Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 38 (1997), No. 1, 91-93.

Cutting Circles and the Morley Theorem

Ludwig Stammler
Fachbereich Mathematik und Informatik
Martin-Luther-Universität Halle-Wittenberg
Theodor-Lieser-Str. 5, D-06120 Halle

Abstract. The midpoints of three 1-cutting circles form a triangle with sides parallel to that of the Morley triangle.

According to [2] a circle k is called 1-*cutting circle* ("1-Schnittkreis") of a triangle ABC iff the three following statements hold: The circle k cuts out of the lines g(BC), g(CA), g(AB) a chord of length $|BC|, |CA|, |AB|$ respectively. In [2] was proved:

For each triangle ABC exist, additionally to the circumscribed circle c, three and only three 1*-cutting circles. Their midpoints U, V, W are the corners of an equilateral triangle UVW, which is circumscribed about c.*

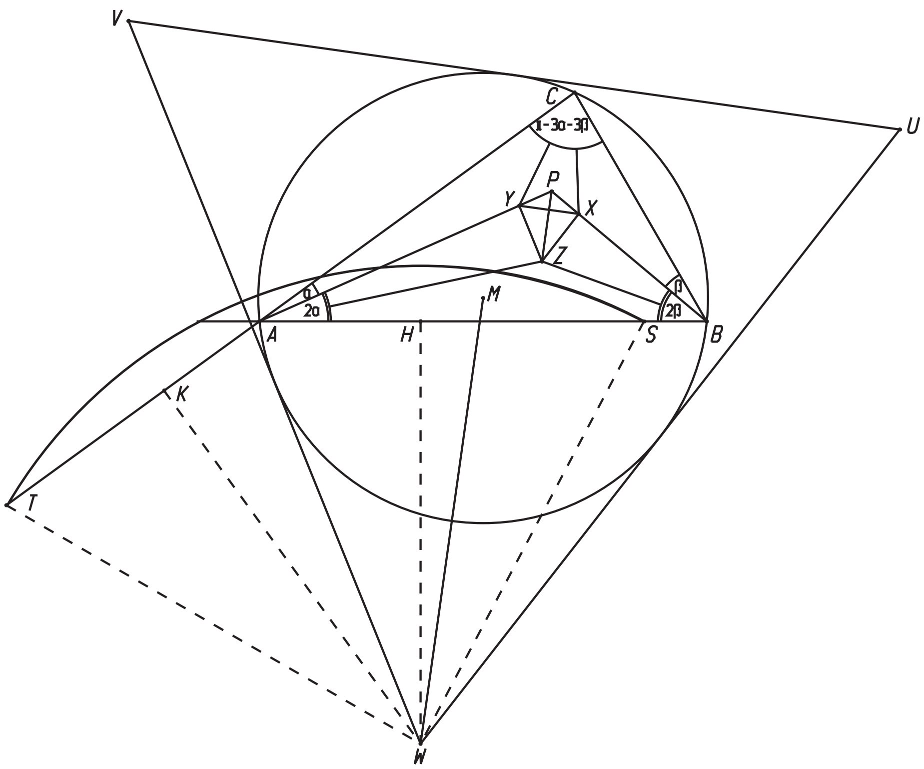
Further in [2] was stated: In general for a given triangle ABC the points U, V, W are not constructable, using lines and circles only. But they are obtainable, if one permits to solve cubic equations, e.g. to construct angle trisecting rays, too. Therefore the theorem mentioned above was called a "pendant" to the well-known theorem of Morley. The question whether there is a geometric construction (including trisections) which is practicable, has not been answered yet.

A positive answer is given by the following theorem now. It shows more than a pure analogy, namely a genuine connection between the triangle UVW and the Morley triangle XYZ $\big[$defined as known: Let y_a, z_a be the trisecting rays of $\angle A$ (where y_a is nearer to AC than z_a), analogously z_b, x_b the trisecting rays of $\angle B$, and x_c, y_c those of $\angle C$; then let $X := x_b \cap x_c, Y := y_c \cap y_a, Z := z_a \cap z_b\big]$.

Theorem. *For each triangle ABC the triangle UVW of the midpoints of* 1*-cutting circles and the Morley triangle XYZ have pairwise parallel sides with equal orientation.*

Proof. Let M be the midpoint and r the radius of the circle c circumscribed about ABC. Using a symmetry argument, it is sufficient to verify the following statement: If MW is the segment anti-orientated parallel to the segment from Z perpendicular to XY, and having the length $|MW| = 2r$, then W is the midpoint of a 1-cutting circle. Once more by symmetry, it is sufficient to prove: If k is the circle which has the midpoint W as defined

0138-4821/93 \$ 2.50



now, and cuts out of g(AB) a chord of length $|AB|$, then the same circle k is cutting out of g(AC) a chord of length $|AC|$. This may be proved as follows:
Let $\alpha := \frac{1}{3}|\angle A|, \beta := \frac{1}{3}|\angle B|$, further let $H \in$ g$(AB), S \in$ g$(AB), K \in$ g$(AC), T \in$ g(AC) with $WH \perp AB, |HS| = \frac{1}{2}|AB|, WK \perp AC, |KT| = \frac{1}{2}|AC|$, and let $P :=$ g$(AY) \cap$ g(BX). Then Z is the intersection point of the angle bisectors at the triangle ABP. Further as a consequence of the proof for the theorem of Morley, given in [1], $|XP| = |YP|$ holds, this implies $PZ \perp XY$, e.g.: By definition of W, the segments PZ and MW are equi-orientated parallel to each other.

Using now, for instance, a coordinate system in which $A = (-1; 0), B = (1; 0)$ hold, and C has a positive ordinate, it can be deduced successively: From $|\angle C| = \pi - 3\alpha - 3\beta$ and the fact that M is the midpoint of c follows

$$M = (0; -\cot(3\alpha + 3\beta), \quad r = \frac{1}{\sin(3\alpha + 3\beta)}.$$

From $\angle APZ = \frac{1}{2}|\angle APB| = \frac{\pi}{2} - \alpha - \beta, |\angle BAP| = 2\alpha$, and $MW \parallel PZ$ follows: g(WM) has the inclination angle $\frac{\pi}{2} + \alpha - \beta$. From this and $|MW| = 2r$, in $W = (\xi; \eta)$ are

$$\xi = \frac{-2\sin(\beta - \alpha)}{\sin(3\alpha + 3\beta}, \quad \eta = \frac{-\cos(3\alpha + 3\beta) - 2\cos(\beta - \alpha)}{\sin(3\alpha + 3\beta)}.$$

An equation of g(AC) is $(x + 1) \cdot \sin(3\alpha) - y \cdot \cos(3\alpha) = 0$; hence, by definition of K,

$$l := |WK| = \big|(\xi + 1) \cdot \sin(3\alpha) - \eta \cdot \cos(3\alpha)\big|.$$

In the triangle ABC, from $|AB| = 2$ and $|\angle B| = 3\beta, |\angle C| = \pi - 3\alpha - 3\beta$ follows

$$b := |AC| = \frac{2\sin(3\beta)}{\sin(3\alpha + 3\beta)}.$$

So, by definition of S and T, and with the values found here, we obtain the length

$$|WS| = \sqrt{1 + \eta^2}, \quad |WT| = \sqrt{l^2 + \left(\frac{b}{2}\right)^2},$$

and it remains to be verified $|WS| = |WT|$, e.g. the identity

$\sin^2(3\alpha + 3\beta) + \big(\cos(3\alpha + 3\beta) + 2\cos(\beta - \alpha)\big)^2 =$

$\Big(\big((\sin(3\alpha + 3\beta) - 2\sin(\beta - \alpha)\big)\sin(3\alpha) + \big(\cos(3\alpha + 3\beta) + 2\cos(\beta - \alpha)\big)\cos(3\alpha)\Big)^2 + \sin^2(3\beta).$

This trigonometrical calculation may be done, and the theorem is proved.

References

[1] Coxeter, H.S.M.: *Unvergängliche Geometrie.* Birkhäuser Verlag Basel und Stuttgart 1993.

[2] Stammler, L.: *Dreiecks-Proportionalschnittkreise, ihre Mittenhyperbel und ein Pendant zum Satz von Morley.* Elem. Math. **47** (1992), 158–168.

Received December 12, 1995