Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 38 (1997), No. 1, 135-143.

Commutativity of the Schur Algebra

Mina Teicher
Department of Mathematics and Computer Science
Bar-Ilan University
Ramat-Gan 52900, Israel

Abstract. In this paper we study the Schur algebra of the induced representation of the Weil representation of a certain subgroup of $\mathrm{GL}_2(C)$ and prove that it is commutative. It thus implies that the representation is multiplicity free.

Introduction

This paper is based on my talk given at the Schur-Gedenkkolloquium which was held at Humboldt Universität in Berlin.

Let G be an algebraic group, U a maximal unipotent subgroup of G.

$$U \simeq \begin{pmatrix} 1 & * & \dots & * \\ & \ddots & 0 & \vdots \\ & 0 & \ddots & * \\ & & & 1 \end{pmatrix}$$

For almost all G, every irreducible representation π of G is isomorphic to some component of the induced representation of a character ψ of U, (i.e., $\pi \subset \operatorname{Ind}_U^G \psi$). Therefore, to know all the irreducible components of $\operatorname{Ind}_U^G \psi$ is like knowing all the irreducible representations of G. Thus, studying $\operatorname{Ind}_U^G \psi$ is very important.

An irreducible component of $\operatorname{Ind}_U^G \psi$ which is isomorphic to π is called the Whittaker Model of π. If we would know that all irreducible components appear only once in $\operatorname{Ind}_U^G \psi$ (i.e., $\operatorname{Ind}_U^G \psi$ is multiplicity free), then we could easily find all irreducible representations of G and conclude that the Whittaker Model is unique.

In 1962, Gelfand and Graev (see [1]) proved that for an algebraic group G and for U and ψ, as above, $\operatorname{Ind}_U^G \psi$ is multiplicity free. In 1974, Shalika proved it for $GL(n,k)$ (see [5]) where k is a local field. In 1977, Piatetski-Shapiro proved it for $GL(n,\mathbb{A})$ over a global field (see [3], [4]).

0138-4821/93 \$ 2.50

In this manuscript we shall discuss a generalized Whittaker Model, where U is replaced by $D = M_1U$ and ψ is replaced by ω_ψ, the Weil representation of M_1U. We shall outline the proof that $\mathrm{Ind}_D^G\,\omega_\psi$ is multiplicity free. We shall do it by proving that the Schur algebra of $\mathrm{Ind}_D^G\,\omega_\psi$ is multiplicity free.

We shall start by recalling a few definitions. Then we shall explain how the Schur lemma connects to multiplicity problems. We shall discuss the Schur algebra of the induced representation. Then we shall present the group G which we are studying, and the group $D = M_1U$. We shall construct the Weil representation W_ψ of D and then outline the proof of "$\mathrm{Ind}_D^G\,\omega_\psi$ is multiplicity free."

The paper is divided as follows:

Introduction
1. Definitions and notations
2. The Schur algebra and multiplicity problems
3. The induced representation
4. Introducing the groups $D < G$
5. The Weil representation of D
6. The main theorem

1. Definitions and notations

In this paper G will denote a group. We recall basic definitions and notations:

$T : (\pi_1, V_1) \to (\pi_2, V_2)$ *is a homomorphism of representations if:* $T : V_1 \to V_2$ *is a homomorphism of vector spaces, and* $T\pi_1(g) = \pi_2(g)T\ \forall\ g \in G$.

$(\pi_1, V_1) \simeq (\pi_2, V_2)$ *are equivalent representations if: there exist* $T : (\pi_1, V_1) \xrightarrow{\sim} (\pi_2, V_2)$ *isomorphism of representations.* $(\pi_1 \underset{T}{\sim} \pi_2)$.

V_1 *is an invariant subspace of* V *if:* V_1 *is a subspace of* V *and* $\pi(g)v_1 \in V_1\ \forall v_1 \in V_1$. *We then have a subrepresentation,* $\pi\big|_{V_1}$.

(π, V) *is irreducible if:* V *has no invariant subspace.*

(π, V) *is decomposable if:* $V = \oplus \sum V_i$, *where* V_i *is an invariant subspace and* $\pi|_{V_i}$ *is irreducible.*

(π, V) *is a finite dimensional representation if:* $\dim V < \infty$. *A one-dimensional representation, a character, is denoted by* χ.

(π, V) *is multiplicity free if:* $V = \oplus \sum_{i=1}^{m} V_i$, V_i *invariant and irreducible,*
$V_i \not\sim V_j \quad \forall\ i \neq j$.

We recall the following classical theorem:

Theorem 1.1 *If* G *is compact and* π *is a finite dimensional representation, then* (π, V) *is decomposable.*

2. The Schur algebra and multiplicity problems

The basic theorem of representation theory is the so-called "Schur Lemma".

Schur Lemma 2.1 *Let (π_1, V_1), (π_2, V_2) be two finite dimensional irreducible representations of G. Let $f : V_1 \to V_2$ be a homomorphism of the representations, $f\pi_1(g) = \pi_2(g)f \quad \forall\ g \in G$. Then:*

$\pi_1 \not\sim \pi_2 \Rightarrow f \equiv 0$

$\pi_1 \underset{T}{\sim} \pi_2 \Rightarrow f = \lambda T \quad \lambda \in \mathbb{C}$.

By the Schur lemma, we transfer multiplicity problems of representations to commutativity properties of algebras. We recall:

Definition. (Schur algebra, $\mathrm{Aut}_G(V)$)

$$\begin{aligned}\mathrm{Aut}_G(V) &= \textit{automorphism ring of a representation } \pi : G \to \mathrm{Aut}(V)\\ &= \{f : V \to V; f \textit{ homomorphism},\ f\pi(g) = \pi(g)f\ \forall\ g \in G\}.\end{aligned}$$

The following is an immediate result of the Schur lemma, which we shall use in the proof of the main result:

Theorem 2.2 *Let (π, V) be a finite dimensional decomposable representation of G, $V = \sum\limits_{i=1}^{m} n_i V_i$. Then π is multiplicity free $\Leftrightarrow$ $\mathrm{Aut}_G(V)$ is commutative.*

The proof follows from the fact that $\mathrm{Aut}_G\ (V) = \sum M_{n_i}\ (\mathbb{C})$. Thus $\mathrm{Aut}_G(V)$ is commutative $\Leftrightarrow \sum\ M_{n_i}\ (\mathbb{C})$ is commutative $\Leftrightarrow n_i = 1\ \forall i \Leftrightarrow (\pi, V)$ is multiplicity free.

3. The induced representation

The result in this paper deals with induced representation. We recall:

Definition. (Induced Representation.) *Let D be a subgroup of G, let (π, W) be a representation of D, $\dim W < \infty$. The induced representation of π from D to G, denoted $\mathrm{Ind}_D^G\, \pi$ is: (V, τ) where*

$V = \{f : G \to W | f(\delta g) = \pi(\delta) f(g)\ V\delta \in D,\ \forall g \in G\}$

$\tau(g_0) f(g) = f(g g_0)$.

The Schur algebra of an induced representation has a model which is more convenient for us. We prove:

Theorem 3.1 *Let G be a finite group $D < G$, (π, W) a representation of D. $(V, \tau) = \mathrm{Ind}_D^G\, \pi$. Then*

$$\operatorname{Aut}_g V \overset{S}{\simeq} \{K : G \to \operatorname{End}_C W \mid K(\delta_1 g \delta_2) = \pi(\delta_1)K(g)\pi(\delta_2) \\ \forall\ \delta_1, \delta_2 \in D \quad \forall\ g \in G\}$$

$$K_1 *\ K_2(g) = \sum_{X \in G} K_1(gX^{-1})K_2(x).$$

Proof. By the definition of induced representation $V = \{f : G \to W \mid f(\delta g) = \pi(\delta)f(g)\}$. For $K : G \to \operatorname{End}_C W$ as above we associate $S(K) : V \to V$ which is defined by $S(K)(f)(g) = \sum_{h \in G/D} K(gh^{-1})(f(h))$. S turns out to be an isomorphism.

4. Introducing the groups $D < G$

In the sequel k is a finite field, $\operatorname{char}(k) \neq 2$, and $\langle , \rangle$ is an antisymmetric form on k^*.

Let $G = \{g \in GL_{2n}(\mathbb{C}) \mid \langle gx, gy \rangle = \langle x, y \rangle \cdot \lambda(g) \quad \forall\ x, y \in k^{2n}\}$.

Take e_1, an isotropic vector, i.e., $\langle e_1, e_1 \rangle = 0$. Let $Q = \{g \in G \mid g(e_1) = \alpha e_1$ for some $\alpha \in k^*\}$. The group Q is decomposed to a unipotent part and a reductive part (classically known as a Levi decomposition), i.e., $Q = MU$ where:

$$M \simeq S_p(2n-2) \times k^*$$
$$U \simeq \text{ Heisenberg group of order } 2n-2.$$

Let $M_1 = \{m \in M \mid mu = um \quad \forall\ u \in$ Center $(U)\}$. We define: $D = M_1 U$.

We shall compute G, D for the special case where $2n = 4$ and

$$\langle x, y \rangle = {}^t x J y \qquad J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \qquad x, y \in k^4.$$

In this case $G = \{g \in GL_n(\mathbb{C}) \mid {}^t gJg = \lambda(g)J$ for some $\lambda(g) \in k^*\}$ and is denoted by $GS_p(4, k)$.

Let $\{e_i \mid i = 1, \ldots, 4\}$ be the standard basis of k^4. Then $Q = \{g \in G \mid g(e_1) = \alpha e_1\}$ is equal to

$$Q = \left\{ \begin{pmatrix} \alpha & & & \\ 0 & & * & \\ 0 & & & \\ 0 & 0 & 0 & \beta \end{pmatrix} \right\}$$

Let $E = \operatorname{span}\{e_2, e_3\}$. Let $M = \{g \in Q \mid g(E) = E$ and $g(e_1) = \beta e_4$ for some $\beta \in k^*\}$. Thus:

$$M = \left\{ \begin{pmatrix} \alpha & & \\ & \boxed{h} & \\ & & \beta \end{pmatrix} \middle| \, h \in GS_p(2,k) \quad \alpha\beta = \lambda(h) = \lambda(g), \right\}$$

Clearly, $M = GS_p(2,k) \times k^*$.

Let $U = \{g \in G | g(e_1) = e_1, g(e) = e + \alpha e_1 \text{ for some } \alpha \in k \ \forall\ e \in E\}$. Thus

$$U = \left\{ \begin{pmatrix} 1 & x & y & z \\ & 1 & 0 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \middle| \, x, y, z \in k \right\}.$$

Let us denote $(x,y,z) = \begin{pmatrix} 1 & x & y & z \\ & 1 & 0 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix}$. By matrix multiplication we get:

$$(x,y,z) \cdot (x_1,y_1,z_1) = (x + x_1,\ y + y,\ xy_1 - x_1 y + z + z_1).$$

Evidently: $Z(U) = \{(0,0,k)\}$. Let $M_1 = \{m \in M | mu = um \ \forall\ u \in Z(U)\}$. Then:

$$M_1 = \left\{ \begin{pmatrix} \alpha & & \\ & h & \\ & & \alpha \end{pmatrix} \middle| \, h \in GS_p(2,k), \lambda(h) = \alpha^2 \right\}$$

$$= \left\{ \alpha \begin{pmatrix} 1 & & \\ & h' & \\ & & 1 \end{pmatrix} \middle| \, h' \in S_p(2,k), \alpha \in k^* \right\}.$$

Clearly, $M_1 \sim S_p(2,k) \times k^*$. Since $M_1 U = U M_1$, $M_1 U$ is actually a subgroup and we denote $D = M_1 U$.

5. The Weil representation of D

We construct the Weil representation of D in several steps:

a) *The Weil representation of U*

Let ψ be a nontrivial additive character of k, $\psi \not\equiv 1$. Let $X = \{(x,0,z)\} \subseteq U$. Clearly, X is commutative. We extend ψ to X by $\psi(x,0,z) = \psi(z)$. Its induced representation to U, $\omega_\psi = \operatorname{Ind}_X^U \psi$ is called the Weil representation of U.

It is well-known that ω_ψ is the *only* irreducible representation of U whose restriction to $Z(U)$ is ψ. Let us denote its vector space by V.

b) *The Weil representation of $S_p(2n-2,k)$*

The group $S_p(2n-2,k)$ acts on $U/Z(U) \simeq k^{2n-2}$ in a natural way. This action defines $\forall\ g \in S_p(2n-2,k)$ a new representation of $U : u \to \omega_\psi(gu)$. By the above theorem,

$\omega_\psi(\cdot)$ and $\omega_\psi(g\cdot)$ are equivalent. Then $\forall\ g \in S_p(2n-2,k)\ \exists\ A(g)$ equivalence operator $A(g) : V \to V$, s.t. $\omega_\psi(gu) = A(g)^{-1}\omega_\psi(u)A(g)$.

It turns out that generally $A(g_1g_2) = C(g_1g_2)A(g_1)A(g_2)$ (i.e., $A(g)$ is a projective representation). But for k finite, $C(g_1g_2) \equiv 1$ and thus, $A(g)$ is a representation. We call $A : S_p(2n-2,k) \to \mathrm{Aut}(V)$ the Weil representation of $S_p(2n-2,k)$.

c) *The Weil representation of* M_1

Let μ be a multiplicative character of k^*. Recall that $M_1 = S_p(2n-2,k) \times k^*$. We define $\omega_\psi : M_1 \to \mathrm{Aut}(V)$ by: $\omega_\psi(m,\alpha) = \mu(\alpha)A(m)$.

d) *The Weil representation of* D

The Weil representations of M_1 and U are compatible (see [2]), so they glue to a representation of D, $\omega_\psi : D \to \mathrm{Aut}(V)$.

In what follows we need precise formulas for ω_ψ and thus we compute:

Formulas of $\omega_\psi : U \to \mathrm{Aut}(V)$ **for** $2n = 4$ **and** $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$

$$V = \{f : U \to \mathbb{C} | f((x,0,z)u) = \psi(z)f(V)\}$$
$$V \sim \{f : \{(0,y,0)\} \to \mathbb{C}\} \sim \{f : k \to \mathbb{C}\}.$$

The action:

$$\begin{aligned}
\omega_\psi(0,y,0)f(w) &= f(w+y) \\
\omega_\psi(x,0,0)f(w) &= f((0,w,0)\cdot(x,0,0)) \\
&= f(x,w,-xw) \\
&= f(x,0-2xw)\cdot(0,w,0) \\
&= \psi(-2xw)f(w) \\
\omega_\psi(0,0,z)f(w) &= f((0,w,0)\cdot(0,0,z)) \\
&= f(0,w,z)) \\
&= f((0,0,z)\cdot(0,w,0)) \\
&= \psi(z)f(w).
\end{aligned}$$

Formulas of ω_ψ **on** $S_p(2,k)$

$$\begin{aligned}
\omega_\psi \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} f(w) &= \psi\left(\frac{1}{2}cw^2\right) f(\omega) \\
\omega_\psi \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f(w) &= f(wa^{-1}) \\
\omega_\psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(w) &= \sum_{c\in k}(cw)f(c)
\end{aligned}$$

In [6], A. Weil, who constructed this representation, proved that it is irreducible. Thus we have:

Theorem 5.1 (1964, Weil) *The representation ω_ψ of $D = M_1U$ is irreducible.*

We are ready now to formulate and prove our main result.

6. The main theorem

We know (see §I) that the induced representation of the Weil representation of D, $\mathrm{Ind}_D^G\,\omega_\psi$, is decomposable. We prove here that it is multiplicity free.

Theorem. $\mathrm{Ind}_D^G\,\omega_\psi$ *is multiplicity free.*

Proof. Let us denote the space of ω_ψ by V, $V = \{f : k \to c\}$. By Theorem 2.2 it is enough to prove that the Schur algebra of $\mathrm{Ind}_D^G\,\omega_\psi$ is commutative. By Theorem 3.1, we have another presentation of the Schur algebra of an induced representation, i.e., the Schur algebra $\simeq \{K : G \to \mathrm{Aut}(V) \mid K(\delta_1 g \delta_2) = \omega_\psi(\delta_1)K(g)\omega_\psi(\delta_2)\ \forall\ \delta_1, \delta_2 \in D\ \ \forall\ g \in G\}$. We denote this algebra by $\mathcal{K}$, and we shall prove that $\mathcal{K}$ is commutative.

To prove that $\mathcal{K}$ is commutative we introduce an involution on $\mathcal{K}$, i.e., a function τ, $\tau : \mathcal{K} \to \mathcal{K}$ such that: $\tau\tau(K) = K\quad \forall\ K \in \mathcal{K}$ and $\tau(K_1K_2) = \tau(K_2)\tau(K_1)\quad \forall K_1, K_2 \in \mathcal{K}$ and we prove that it is actually the identity. Denote ${}^\tau K = \tau(K)$. We shall prove ${}^\tau K = K\quad \forall\ K \in \mathcal{K}$. We do it in the following steps:

1) Introduce an involution on G that stabilizes D and stabilizes $Z(U)$ elementwise.
2) Choose a base B of V s. t. $\omega_\psi({}^\tau\delta) = {}^t(\omega_\psi(\delta))$ for t transposes w.r.t. B.
3) Define ${}^\tau K(g) = {}^tK({}^\tau g)$ and prove that ${}^\tau K \in \mathcal{K}$, (thus τ is an involution on $\mathcal{K}$).
4) Find representatives for $Q\backslash G/Q$.
5) Find representatives for $M_1\backslash M$ and M/M_1.
6) Find representatives for $D\backslash G/D$.
7) Prove that ${}^\tau K(\eta) = K(\eta)$ for every representative from step 6.
8) Deduce that ${}^\tau K = K\ \forall K \in \mathcal{K}$ and thus $\mathcal{K}$ is commutative.

1) Let $T = T_4 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$. Clearly T is symmetric of order 2. We define an involution ${}^\tau g$ on G as follows: ${}^\tau g = {}^tTgT$. Since ${}^tT = T$, $T^2 = T$ and t is an involution, we get that ${}^\tau({}^\tau g) = g$, ${}^\tau(g_1g_2) = {}^\tau g_2\,{}^\tau g_1$, and τ is an involution. Clearly, ${}^\tau(x, y, z) = (-x, y, z)$ and ${}^\tau\begin{pmatrix} \alpha & & \\ & \alpha h & \\ & & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & & \\ & \alpha T_2^t h T_2 & \\ & & \alpha \end{pmatrix}$. Thus, τ stabilizes D and stabilizes $Z(U)$ elementwise.

2) There exists the following theorem:

Theorem. *Let G be a group, (π, V) an irreducible representation of G, τ an involution on G. Then there exists a base B of V such that the corresponding transpose t satisfies* $\pi({}^\tau g) = {}^t\pi(g)$.

As quoted in Theorem 5.1, ω_ψ is irreducible and thus we can apply it to the above theorem to get a transpose t such that: $\omega_\psi({}^\tau\delta) = {}^t\omega_\psi(\delta) \quad \forall\, \delta \in D$.

3) We define ${}^\tau K$ by ${}^\tau K(g) = {}^t(K({}^\tau g))$. Now: ${}^\tau K(\delta_1 g \delta_2) = {}^t\left(K\left({}^\tau\delta_2\, {}^\tau g\, {}^\tau\delta_1\right)\right) = {}^t\left(\omega_\psi({}^\tau\delta_2)\, {}^\tau K(g)\omega_\psi({}^\tau\delta_1)\right) = {}^t\left({}^t\omega_\psi(\delta_2)\, {}^tK({}^\tau g)\right)\, {}^t\omega_\psi(\delta_1)) = \omega_\psi(\delta_1)\, {}^\tau K(g)\omega_\psi(\delta_2)$. Thus, ${}^\tau K \in \mathcal{K}$. It is easy to check that τ is an involution on $\mathcal{K}$.

4) Consider $\mathbb{P}^3 = \mathbb{P}(k^4) = \left\{\ell(x) | x \in k^4 - \{0\}\right\}$. G acts transitively on $\mathbb{P}^3$ by $g(\ell(x)) = \ell(g(x))$. Let Q be the stabilizer of $\ell(e_1)$. Q also acts on $\mathbb{P}^3$ and there are 3 orbits to this action:

(1) $\{\ell(e_1)\}$.
(2) $\{\ell(e) | \langle e, e_4\rangle \neq 0 \;\; \ell(e) \neq \ell(e_1)\} = \text{orbit } (\ell(e_3))$.
(3) $\{\ell(e) | \langle e, e_4\rangle = 0\} = \text{orbit } (\ell(e_4))$.

Since G acts transitively on $\mathbb{P}^3$, $\mathbb{P}^3 \simeq G/Q$. Since there are 3 orbits to the action of Q on $\mathbb{P}^3$ we have $Q\backslash G/Q \simeq \{3 \text{ orbits}\}$. In this correspondence an orbit of $\ell(e)$ corresponds to $\overline{g} \in Q\backslash G/Q$ such that $g(e_1) = e$. Thus, $Q\backslash G/Q$ has 3 cosets whose representatives are: $\begin{pmatrix} I_2 & \\ & I_2 \end{pmatrix}, \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix}, \begin{pmatrix} & T_2 \\ T_2 & \end{pmatrix}$.

5) We shall prove that $\left\{\begin{pmatrix} I_2 & \\ & \alpha I_2 \end{pmatrix} \;\; \alpha \in k^*\right\}$ are representatives of M/M_1. The proof that they are also representatives for $M_1\backslash M$ is similar.

$$\begin{pmatrix} \alpha & & \\ & g & \\ & & \beta \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\beta} I_2 & \\ & I_2 \end{pmatrix} \begin{pmatrix} \beta & & \\ & g^1 & \\ & & \beta \end{pmatrix}.$$

$\lambda\begin{pmatrix} \alpha & & \\ & g & \\ & & \beta \end{pmatrix} = \alpha\beta$, $\lambda\begin{pmatrix} \frac{\alpha}{\beta} I_2 & \\ & I_2 \end{pmatrix} = \dfrac{\alpha}{\beta}$. Thus, $\begin{pmatrix} \beta & & \\ & g^1 & \\ & & \beta \end{pmatrix}$ is also an element of M and $\lambda\begin{pmatrix} \beta & & \\ & g^1 & \\ & & \beta \end{pmatrix}$ $= \beta^2$. Thus, $\begin{pmatrix} \beta & & \\ & g^1 & \\ & & \beta \end{pmatrix} \in M_1$. Thus, $\left\{\begin{pmatrix} \frac{\alpha}{\beta} I_2 & \\ & I_2 \end{pmatrix}\right\}$ are representatives for M/M_1. We replace $\frac{\alpha}{\beta}$ by α and we get that $\left\{\begin{pmatrix} \alpha I_2 & \\ & I_2 \end{pmatrix} \;\middle|\; \alpha \in k^*\right\}$ is a set of representatives for M/M_1.

6) From 4) and 5) we get that we have 3 types of representatives for $Q\backslash G/Q$.

type I: $\begin{pmatrix} \alpha I_2 & \\ & I_2 \end{pmatrix} \quad \alpha \in k^*$.

type II: $\begin{pmatrix} 0 & \alpha I_2 \\ I_2 & 0 \end{pmatrix} \quad \alpha \in k^*$.

type III: $\begin{pmatrix} 0 & \alpha T_2 \\ T_2 & 0 \end{pmatrix} \quad \alpha \in k^*$.

7) We shall only prove here that ${}^{\tau}K(\eta) = {}^{\tau}K$ for every η of type I. Let $\eta = \begin{pmatrix} \alpha I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ $\alpha \in k^*$. We want to prove that ${}^{\tau}K(\eta) = K(\eta)$ for every $K \in \mathcal{K}$.

Case I: $\alpha \neq 1$
Since $\alpha \neq 1$ and ψ is not trivial, there exists $c \in k$ such that $\psi(c) \neq \psi\left(\frac{c}{\alpha}\right)$. Let $K \in \mathcal{K}$.
Since $\begin{pmatrix} 1 & & c & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \eta = \eta \begin{pmatrix} 1 & & \frac{c}{\alpha} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, we have $\omega_\psi \begin{pmatrix} 1 & & c & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} K(\eta) = K(\eta)\omega_\psi \begin{pmatrix} 1 & & \frac{c}{\alpha} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$.
Thus $\psi(c)K(\eta) = K(\eta)\psi\left(\frac{c}{\alpha}\right)$. Since $\psi(c) \neq \psi\left(\frac{c}{\alpha}\right)$, we get $K(\eta) = 0$. This is true for every $K \in \mathcal{K}$ and thus also ${}^{\tau}K(\eta) = 0$. In particular, ${}^{\tau}K(\eta) = K(\eta)$.

Case II: $\alpha = 1$
In this case $\eta = I$ and $\delta\eta = \eta\delta \quad \forall\ \delta \in D$. Let $K \in \mathcal{K}$. We have $\omega_\psi(\delta)K(\eta) = K(\eta)\omega_\psi(\delta) \quad \forall\ \delta \in D$, and thus by the Schur lemma, 2.1, $K(\eta) = c \cdot I$. Since ${}^{\tau}\eta = \eta$ we get ${}^{\tau}K(\eta) = {}^{t}K(\eta) = {}^{t}(cI) = cI = K(\eta)$.

8) Take $K \in \mathcal{K}$. Since ${}^{\tau}K(\eta) = K(\eta)$ for every representative of $Q\backslash G/Q$ by 7), it is true for every element of G thus ${}^{\tau}K = K$. Thus, the involution τ on $\mathcal{K}$ is the identity, therefore, $\mathcal{K}$ is commutative. $\square$

References

[1] Gelfand, I. M.; Graev, M. I: *Construction of irreducible representations of simple algebraic groups over finite field.* Acad. Sci. USSR **147** (1962), no. 3, 529. Russian.

[2] Gerardin, P.: *Weil's representations associated to finite fields.* J. Algebra **46** (1977), 54–101.

[3] Piatetski-Shapiro, I. I.: *Euler Subgroups in Lie Groups and their Representations.* (I. M. Gelfand ed.) John Wiley, New York-Toronto 1971, 597–620.

[4] Piatetski-Shapiro, I. I.: *Multiplicity one theorems.* Proc. Symp. Pure Math. **33** (1979), 209–212.

[5] Shalika, J. A.: *The multiplicity one theorem for GL(n).* Ann. Math. **100** (1974), 171–193.

[6] Weil, A.: *Sur certain groupe d'opérateurs unitaires.* Acta Math. (Upsala) **11** (1964), no. 1–4, 143–211.

Received January 3, 1996