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Radicals Induced by the Total of Rings

Dedicated to Prof. Dr. F. Kasch on his 75^{th} birthday

K. I. Beidar **R. Wiegandt***

Department of Mathematics, National Cheng Kung University
Tainan 70101, Taiwan, ROC
t14270@sparc1.cc.ncku.edu.tw

Mathematical Institute, Hungarian Academy of Sciences
P.O.Box 127, H-1364 Budapest
wiegandt@math-inst.hu

Abstract. The upper radicals $\mathcal{K}$ and $\mathcal{K}_p$ of the classes of rings with zero total and of prime rings with zero total, respectively, are introduced and discussed. $\mathcal{K}$ is a supernilpotent normal but nonspecial radical and $\mathcal{K}_p$ is a special normal radical. Crucial is the existence of the ring constructed in Theorem 3.1 which shows that $\mathcal{K} \neq \mathcal{K}_p$. Also the position of $\mathcal{K}$ and $\mathcal{K}_p$ is localized in the lattice of all radicals.

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1. Introduction

The total $\mathrm{Tot}(A)$ of a ring A has been introduced by Kasch [8] and it reminds to a radical property, though the shortcoming is that $\mathrm{Tot}(A)$ need not be an ideal because it is not closed under addition. To overcome this difficulty, the following idea is plausible: the upper radical $\mathcal{K}$ of all rings A with $\mathrm{Tot}(A) = 0$, may be a decent radical. We call $\mathcal{K}$ the *Kasch radical*, and show that $\mathcal{K}$ is a supernilpotent normal radical. Simultaneously we consider also the upper radical $\mathcal{K}_p$ of all prime rings A with $\mathrm{Tot}(A) = 0$. $\mathcal{K}_p$ is a special normal radical containing the Kasch radical $\mathcal{K}$. One of the benefits of radical theoretical investigations is to explore

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rings with interesting but unusual properties. This is featured in this note by constructing a biregular ring G such that $\mathrm{Tot}(G) = 0$ and $\mathrm{Tot}(G/P) \neq 0$ for each prime ideal P of G. The existence of such a ring is crucial, it shows that $\mathcal{K} \neq \mathcal{K}_p$. Thus, in a natural way we have got a nonspecial supernilpotent normal radical, namely $\mathcal{K}$. We also position the radicals $\mathcal{K}$ and $\mathcal{K}_p$ in the lattice of all radicals. The Kasch radical is bigger than the Behrens and the Jacobson radicals, and is incomparable with the Brown–McCoy radical, the generalized nil-radical, and the strongly prime radical. Rings with zero total have been studied in the forthcoming paper [2].

2. The radicals $\mathcal{K}$ and $\mathcal{K}_p$

All rings considered will be associative, not necessarily with unity element. We shall make use of the following notation: $[x, y] = xy - yx$. As usual, $I \lhd A$ will denote the fact that I is an ideal in a ring A. An ideal I of a ring A is said to be *essential* (or *large*) in A, if $I \cap K \neq 0$ for every nonzero ideal K of A. This fact will be denoted by $I \lhd\cdot A$. A nonzero element $a \in A$ is said to be *partially invertible*, if there exists an element $b \in A$ such that ab is a nonzero idempotent. Following Kasch [8] the total $\mathrm{Tot}(A)$ of a ring A may be defined as the set

$$\begin{aligned}\mathrm{Tot}(A) &= \{a \in A \mid aA \text{ does not contain a nonzero idempotent}\}\\ &= \{a \in A \mid a \text{ is not partially invertible}\}.\end{aligned}$$

Although $\mathrm{Tot}(A)$ is regretfully not closed under addition, it enjoys interesting properties, for instance

(a) $\mathrm{Tot}(A)$ is left and right symmetric,

(b) $\mathrm{Tot}(A)$ is a semi-ideal: $a\mathrm{Tot}(A)b \subseteq \mathrm{Tot}(A)$ for every $a, b \in A$,

(c) $J(A) + \mathrm{Tot}(A) \subseteq \mathrm{Tot}(A)$ where $J(A)$ is the Jacobson radical of A.

A class $\mathcal{C}$ of rings is said to be a *weakly special class*, if

(i) $\mathcal{C}$ consists of semiprime rings,

(ii) $\mathcal{C}$ is hereditary: $I \lhd A \in \mathcal{C}$ implies $I \in \mathcal{C}$,

(iii) $\mathcal{C}$ is closed under essential extensions: $I \lhd\cdot A$ and $I \in \mathcal{C}$ imply $A \in \mathcal{C}$.

As is well-known (see [11] and [5]) the upper radical

$$\mathcal{UC} = \{A \mid A \text{ has no homomorphic image in } \mathcal{C}\}$$

of a weakly special class $\mathcal{C}$ is a supernilpotent radical, and every supernilpotent radical is the upper radical of a weakly special class.

A *special class* $\mathcal{C}$ of rings is a weakly special class consisting of prime rings. The upper radical $\mathcal{UC}$ of a special class $\mathcal{C}$ is called a *special radical*. As known from [1], for a special radical $\gamma = \mathcal{UC}$ a ring A has zero γ-radical, $\gamma(A) = 0$, exactly when A is a subdirect sum of rings from $\mathcal{C}$.

For details on radical theory the reader is referred to the books [4] and [14].

Given a ring A with a subset S, we set

$$\begin{aligned}l_A(S) &= \{x \in A \mid xS = 0\},\\ r_A(S) &= \{x \in A \mid Sx = 0\}.\end{aligned}$$

Proposition 2.1 *The class $\mathcal{M} = \{A \mid \mathrm{Tot}(A) = 0\}$ is a weakly special class, and hence the Kasch radical $\mathcal{K} = \mathcal{U}\mathcal{M}$ is a supernilpotent radical.*

Proof. Since $J(A) \subseteq \mathrm{Tot}(A)$ for every ring, $\mathrm{Tot}(A) = 0$ implies $J(A) = 0$. Thus a ring A with $\mathrm{Tot}(A) = 0$ is semiprimitive, and so also is semiprime.

Next, we show that the class $\mathcal{M}$ is hereditary. Assume that $I \triangleleft A$ and $\mathrm{Tot}(A) = 0$. For any element $a \in I$, $a \neq 0$, there exists an element $b \in A$ such that $ab = (ab)^2 \neq 0$. Choosing $b' = bab$, we get

$$(ab')^2 = (ab)^4 = ab = (ab)^2 = ab'$$

and so the element a is partially invertible also in I.

To prove that the class $\mathcal{M}$ is closed under essential extensions, let us consider $I \triangleleft \cdot A$ such that $\mathrm{Tot}(I) = 0$, and let us assume that there exists a nonzero element $a \in \mathrm{Tot}(A)$. Then for any nonzero element $b \in I$ we have $ab \in I$, and hence by $\mathrm{Tot}(I) = 0$ there exists an element $b' \in I$ such that $abb' = (abb')^2 \neq 0$, provided that $ab \neq 0$. This is, however, impossible because $0 \neq a \in \mathrm{Tot}(A)$. Thus $ab = 0$ for all $b \in I$, that is, $0 \neq a \in l_A(I)$. Since $0 \neq l_A(I) \triangleleft A$ and $I \triangleleft \cdot A$, we conclude that $I \cap l_A(I) \neq 0$ which contradicts the semiprimeness of $I \in \mathcal{M}$.

Lemma 2.2 *Let R be a right ideal of a ring A. Suppose that P is a semiprime proper ideal of the ring R. Then:*

(i) $P^4 \subseteq AP^2A \cap R \subseteq P$.

(ii) *If Q is an ideal of A containing AP^2A and maximal with respect to the property $Q \cap R \subseteq P$, then:*

   (a) *Q is a semiprime ideal of the ring A,*

   (b) *if $\overline{A} = A/Q$, $\overline{R} = (R + Q)/Q$, and $\overline{P} = (P + Q)/Q$, then $r_{\overline{A}}(\overline{R}) = 0$, $l_{\overline{R}}(\overline{R}) = \overline{P}$ and $\overline{R}/\overline{P} \cong R/P$.*

Proof. Set $I = AP^2A$. Then

$$(I \cap R)^2 \subseteq IR = (AP)P(AR) \subseteq P.$$

Since P is a semiprime ideal of R, we conclude that $I \cap R \subseteq P$. Obviously $P^4 \subseteq AP^2A$ and so $P^4 \subseteq I \cap R$ which proves (i). Further, by Zorn's Lemma there exists an ideal Q of A containing I and maximal with respect to the property $Q \cap R \subseteq P$. If Q is not a semiprime ideal of A, then there exists an ideal J of A such that $J \supset Q$ and $J^2 \subseteq Q$. By the choice of Q we have that $J \cap R \not\subseteq P$. Further,

$$(J \cap R)^2 \subseteq J^2 \cap R \subseteq Q \cap R \subseteq P,$$

which contradicts the semiprimeness of the ideal P of the ring R. Therefore Q is a semiprime ideal of the ring A and so $\overline{A}$ is a semiprime ring. Now one can easily show that $l_{\overline{R}}(\overline{R})$ is a semiprime ideal of the ring $\overline{R}$. Since $\overline{P}^2 = 0$, we conclude that $\overline{P} \subseteq l_{\overline{R}}(\overline{R})$. Next, the relation $Q \cap R \subseteq P$ implies that $\overline{R}/\overline{P} \cong R/P$ and so $\overline{P}$ is a semiprime ideal of the ring $\overline{R}$. As $(l_{\overline{R}}(\overline{R}))^2 = 0$, it follows that $l_{\overline{R}}(\overline{R}) \subseteq \overline{P}$ and hence $l_{\overline{R}}(\overline{R}) = \overline{P}$. Finally, let $K = r_{\overline{A}}(\overline{R})$. If $K \neq 0$, we conclude from the choice of Q that $K \cap \overline{R} \neq 0$. Hence $K \cap \overline{R}$ is a nonzero nilpotent right ideal of the semiprime ring $\overline{A}$, a contradiction. Thus $r_{\overline{A}}(\overline{R}) = 0$.

Lemma 2.3 *Let A be a semiprime ring with right ideal R. Then:*
(i $A \in \mathcal{M}$ *implies that* $R/l_R(R) \in \mathcal{M}$.
(ii) $R/l_R(R) \in \mathcal{M}$ *and* $r_A(R) = 0$ *imply that* $A \in \mathcal{M}$.

Proof. (i) Let $a \in R \setminus l_R(R)$. We claim that aR contains a nonzero idempotent. Indeed, since $a \notin l_R(R)$, $ar \neq 0$ for some $r \in R$. As $\mathrm{Tot}(A) = 0$, there exists $b \in A$ such that $arb = (arb)^2 \neq 0$. Clearly $rb \in R$ and so our claim is proved. It follows that $R/l_R(R) \in \mathcal{M}$.

(ii) Suppose that $A \notin \mathcal{M}$. Then $\mathrm{Tot}(A) \neq 0$. Choose any nonzero element $a \in \mathrm{Tot}(A)$. If $aAR = 0$, then $(RaA)^2 = 0$ which implies that $RaA = 0$ and so $aA \subseteq r_A(R) = 0$. Therefore $aA = 0$ which contradicts the semiprimeness of A. Hence $axR \neq 0$ for some $x \in A$ and so $(axR)^3 \neq 0$ because axR is a nonzero right ideal of the semiprime ring A. It follows that $rax \notin l_R(R)$ for some $r \in R$. As $R/l_R(R) \in \mathcal{M}$, we conclude that there exists an element $b \in R$ such that $raxb + l_R(R)$ is a nonzero idempotent of the ring $R/l_R(R)$. In particular

$$(raxb)^3 - raxb \in l_R(R) \ \text{ and } \ (raxb)^2 \neq 0.$$

It follows that $(raxb)^4 - (raxb)^2 = [(raxb)^3 - raxb](raxb) = 0$. Setting $s = xbraxb$, we conclude that $(ras)^2 = ras$ and $ras \neq 0$. We have that $(asr)^4 = (asr)^2$ and $r(asr)^2as = (ras)^3 = ras \neq 0$, a contradiction to $a \in \mathrm{Tot}(A)$. Hence $\mathrm{Tot}(A) = 0$ and $A \in \mathcal{M}$.

A radical α is said to be *right strong* if for every right ideal R of any ring A the relation $\alpha(R) = R$ implies $R \subseteq \alpha(A)$. Next, a radical α is called *right hereditary* if for every right ideal R of any ring A the relation $\alpha(A) = A$ implies that $\alpha(R) = R$. The concepts of a left strong radical and a left hereditary radical are defined analogously (see [13]). Normal radicals were introduced in [6] in relation to Morita contexts. For us it is enough to know that the right strong and right hereditary supernilpotent radicals are precisely the supernilpotent normal radicals (see the remark after Theorem 4 in [12]).

Theorem 2.4 *The Kasch radical $\mathcal{K}$ is a right and left strong, right and left hereditary radical, hence a normal radical.*

Proof. Let R be a right ideal of a ring A such that $\mathcal{K}(R) = R$. We show that $R \subseteq \mathcal{K}(A)$. Suppose that $R \not\subseteq \mathcal{K}(A)$. Then there exists an ideal I of A such that $A/I \in \mathcal{M}$ and $R \not\subseteq I$. Replacing A and R by A/I and $(R+I)/I$ respectively, we may assume that $A \in \mathcal{M}$ and R is a nonzero $\mathcal{K}$-radical right ideal of A. Taking into account that $R/l_R(R) \in \mathcal{M}$ by Lemma 2.3**(i)**, we conclude that $R = l_R(R)$ and so $R^2 = 0$. Since A is semiprime, we infer that $R = 0$, a contradiction. Hence $\mathcal{K}$ is a right strong radical.

Next, let R be a right ideal of a $\mathcal{K}$-radical ring A. We show that R is a $\mathcal{K}$-radical ring as well. Suppose that $\mathcal{K}(R) \neq R$. Then there exists an ideal P of the ring R such that $0 \neq R/P \in \mathcal{M}$. Note that P is a semiprime ideal of R. Let Q, $\overline{A}$, $\overline{R}$, and $\overline{P}$ be as in Lemma 2.2. Then $\overline{R}/l_{\overline{R}}(\overline{R}) \cong R/P \in \mathcal{M}$ and $r_{\overline{A}}(\overline{R}) = 0$. It follows from Lemma 2.3**(ii)** that $\overline{A} \in \mathcal{M}$ contradicting $\overline{A} \in \mathcal{K}$. Thus $\mathcal{K}$ is a right hereditary radical.

By symmetry $\mathcal{K}$ is a left strong left hereditary radical, and by the remark done before Theorem 2.4, $\mathcal{K}$ is a normal radical.

Proposition 2.5 *The class*

$$\mathcal{M}_p = \{A \mid \mathrm{Tot}(A) = 0 \text{ and } A \text{ is a prime ring}\}$$

is a special class. Hence $\mathcal{K}_p = \mathcal{U}\mathcal{M}_p$ is a special radical and $\mathcal{K} \subseteq \mathcal{K}_p$.

Proof. The class $\mathcal{P}$ of all prime rings is a special class, as is well-known. Thus so is the class $\mathcal{M}_p = \mathcal{M} \cap \mathcal{P}$. Since $\mathcal{M}_p \subseteq \mathcal{M}$, it follows $\mathcal{K} \subseteq \mathcal{K}_p$.

An analogous proof to that of Theorem 2.4 yields

Proposition 2.6 *$\mathcal{K}_p$ is a right and left strong, right and left hereditary radical, and also a normal radical.*

3. An example showing $\mathcal{K} \neq \mathcal{K}_p$

Let R be a ring with a family of subrings $\{R_i \mid i \in I\}$ such that
(i) $R = \cup_{i \in I} R_i$;
(ii) Each R_i is a simple Ore domain;
(iii) For any $i, j \in I$ there exists $k \in I$ such that $R_i \cup R_j \subseteq R_k$.
Then one can easily show that

$$R \text{ is a simple Ore domain.} \tag{1}$$

Recall that a ring A is called *biregular*, if each principal ideal of A is generated by a central idempotent. Further, a ring A is called *reduced* if A has no nonzero nilpotent elements.

Theorem 3.1 *There exists a biregular reduced ring G such that $\mathrm{Tot}(G) = 0$ but $\mathrm{Tot}(G/P) \neq 0$ for all prime ideals P of G. Moreover, G/P is a simple Ore domain which is not a division ring.*

Proof. Let Z_2 be a two element field. We set $U = \prod_{i=1}^{\infty} U_i$ and $V = \oplus_{i=1}^{\infty} U_i$ where $U_i = Z_2$ for all $i = 1, 2, \ldots$. Next we denote by B the factor ring U/V. Given $u, v \in B$, we write $u < v$ if $u = uv \neq v$. Clearly $<$ is a partial order on B, and B has no nonzero minimal elements (i.e., for each $0 \neq v \in B$ there exists $0 \neq u \in B$ with $u < v$). Given a maximal ideal M of B, the subset $\tau = B \setminus M$ is called an ultrafilter. We denote by T the set of all ultrafilters of B. For $e \in B$, we set

$$T(e) = \{\tau \in T \mid e \in \tau\}.$$

Note that $T(e) = \emptyset$ if and only if $e = 0$. It is known that T is a compact Hausdorff space with base of closed-open subsets $\{T(e) \mid e \in B\}$. Furthermore, every closed-open subset of T is of the form $T(e)$, $e \in B$. Clearly $T(u) \cap T(v) = T(uv)$ for all $u, v \in B$. Since B has no nonzero minimal idempotents, T has no isolated points (i.e., points which are open subsets).

A finite subset $e_1, e_2, \dots, e_n \in B$ is said to be a partition of 1 if $e_1 + e_2 + \dots + e_n = 1$ and $e_ie_j = 0$ for all $i \neq j$.

Let Q be the rational number field. Given any division Q-algebra D, the Q-algebra $D\{x_i, y_i \mid i = 1, 2, \dots, n\}$ generated by D and x_i, y_i, $i = 1, 2, \dots, n$, with relations

$$[x_i, x_j] = [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij}$$

is a simple Noetherian domain by [9, Theorem 3.8].

We set $R = Q\{x_\tau, y_\tau \mid \tau \in T\}$. Clearly R is a simple Ore domain according to the above result and (1). Note that R is a Weyl algebra in $\{x_\tau, y_\tau \mid \tau \in T\}$. Given a subset $S \subseteq T$, we denote by R_S the subalgebra of R generated by $\{x_\tau, y_\tau \mid \tau \in T \setminus S\}$. Obviously $R_S = Q\{x_\tau, y_\tau \mid \tau \in T \setminus S\}$. Let Δ be the classical division ring of quotients of R. Clearly $\Delta \supseteq \Delta_S$, the classical division ring of quotients of R_S. We denote by A_S the subalgebra of Δ generated by Δ_S and $\{x_\tau, y_\tau \mid \tau \in S\}$. Clearly $A_S = \Delta_S\{x_\tau, y_\tau \mid \tau \in S\}$ and so A_S is a simple Ore domain. Note that A_S is a division ring if and only if $S = \emptyset$. Further, if $U, V \subseteq T$, then the subalgebra of Δ generated by A_U and A_V is contained in $A_{U \cap V}$. In particular $A_U \subseteq A_V$ if and only if $V \subseteq U$.

Consider Δ as a discrete topological space. Now let H be the Q-algebra of all continuous functions $f : T \to \Delta$. Given $f \in H$, $f(T)$ is a compact subspace of the discrete topological space Δ because T is a compact space and f is continuous. Therefore $|f(T)| < \infty$ and so there exists a partition $e_1, e_2, \dots, e_n$ of 1 such that $f|_{T(e_i)} = const$. The function f is said to be *special* if $f(T(e_i)) \subseteq A_{T(e_i)}$ for all $i = 1, 2, \dots, n$. We denote by G the subset of all special functions and claim that G is a subalgebra. Indeed, let $g, h \in G$. Then there exist partitions $u_1, u_2, \dots, u_m$ and $v_1, v_2, \dots, v_n$ such that $g(T(u_i)) \subseteq A_{T(u_i)}$ and $g(T(v_j)) \subseteq A_{T(v_j)}$ for all i, j. Clearly $(g+h)|_{T(u_i) \cap T(v_j)} = const$ and $(g \cdot h)|_{T(u_i) \cap T(v_j)} = const$ for all i, j such that $u_iv_j \neq 0$. Setting $K = \{(i, j) \mid u_iv_j \neq 0\}$, we note that $\{u_iv_j \mid (i, j) \in K\}$ is a partition. Since $T(u_i) \cap T(v_j) = T(u_iv_j)$, the subalgebra of Δ generated by $A_{T(u_i)}$ and $A_{T(v_j)}$ is contained in $A_{T(u_iv_j)}$ for all $(i, j) \in K$. Therefore $(g+h)(T(u_iv_j)) \subseteq A_{T(u_iv_j)}$ and $(g \cdot h)(T(u_iv_j)) \subseteq A_{T(u_iv_j)}$ which proves our claim. Given $v \in B$, we denote by $\overline{v}$ the function given by the rule

$$\overline{v}(\tau) = \begin{cases} 1 & \text{if} \quad v \in \tau \\ 0 & \text{if} \quad v \notin \tau. \end{cases}$$

Clearly $\overline{v}$, $v \in B$, are the only central idempotents of G. Furthermore, if $e_1, e_2, \dots, e_n$ is a partition of 1 in B, then $\overline{e_1}, \overline{e_2}, \dots, \overline{e_n}$ is a partition of 1 in G.

We claim that $\mathrm{Tot}(G) = 0$. Indeed, let $0 \neq f \in G$. Then $a = f(\tau) \neq 0$ for some $\tau \in T$. Clearly $f^{-1}(a)$ is a closed-open subset of T and so $f^{-1}(a) = T(e)$ for some $e \in B^*$. Since $a \in \Delta$ involves only a finite number of the x_σ and y_σ, there exists a finite subset $S \subseteq T$ such that $a \in \Delta_{T \setminus S} \subseteq A_{T \setminus S}$. Recalling that T has no isolated points, we conclude that $T(e) \not\subseteq S$. Let $\sigma \in T(e) \setminus S$. Choose $v \in B^*$ such that $\sigma \in T(v) \subseteq T(e)$ (i.e., $v \in \sigma$ and $v \leq e$) and $T(v) \cap S = \emptyset$. Therefore $a \in \Delta_{T \setminus S} \subseteq A_{T(e)}$, and it follows that $a^{-1} \in A_{T(e)}$. Define the function $g : T \to \Delta$ by the rule

$$g(\tau) = \begin{cases} a^{-1} & \text{if} \quad \tau \in T(e) \\ 0 & \text{if} \quad \tau \notin T(e). \end{cases}$$

Then $g \in G$ and $fg = \overline{e} \in fG$. Hence $f \notin \mathrm{Tot}(G)$ and thus $\mathrm{Tot}(G) = 0$.

Let P be any prime ideal of G. We set

$$M_P = \{v \in B \mid \overline{v} \in P\}$$

and claim that M_P is a maximal ideal of B and $G/P \cong A_{\tau_P}$ where $\tau_P = B \setminus M_P$. Indeed, the first statement follows from the fact that $\{\overline{v} \mid v \in B\}$ is the Boolean ring of central idempotents of G which is isomorphic to B. Consider now the map $\alpha : G \to \Delta$ given by the rule $\alpha(f) = f(\tau_P)$, $f \in G$. First we show that $\alpha(G) \subseteq A_{\tau_P}$. To this end, let $f \in G$ and $e_1, e_2, \ldots, e_n$ be a partition of 1 such that $f|_{T(e_i)} \subseteq A_{T(e_i)}$. Clearly $e_j \in \tau_P$ for some j. Therefore,

$$\alpha(f) = f(\tau_P) \in A_{T(e_j)} \subseteq A_{\tau_P}$$

because $\tau_P \in T(e_i)$ and so $\alpha(G) \subseteq A_{\tau_P}$. Next we show that $\alpha(G) = A_{\tau_P}$. Let $g \in A_{\tau_P}$. Since g involves only a finite number of x_τ and y_τ, there exists a finite subset $S \subseteq T$ such that $\tau_P \notin S$ and $g \in A_{T\setminus S} \subseteq A_{\tau_P}$. Since T is a compact Hausdorff space, there exists $e \in B$ such that $\tau_P \in T(e)$ and $T(e) \cap S = \emptyset$. Define the function $\overline{g} : T \to \Delta$ by the rule

$$\overline{g}(\tau) = \begin{cases} g & \text{if} \quad \tau \in T(e) \\ 0 & \text{if} \quad \tau \notin T(e). \end{cases}$$

Since $S \subseteq T \setminus T(e)$, $T(e) \subseteq T \setminus S$ and so $A_{T\setminus S} \subseteq A_{T(e)}$. Hence $\overline{g} \in G$. Clearly $\overline{g}(\tau_P) = g$ which proves our statement. Finally we show that $\ker(\alpha) = P$. Indeed, let $f \in \ker(\alpha)$. Then $f(\tau_P) = 0$ and so $\tau_P \in f^{-1}(0)$. Being a closed-open subset, $f^{-1}(0) = T(v)$ for some $v \in B$. Since $\tau_P \in f^{-1}(0)$, we conclude that $v \in \tau_P$ and so $v \notin M_P$. It follows that $\overline{v} \notin P$. We already know that $\overline{v}$ is a central idempotent of G. As $\overline{v}f(\tau) = 0$ for all $\tau \in T$, we see that $\overline{v}f = 0$ and so $\overline{v}(GfG) = 0$. Recalling that P is prime, we conclude that $f \in P$. Hence $\ker(\alpha) \subseteq P$. On the other hand let $f \in P$. Suppose that $f \notin \ker(\alpha)$. Then $a = f(\tau_P) \neq 0$. We already know that $f^{-1}(a) = T(e)$ for some $e \in B$ and $a \in A_{T(e)}$. Note that

$$e \in \tau_P. \tag{2}$$

Since $A_{T(e)}$ is a simple ring with 1, there exist $b_1, c_1, \ldots, b_m, c_m \in A_{T(e)}$ such that $\sum_{i=1}^m b_i a c_i = 1$. Define functions $g_i, h_i : T \to \Delta$ by the rules

$$\begin{aligned} g_i(\tau) &= \begin{cases} b_i & \text{if} \quad e \in \tau \\ 0 & \text{if} \quad e \notin \tau. \end{cases} \\ h_i(\tau) &= \begin{cases} c_i & \text{if} \quad e \in \tau \\ 0 & \text{if} \quad e \notin \tau. \end{cases} \end{aligned}$$

Clearly $g_i, h_i \in G$, $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m g_i f h_i = \overline{e} \in P$. Hence $e \in M_P$ and so $e \notin \tau_P$ contradicting (2). Hence it follows that $\ker(\alpha) = P$. Since $G/P \cong A_{\tau_P}$ is a simple Ore domain which is not a division ring, we conclude that $\mathrm{Tot}(G/P) \neq 0$ for all prime ideals P of G. Thus G is a ring without nonzero nilpotent ideals all of whose prime ideals are maximal. Hence it follows from [3, Corollary 4.4] that G is a biregular ring. The proof is complete.

Corollary 3.2 *$\mathcal{K} \neq \mathcal{K}_p$ and $\mathcal{K}$ is a supernilpotent nonspecial radical.*

Proof. The ring G in Theorem 3.1 has no nonzero homomorphic image in $\mathcal{M}_p$, whence $G \in \mathcal{K}_p$. Since $\mathrm{Tot}(G) = 0$ and the class $\mathcal{M}$ is hereditary by Proposition 2.1, it follows that $\mathcal{K}(G) = 0$. This proves $\mathcal{K} \neq \mathcal{K}_p$. According to Theorem 3.1, G is a semiprime ring. Thus G is a subdirect sum of the prime rings G/P where P ranges through all prime ideals of G. By Theorem 3.1, G/P is a simple prime ring, so either $\mathcal{K}(G/P) = 0$ or $\mathcal{K}(G/P) = G/P$. If $\mathcal{K}(G/P) = 0$, then by $\mathcal{K} = \mathcal{U}\mathcal{M}$ the ring G/P has a nonzero homomorphic image in $\mathcal{M}$. Hence the simplicity of G/P implies $G/P \in \mathcal{M}$, that is, $\mathrm{Tot}(G/P) = 0$, contradicting Theorem 3.1. Thus $\mathcal{K}(G/P) = G/P$ holds for every prime ideal P of G. This means that G cannot be represented as a subdirect sum of prime rings with zero $\mathcal{K}$-radical, and so $\mathcal{K}$ is a nonspecial radical.

The first example for a supernilpotent nonspecial radical was given by Rjabuhin [11]. An example for a supernilpotent normal nonspecial radical was constructed by Jaegermann and Sands (see p.345 in [7]).

4. The position of the radicals $\mathcal{K}$ and $\mathcal{K}_p$

In this section we shall make clear the relation of the radicals $\mathcal{K}$ and $\mathcal{K}_p$ to other well-known radicals.

Theorem 4.1 *The radical $\mathcal{K}_p$ is properly contained in the upper radical $\mathcal{T}$ of all matrix rings over division rings and in the upper radical $\mathcal{U}\nu$ of the class ν of all von Neumann regular rings.*
The radicals $\mathcal{K}$ and $\mathcal{K}_p$ are incomparable with the Brown–McCoy radical $\mathcal{G}$, with the generalized nil-radical $\mathcal{N}_g$, with the Groenewald-Heyman strongly prime radical s.
The Kasch radical $\mathcal{K}$ contains properly the Behrens radical $\mathcal{B}$ and the Jacobson radical J.

Proof. As is well-known, the class ν of all von Neumann regular rings is a *hereditary* radical class, so we may talk about the upper radical class $\mathcal{U}\nu$. If A is a von Neumann regular ring, then the elements of A are obviously partially invertible, whence $\mathrm{Tot}(A) = 0$. Hence $\nu \subseteq \mathcal{M}$ follows implying

$$\mathcal{K}_p = \mathcal{U}\mathcal{M}_p \subseteq \mathcal{U}(\nu \cap \mathcal{P})$$

where $\mathcal{P}$ stands for the class of all prime rings. Clearly $\mathcal{U}\nu \subseteq \mathcal{U}(\nu \cap \mathcal{P})$. Let $A \in \nu$. Being a von Neumann regular ring, A is semiprime and so is a subdirect sum of prime rings. As ν is homomorphically closed, each of these prime rings is von Neumann regular and hence A is $\mathcal{U}(\nu \cap \mathcal{P})$-semisimple. Therefore $\mathcal{U}\nu \supseteq \mathcal{U}(\nu \cap \mathcal{P})$ and so $\mathcal{U}\nu = \mathcal{U}(\nu \cap \mathcal{P})$. Thus it follows that $\mathcal{K}_p \subseteq \mathcal{U}\nu$.

Let V be a countably infinite dimensional vector space with a basis $\{x_i \mid i=1,2\ldots,n,\ldots\}$, T the ring of all linear transformations of V. Let us consider the subring S of T generated by the subring H of all linear transformations having finite rank and by the linear transformation t defined by $tx_i = x_{i+1}$, $i = 1, 2, \dots, n, \dots$. It is well-known that H is a simple von Neumann regular ring (without unity element) and H is the only proper ideal of T. Clearly H is a minimal proper ideal of S and thus S is a subdirectly irreducible prime ring. Moreover, the factor ring S/H and also S are not von Neumann regular. Thus $\nu(S) = H$ and $S \in \mathcal{U}\nu$ are valid. For any nonzero element $a \in S$ there exists obviously an element $b \in H$ such that $0 \neq ab \in H$. Since H is von Neumann regular, there exists an element $x \in H$ such that $ab = (ab)x(ab)$, and so $0 \neq abx = (abx)^2$. Thus every nonzero element $a \in S$ is partially invertible, that is, $\mathrm{Tot}(S) = 0$. Since S is a prime ring, also $S \in \mathcal{M}_p$ holds. Hence we have established that $S \in \mathcal{U}\nu \setminus \mathcal{K}_p$, proving $\mathcal{K}_p \neq \mathcal{U}\nu$.

The containment $\mathcal{U} \subseteq \mathcal{T}$ is obvious, and the ring H is in $\mathcal{T}$ but not in $\mathcal{U}\nu$.

In [10] it has been shown that the ring H is in the strongly prime radical class which is properly contained in the class $\mathcal{G}$ of all Brown–McCoy radical rings. Hence $H \in s \setminus \mathcal{U}\nu$ is valid implying $s \not\subseteq \mathcal{U}\nu$ and $\mathcal{G} \not\subseteq \mathcal{U}\nu$.

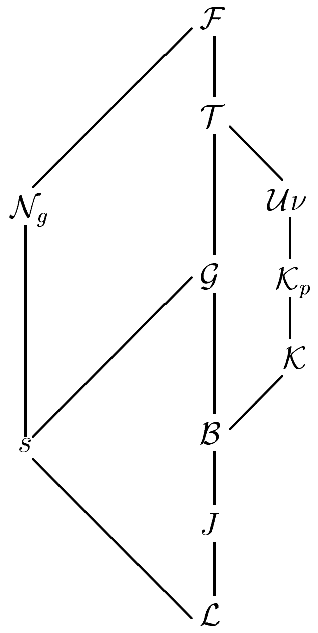
In [4, Example 8] a ring A is given such that A is a simple ring without zero divisors, has a unity element e, and e is its only nonzero idempotent, but A is not a division ring. Thus $A \notin \mathcal{G}$. Assume that $A \notin \mathcal{K}$. Since A is simple, necessarily $\mathrm{Tot}(A) = 0$, that is, for every nonzero element $a \in A$ there exists an element $x \in A$ such that $ax = e$. Hence A is a division ring, contradicting the properties of A. Thus $A \in \mathcal{K}$ holds proving that $A \in \mathcal{K} \setminus \mathcal{G}$, that is, $\mathcal{K} \not\subseteq \mathcal{G}$.

Since $s \not\subseteq \mathcal{U}\nu$, $s \subseteq \mathcal{G}$, $\mathcal{K} \subseteq \mathcal{K}_p \subseteq \mathcal{U}\nu$ and $\mathcal{K} \not\subseteq \mathcal{G}$, we conclude that the radicals s and $\mathcal{G}$ are not comparable with $\mathcal{K}$, $\mathcal{K}_p$, and $\mathcal{U}\nu$.

Since the ring A is simple and has no zero divisors, it follows that A is not in the upper radical determined by the class of all rings without zero divisors, that is, not in the generalized nil-radical $\mathcal{N}_g$. Hence $A \in \mathcal{K} \setminus \mathcal{N}_g$ which yields $\mathcal{K} \not\subseteq \mathcal{N}_g$. It is well-known (cf. [10]) that $s \subset \mathcal{N}_g$, so by $s \not\subseteq \mathcal{U}\nu$ we get that $\mathcal{N}_g \not\subseteq \mathcal{U}\nu$. Thus also $\mathcal{N}_g$ is not comparable with $\mathcal{K}$, $\mathcal{K}_p$, and $\mathcal{U}\nu$.

Consider now the Behrens radical $\mathcal{B}$, that is, the upper radical determined by the class of all subdirectly irreducible rings whose hearts contain nonzero idempotents. If $\mathcal{B} \not\subseteq \mathcal{K}$, then there exists a ring D which is in $\mathcal{B}$ but not in $\mathcal{K}$. Then D has a nonzero homomorphic image E in $\mathcal{M}$. Clearly $E \in \mathcal{B}$ and $\mathrm{Tot}(E) = 0$. In particular E contains a nonzero idempotent e. Let I be a ideal of E maximal with respect to the property $e \notin I$. Then E/I is a subdirectly irreducible ring whose heart contains a nonzero idempotent (namely, $e + I$), contradicting $E \in \mathcal{B}$. Thus $\mathcal{B} \subseteq \mathcal{K}$. Since $\mathcal{B} \subseteq \mathcal{G}$ necessarily $\mathcal{B} \neq \mathcal{K}$.

The place of the Kasch radical in the lattice of radicals is summarized in the following diagram where $\mathcal{L}$ and $\mathcal{F}$ stand for the Levitzki locally nilpotent radical and for the upper radical determined by the class of all division rings, respectively. The containments not explained in the proof of Theorem 4.1 are well-known.



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