). By [8] $T\mathbf{P}^n(-1)|D$ is indecomposable. Hence ([1]) it is stable and for every integer i with $1 \le i < n$ a rank i subbundle of $T\mathbf{P}^n(-1)|D$ has degree i.

Lemma 1.7. *Let $Z \subset \mathbf{P}^n$ be an integral nondegenerate curve. Then $T\mathbf{P}^n(-1)|Z$ has not $\boldsymbol{O}_Z$ as quotient.*

Proof. Consider the Euler exact sequence of $T\mathbf{P}^n$

$$0 \to \boldsymbol{O}_{\mathbf{P}^n}(z) \to (n+1)\boldsymbol{O}_{\mathbf{P}^n}(z+1) \to T\mathbf{P}^n(z) \to 0. \tag{3}$$

By the restriction to Z of (3) the bundle $T\mathbf{P}^n(-1)|Z$ is spanned. Hence $T\mathbf{P}^n(-1)|Z$ has $\boldsymbol{O}_Z$ as quotient if and only if it has $\boldsymbol{O}_Z$ as a direct factor. Choose homogeneous coordinates $z_0, \dots, z_n$ on $\mathbf{P}^n$. By the restriction of the Euler sequence every nonzero morphism $T\mathbf{P}^n(-1)|Z \to \boldsymbol{O}_Z$ is induced by $n+1$ constants $a_0, \dots, a_n$, not all 0. It is easy to see that Z is contained in the hyperplane $\{\sum a_i z_i = 0\}$. $\square$

Note that for every subscheme Z of $\mathbf{P}^n$ the Euler sequence (3) induces a map $H^0(\mathbf{P}^n, \boldsymbol{O}(1)) \to H^0(Z, T\mathbf{P}^n(-1)|Z)$.

Lemma 1.8. *Let $Z \subset \mathbf{P}^n$ be a smooth nondegenerate curve and F a rank $n-1$ quotient bundle of $T\mathbf{P}^n(-1)|Z$. Assume $\deg(F) \le \deg(Z) - 2$. Then the composition q of the maps $H^0(\mathbf{P}^n, \boldsymbol{O}(1)) \to H^0(Z, T\mathbf{P}^n(-1)|Z) \to H^0(Z, F)$ is injective and in particular we have $h^0(Z, F) \ge n+1$.*

Proof. Assume by contradiction the existence of $s \in \mathrm{Ker}(q)$, $s \ne 0$. Let D be the zero locus of s as section of $T\mathbf{P}^n(-1)|Z$. Hence we have an inclusion $(T\mathbf{P}^n(-1)|Z)/\boldsymbol{O}_Z(D) \to F$. Since s vanishes on a unique point of $\mathbf{P}^n$, we have $\deg(D) \le 1$. Hence $\deg(F) \ge \deg(Z) - 1$, contradiction. $\square$

Lemma 1.9. *Let $Z \subset \mathbf{P}^3$ be a smooth canonical curve of genus 4 (hence $\deg(Z) = 6$). Then $T\mathbf{P}^3(-1)|Z$ is stable, i.e. every rank 1 quotient bundle of $T\mathbf{P}^3(-1)|Z$ has degree ≥ 3 and every quotient rank 2 bundle of $T\mathbf{P}^3(-1)|Z$ has degree ≥ 5.*

Proof. Since Z is not a plane curve , $T\mathbf{P}^3(-1)|Z$ has not $\boldsymbol{O}_Z$ as a quotient (Lemma 1.7). Since Z is canonically embedded, Z is not hyperelliptic, i.e. every spanned nontrivial line bundle on Z has degree ≥ 3. Let F be a rank 2 quotient bundle of $T\mathbf{P}^3(-1)|Z$ and assume by contradiction $\deg(F) \le 4$. Since $h^0(Z, F) > \mathrm{rank}(F)$, for every $P \in Z$ there is $s \in H^0(Z, F)$, $s \ne 0$, with $s(P) = 0$. Hence $D := (s)_0$ has $\deg(D) \ge 1$. Thus the quotient line bundle $M := F/\boldsymbol{O}_Z(D)$ has degree $\deg(F) - \deg(D) \le 3$. Hence we have $\deg(F) = 4$ and $D = \{P\}$. Thus we obtain $h^0(Z, M) \le 2$. By construction we have $h^0(Z, M) \ge h^0(Z, F) - 1 \ge 3$ (Lemma 1.8), contradiction. $\square$

Here is our main result. For a description of the curves on the quartic rational surface Y with a double line considered in the statement of 1.10, see the first part of the proof of 1.10 or [5], §1. For all integers d, g with $d \geq 4$ and $0 \leq g < (d-1)^2/8$ there is an integer $r \geq 0$ such that $d' := d - 4r \geq 0$ and $0 \leq g' := g + r(2r - d + 1) \leq d' - 3$ ([5]).

Theorem 1.10. *Fix integers d, g with $d \geq 4$, $0 \leq g < (d-1)^2/8$. Fix an integer $r \geq 0$, such that $d' := d - 4r \geq 3$, and $0 \leq g' := g + r(2r - d + 1) \leq d' - 3$. Set $k := [(g'+1)/2]$. Let $Y \subset \mathbf{P}^3$ be the quartic rational surface with double line described in [5], §1. Then there is a smooth connected curve $Z \subset Y$ with* $\deg(Z) = d$, *$p_a(Z) = g$ and such that:*

(i) *Every rank 1 subbundle of $T\mathbf{P}^3(-1)|Z$ has degree $\leq d/3 - r/3 - k/3 + 3$. Every rank 2 subbundle of $T\mathbf{P}^3(-1)|Z$ has degree $\leq 2d/3 - 2r/3 - 2k/3 + 6$.*

(ii) *If $r + k \geq 10$, then $T\mathbf{P}^3(-1)|Z$ is stable.*

(iii) *If $(d+13)^2/9 \leq g < (d-1)^2/8$, then $T\mathbf{P}^3(-1)|Z$ is stable.*

Proof. Part (ii) follows from part (i). Part (iii) follows from part (ii) and trivial arithmetic computations; note that the inequality in (iii) implies $d \geq 210$. Here we will prove part (i). We use the proof of [5], Th. 1.1. Let S be the blowing up of $\mathbf{P}^2$ at 9 general points $P_i, 1 \leq i \leq 9$. Let $\Gamma \subset S$ be the smooth elliptic curve whose image Γ' in $\mathbf{P}^3$ is the double line of Y (hence $\omega_{S^*} \cong \boldsymbol{O}_S(\Gamma)$). Let c be the curve of S which is the strict transform of a line through the first exceptional point P_1 of the blowing up $S \to \mathbf{P}^2$ (hence a conic contained in Y). Let b be the disjoint union of the two exceptional divisors corresponding to the last 2 points P_8 and P_9 of the blowing up $S \to \mathbf{P}^2$ (hence two disjoint lines of Y). Hence $c^2 = c \cdot b = 0$, $c \cdot \Gamma = b \cdot \Gamma = 2$, $\Gamma^2 = 0$. Let $H := c + \Gamma$ be the hyperplane divisor of $S \to Y \subset \mathbf{P}^3$ (hence with the line $\mathrm{Sing}(Y)$ as image of Γ). We denote by $G \subseteq \mathrm{Aut}(\mathrm{Pic}(S)) \cong \mathrm{Aut}(\mathbf{Z}^{10})$ the group of isometries (with respect to the intersection form) preserving ω_S.

Sublemma 1.10.1. *Let A be a general quadric section of Y (hence* $\deg(A) = 8$*). Then every line subbundle of $T\mathbf{P}^3(-1)|A$ has degree ≤ 2 and every rank 2 subbundle of $T\mathbf{P}^3(-1)|D$ has degree ≤ 4.*

Proof. By Remark 1.1 it is sufficient to find a reducible but reduced quadric section A' of X with the same properties. By Remark 1.6 and Lemma 1.2 it is sufficient to show the existence of a nodal quadric section $D \cup D'$ with D and D' smooth degree 4 elliptic curves. By [5], Prop. 1.11, there is a smooth degree 4 elliptic curve $D \subset Y$. Since D is the complete intersection of two quadrics, we may find a quadric Q such that $Q \cap Y$ is a nodal curve $D \cup D'$ with D' smooth. Since D' is linked to D by the surfaces Q and Y we see that D' is a connected linearly normal elliptic curve, as wanted. □

Now we continue the proof of 1.10. First, we assume g' even. We take $\sigma \in G$ with $c \cdot \sigma(c) = d' - g' - 2 = d' - 2k - 2$ ([5], Prop. 1.9). By the proof of [5], Th. 1.1, a general member Z of $|\sigma(c) + k\Gamma + rH|$ is a smooth curve of degree d and genus g. By assumption we have $r \geq k + 3$. To show that Z solves part (i) of the statement we use 1.1, 1.2, 1.3 and the following reducible curve $Z' \cup Z'' \cup M \in |\sigma(c) + k\Gamma + rH|$. If $r - k - 1$ is even we take $Z' \in |\sigma(c) + \Gamma|$, $Z'' \in |(k-1)(H + \Gamma)|$ and $M \in |(r - k + 1)(H)|$. Since $r - k + 1$ is even, we may apply the Sublemma 1.10.1 to M. The image in Y of a general element of $|H+\Gamma|$ is a canonical curve of genus 4. We apply Lemma 1.9 $k-1$ times to handle Z''. Note that $(\sigma(c)+\Gamma)^2 = (c+\Gamma)^2 = 4$ and that $\sigma(c)+\Gamma$ is ample and spanned and very ample except along Γ ([5], Lemma 1.6). This is sufficient to apply the constructions and proofs of [3], Th. 1.4, to $\sigma(c)+\Gamma$. We obtain that either $T\mathbf{P}^3(-1)|Z'$ is semistable or the maximal rank 1 (resp. rank 2) subbundle of $T\mathbf{P}^3(-1)|Z'$ has degree $\leq (\deg(T\mathbf{P}^3(-1)|Z')/3+2$ (resp. $2(\deg(T\mathbf{P}^3(-1)|Z')/3+2)$. In summary, if $g'=2k$ is even and $r-k+1$ is even, then every rank 1 subsheaf of $T\mathbf{P}^3(-1)|(Z'\cup Z''\cup M)$ has degree $\leq (r-k+1)+(k-1)+(\sigma(c)+\Gamma)\cdot(c+\Gamma)+2 = r+(d'-2k+2)/3 = d/3-r/3-k/3+2/3$. If $g'=2k$ is even and $r-k+1$ is even, then every rank 2 subsheaf of $T\mathbf{P}^3(-1)|(Z'\cup Z''\cup M)$ has degree $\leq 2d/3-2r/3-2k/3+4/3$. If $r-k$ is even, we use $Z'=c+2\Gamma$, $Z''\in|(k-2)(H+\Gamma)|$ and $M\in|(r-k)H|$ and conclude as before, except that now the maximal rank 1 (resp. rank 2) subbundle of $T\mathbf{P}^3(-1)|Z'$ has degree $\leq \deg(T\mathbf{P}^3(-1)|Z')/3+4$ (resp. $2(\deg(T\mathbf{P}^3(-1)|Z'))/3+4$). Now assume g' odd, i.e. $g'=2k-1$. Take $\sigma\in G$ with $c\cdot\sigma(b)=d'-g'-3=d'-2k-2$. By the proof of [5], Th. 1.1, a general member Z of $|\sigma(b)+k\Gamma+rH|$ is a smooth curve of degree d and genus g. We have $(\sigma(b)+3\Gamma)^2=10$ and $(\sigma(b)+2\Gamma)^2=6$. Now we take a reducible curve $Z'\cup Z''\cup M$ as in the previous case with each time $\sigma(b)+3\Gamma$ instead of $\sigma(c)+\Gamma$ and $\sigma(b)+2\Gamma$ instead of $\sigma(c)+2\Gamma$. Now the maximal rank 1 (resp. rank 2) subbundle of $T\mathbf{P}^3(-1)|Z'$ has degree $\leq \deg(T\mathbf{P}^3(-1)|Z')/3+5$ (resp. $2(\deg(T\mathbf{P}^3(-1)|Z'))/3+5$). □

2. Simple restricted tangent bundle

Here we will study the simpleness of the restricted tangent bundle for curves in $\mathbf{P}^n$ with $n=3,4$ or 5. We will follow the extension of [9] given in an unpublished part of [11] ([11], Th. 0.4 and ch. 4.2) on curves on a smooth $K3$ surface.

Set $E := \mathrm{End}(T\mathbf{P}^n)$. Tensor the dual of the Euler exact sequence (3) by $T\mathbf{P}^n$. We obtain the following exact sequence:

$$0\to E(t)\to(n+1)T\mathbf{P}^n(t-1)\to T\mathbf{P}^n(t)\to 0. \tag{4}$$

Lemma 2.1. *Let $S\subset\mathbf{P}^n$ be a smooth, nondegenerate $K3$ surface with $\deg(S)=2n-2$. Let C be a general member of a complete linear system $|D|$ on S with $C^2>0$. Assume that $|D|$ contains a reducible but reduced member $Y\cup T$ with Y complete intersection of S with a quadric hypersurface. Assume $T\mathbf{P}^n|S$ simple. Assume $h^1(S,(T\mathbf{P}^n|S)(-3))=0$. Then $T\mathbf{P}^n|C$ is simple.*

Proof. By semicontinuity it is sufficient to prove that $T\mathbf{P}^n|(Y\cup T)$ is simple. By the restriction to S of the exact sequence (4) and the exact sequence

$$0\to E(-2)|S\to E|S\to E|Y\to 0 \tag{5}$$

$T\mathbf{P}^n|Y$ is simple. Assume by contradiction $T\mathbf{P}^n|(Y\cup T)$ not simple. Since $Y\cap T\neq\emptyset$, we may find $m\in H^0(Y\cup T,E|(Y\cup T))$ which drops rank at a point of Y. Hence $m|Y=0$. Thus m induces $m'\in H^0(T,\mathrm{Hom}(T\mathbf{P}^n,T\mathbf{P}^n(-2))|T)$. Since $\Omega^1_{\mathbf{P}^n}(2)$ is spanned by global sections, we have $m'=0$. Hence $m=0$, contradiction. □

Theorem 2.2. *Fix integers d, g with $d \geq 9$ and $2d - 7 \leq g < (d-1)^2/8$. Then there are a smooth connected curve C and a smooth quartic surface S with $C \subset S \subset \mathbf{P}^3$, $\deg(C) = d$, $p_a(C) = g$ and $T\mathbf{P}^3|C$ simple.*

Proof. By the proof in [9] (see [11], ch. 4.2) there are a smooth connected curve C and a smooth quartic surface S with $C \subset S \subset \mathbf{P}^3$, $\deg(C) = d$, $p_a(C) = g$. Let H be the hyperplane divisor of S. By assumption $(C - 2H) \cdot H = d - 8 > 0$ and $(C - 2H)^2 = 2g - 2 - 4d + 16 > 0$. Hence by Riemann-Roch the complete linear system $|C|$ contains a reducible curve $Y \cup T$ with Y complete intersection of S with a quadric surface. By Lemma 2.1 or its proof it is sufficient to note that $h^1(\mathbf{P}^3, E(-2)) = h^2(\mathbf{P}^3, E(-4)) = 0$ (use (3) and (4)) and hence that $h^1(S, E(-2)|S) = 0$ (by (5)). $\square$

Theorem 2.3. *Fix integers d, g with $d \geq 25$ and $2d - 15 \leq g < (d-1)^2/16$. Then there are a smooth connected curve C and a smooth $K3$ surface S with $C \subset S \subset \mathbf{P}^5$, $\deg(C) = d$, $p_a(C) = g$, $\deg(S) = 8$ and $T\mathbf{P}^5|C$ simple.*

Proof. By the proof of [11], Th. 4, given in [11], ch. 4.2, there are a smooth connected curve C and a smooth $K3$ surface S with $C \subset S \subset \mathbf{P}^5$, $\deg(S) = 8$, $\deg(C) = d$, $p_a(C) = g$. As in the proof of 2.3 by Riemann-Roch $|C|$ contains a reducible member $Y \cup T$ with Y intersection of S with a quadric hypersurface. Note that S is the complete intersection of 3 quadric hypersurfaces. Hence by standard exact sequences and Lemma 2.1 we reduce the proof of 2.3 (exactly as the proof of 2.2) to the vanishing of the cohomology groups $h^{i-1}(\mathbf{P}^5, E(-2i))$ and $h^i(\mathbf{P}^5, E(-2i))$ for $i = 1, 2, 3$ and 4. Use again the Euler sequence (3). $\square$

Theorem 2.4. *Fix integers d, g with $d \geq 16$ and $2d - 11 \leq g < (d-1)^2/12$. Then there are a smooth connected curve C and a smooth $K3$ surface S with $C \subset S \subset \mathbf{P}^4, \deg(C) = d$, $p_a(C) = g$, $\deg(S) = 6$ and $T\mathbf{P}^4|C$ simple.*

Proof. Note that every canonically embedded genus 4 space curve is the complete intersection of a quadric and a cubic. This implies that every smooth K3 surface $S \subset \mathbf{P}^4$ with $\deg(S) = 6$ is the complete intersection of a quadric hypersurface Q and a cubic hypersurface M. Hence the proof of 2.3 and the corresponding exact sequences for the inclusions $S = Q \cap M \subset Q \subset \mathbf{P}^4$ give easily the result. $\square$

Remark 2.5. The proof of Theorem 2.2 works verbatim if instead of a smooth quartic surface S we take as S any irreducible quartic surface. Hence using [5], Th. 1.1, and Riemann-Roch we obtain the following result.

Theorem 2.6. *Fix integers d, g with $d \geq 9$ and $2d - 7 \leq g < (d-1)^2/8$. Then there are a smooth connected curve C and an integral quartic rational surface S with a double line such that $C \subset S \subset \mathbf{P}^3$, $\deg(C) = d$, $p_a(C) = g$ and $T\mathbf{P}^3|C$ simple.*

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