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On Areas and Integral Geometry in Minkowski Spaces

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Abstract. In a recent paper with J. A. Wieacker it was shown that certain integral-geometric formulas of Crofton type for lower-dimensional areas carry over from Euclidean to Minkowski spaces, if the Holmes-Thompson notion of Minkowskian area is employed and the Minkowski spaces are of a special type, called hypermetric. Here we show first that there exist Minkowski spaces for which, among all axiomatically defined Minkowskian areas, only the Holmes-Thompson area allows such generalizations. Then we extend further results from Euclidean translative integral geometry to Minkowski spaces.

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1. Introduction and results

In the following, we continue the investigation of the integral geometry of areas in Minkowski spaces begun in [13]. In that paper, some integral-geometric intersection formulas involving lower-dimensional areas were extended from Euclidean space to certain Minkowski spaces. One of these Euclidean results is the Crofton formula

$$\int_{\mathcal{E}_j^n} \lambda_{k+j-n}(M \cap E)\, d\mu_j(E) = a_{nkj} \lambda_k(M), \tag{1}$$

where the involved notions are defined as follows. We work in the Euclidean space $\mathbb{R}^n$ ($n \geq 3$), with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. By $\mathcal{E}_j^n$ we denote the space of j-dimensional flats (affine subspaces) in $\mathbb{R}^n$, and μ_j is the (suitably normalized) rigid motion invariant Borel measure on $\mathcal{E}_j^n$. Further, λ_k is the k-dimensional differential-geometric surface area and also denotes the k-dimensional Lebesgue measure for measurable sets in k-flats (λ_0 is the number of elements). With these notations, formula (1) holds for k-dimensional compact submanifolds M of class C^1 (this can be generalized) in $\mathbb{R}^n$ and for $k, j \in \{1, \dots, n-1\}$ with $k+j \geq n$. The constant a_{njk} depends on the normalization of μ_j. Proofs of (1) can be found in Santaló [8], p. 245, or Sulanke and Wintgen [14], p. 252.

By an n-dimensional Minkowski space we understand a pair $(\mathbb{R}^n, \|\cdot\|_B)$, where $\|\cdot\|_B$ is a norm on $\mathbb{R}^n$. Specifying this norm is equivalent to specifying its unit ball,

$$B = \{x \in \mathbb{R}^n : \|x\|_B \leq 1\},$$

which is a convex body with interior points and with the origin as centre of symmetry. When studying such a Minkowski space, we retain also the given Euclidean metric as a convenient tool; all the Minkowskian notions to be considered are independent of the choice of this Euclidean metric (cf. [13], Section 9). The notion of k-dimensional area in a Minkowski space that we adopt here is the one due to Holmes and Thompson. To define it, let

$$B^o := \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } y \in B\}$$

be the polar body of the Minkowskian unit ball B (thus we use the given scalar product to identify $\mathbb{R}^n$ with its dual space). For $E \in \mathcal{L}_k^n$, the space of k-dimensional linear subspaces of $\mathbb{R}^n$, we denote by $M|E$ the image of a set M under orthogonal projection onto E. The function on $\mathcal{L}_k^n$ defined by

$$\sigma_k(E) := \frac{1}{\kappa_k}\lambda_k(B^o|E), \qquad E \in \mathcal{L}_k^n, \tag{2}$$

where κ_k is the k-dimensional volume of the k-dimensional (Euclidean) unit ball, is the *scaling function* of the *Holmes-Thompson k-area* τ_k. This area is defined by

$$\tau_k(A) := \sigma_k(E_0)\lambda_k(A) \qquad \text{for } A \in \mathcal{B}(E),\ E \in \mathcal{E}_k^n.$$

Here $\mathcal{B}$(E) denotes the σ-algebra of Borel sets in E, and $E_0 = E - E$ is the linear subspace parallel to the k-flat E. More generally, the Holmes-Thompson k-area of a compact C^1 submanifold M of dimension k can be defined by

$$\tau_k(M) = \int_M \sigma_k(T_x M)\, d\lambda_k(x), \tag{3}$$

where $T_x M$ denotes the tangent space of M at x, canonically identified with a linear subspace of $\mathbb{R}^n$. We refer to Thompson [15] for a thorough discussion of areas in Minkowski spaces.

The question then arises whether, in a Minkowski space, formula (1) holds in the form

$$\int_{\mathcal{E}_j^n} \tau_{k+j-n}(M \cap E)\, d\varphi_j(E) = a_{nkj}\tau_k(M), \tag{4}$$

with a suitable measure φ_j on $\mathcal{E}_j^n$, called a *Crofton measure* in [13]. In the Euclidean case, the measure μ_j is invariant under rigid motions, that is, Euclidean isometries. We require, therefore, that the measure φ_j be invariant under Minkowskian isometries, in particular, under translations.

The Holmes-Thompson area is only one in a variety of possible notions of area in Minkowski spaces. In Section 2 of the present paper we strengthen a statement made in [13], according to which the Holmes-Thompson area is particularly suited to the needs of integral geometry. After recalling the usual set of axioms, (M1) – (M4), for a Minkowskian $(n-1)$-area and defining an associated Crofton measure, which assures the validity of a basic integral-geometric formula for area, we prove the following.

Theorem 1. *There exist Minkowski spaces for which, among all Minkowskian $(n-1)$-areas satisfying axioms* (M1) – (M4) *below, only the Holmes-Thompson area (up to a factor) admits an associated Crofton measure.*

In other words, if we want a formula of type (4) for $k = n-1$ and $j = 1$ to hold for all Minkowski spaces, we should work with the Holmes-Thompson area.

It turns out that for other values of k and j formula (4) can be achieved only in a special class of Minkowski spaces. The Minkowski space $(\mathbb{R}^n, \|\cdot\|_B)$ is called *hypermetric* if the dual unit ball B^o is a zonoid. (For zonoids, see the survey by Goodey and Weil [7].) The following was proved in [13].

Theorem 2. *A translation invariant measure φ_{n-1} on $\mathcal{E}^n_{n-1}$ satisfying* (4) *for $k = 1$ and $j = n-1$ exists only if $(\mathbb{R}^n, \|\cdot\|_B)$ is hypermetric. Vice versa, if $(\mathbb{R}^n, \|\cdot\|_B)$ is hypermetric, then there exist translation invariant measures $\varphi_1, \dots, \varphi_{n-1}$ so that* (4) *holds for all $k, j \in \{1, \dots, n-1\}$ with $k + j \geq n$.*

In Section 3 of the present paper, we start from the following counterpart to formula (1) in Euclidean space. If M, M' are compact C^1 submanifolds of $\mathbb{R}^n$ of dimensions k and j, respectively, where $k + j \geq n$, then

$$\int_{G_n} \lambda_{k+j-n}(M \cap gM')\, d\mu(g) = b_{nkj} \lambda_k(M) \lambda_j(M'). \tag{5}$$

Here G_n is the group of rigid motions of $\mathbb{R}^n$ and μ is the Haar measure on G_n; b_{nkj} is a constant depending on the normalization of μ. We cannot hope for a Minkowskian analogue of (5) with G_n and μ replaced by the translation group and its invariant measure, since such a formula does not even hold in the Euclidean case. However, the following special case of (5) involves, in effect, only an integration over the translation group. If we take for M' the Euclidean unit sphere,

$$S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\},$$

then (5) reduces to

$$\int_{\mathbb{R}^n} \lambda_{k-1}(M \cap (S^{n-1} + t))\, d\lambda_n(t) = c_{nk} \lambda_k(M), \tag{6}$$

with a constant c_{nk}; here λ_n is Lebesgue measure on $\mathbb{R}^n$. It is now a reasonable question to ask if (6) holds in the Minkowski space $(\mathbb{R}^n, \|\cdot\|_B)$, with λ_{k-1}, λ_k replaced by the Holmes-Thompson areas τ_{k-1}, τ_k and with the sphere S^{n-1} replaced by the boundary ∂K of a suitable convex body K. Thus, we are asking for a convex body K and a constant $c_{nk}(K)$ so that

$$\int_{\mathbb{R}^n} \tau_{k-1}(M \cap (\partial K + t))\, d\lambda_n(t) = c_{nk}(K)\tau_k(M) \qquad \text{for } M \in \mathcal{S}_k, \tag{7}$$

where $\mathcal{S}_k$ is the set of all k-dimensional compact C^1 submanifolds of $\mathbb{R}^n$. In order that λ_n be a Minkowskian notion, we impose on $|\cdot|$ and B the normalization condition

$$\lambda_n(B^o) = \kappa_n. \tag{8}$$

In analogy to Theorem 2, the following result will be proved in Section 3. (Observe that τ_0 is the number of elements.)

Theorem 3. *A convex body K in $\mathbb{R}^n$ satisfying* (7) *for $k = 1$ exists only if $(\mathbb{R}^n, \|\cdot\|_B)$ is hypermetric. Vice versa, if $(\mathbb{R}^n, \|\cdot\|_B)$ is hypermetric, then there exists a convex body K so that* (7) *holds for all $k \in \{1, \dots, n-1\}$.*

For general convex bodies K, and in particular in a Minkowski space which is not hypermetric, intersection integrals of the type occurring in (7) will in general not have a simple expression. By "simple" we mean here that the result is a product of two functions, one depending only on M and the other only on K. In some cases, however, explicit expressions in terms of mixed volumes involving associated zonoids are possible. In Euclidean translative integral geometry, such results are due to Wieacker [16] (see also Section 6 of [12]). One of his theorems ([16], Theorem 4) was extended to Minkowski spaces in [13]. In Section 4 we obtain the following Minkowskian counterpart to another result of Wieacker (loc. cit., Theorem 5 (a)).

Theorem 4. *For $k \in \{1, \dots, n-1\}$, one can associate with each k-surface $M \in \mathcal{S}_k$ a zonoid $\Pi_k M$ so that, for every convex body $K \subset \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} \tau_{k-1}(M \cap (\partial K + t))\, d\lambda_n(t) = 2nV(\Pi_k M, K, \dots, K).$$

Here V denotes the mixed volume in $\mathbb{R}^n$. Due to the normalization (8), it does not depend on the choice of the Euclidean metric.

2. Minkowski areas and Crofton measures

In a Minkowski space there is only one natural notion of curve length and, up to a constant factor, only one natural notion of volume. For $k \in \{2, \dots, n-1\}$, however, it is not canonically clear how a k-dimensional area in an n-dimensional Minkowski space should be defined. Some requirements are plausible (see Busemann and Straus [2], Busemann [1], Thompson [15]). Naturally, any k-area in a Minkowski space should be invariant under Minkowskian isometries and thus, at least, under translations. If α_k is a Minkowskian k-area on $(\mathbb{R}^n, \|\cdot\|_B)$, one also requires that, for any k-dimensional subspace $E \in \mathcal{L}_k^n$, the restriction of α_k to the Borel sets in E is a non-zero, locally finite measure, hence it is a multiple of Lebesgue measure λ_k in E and, therefore, uniquely determined by the value $\alpha_k(B \cap E) =: \alpha$. Since α_k on E

should depend only on the Minkowskian metric in E, we assume that α depends only on $B \cap E$ and is the same for every linear transform of this set. Together with plausible continuity and normalization assumptions, these considerations lead to a set of axioms. For the rest of this section, we restrict ourselves to the case $k = n-1$.

Let $\mathcal{C}^{n-1}$ denote the set of all $(n-1)$-dimensional convex bodies in $\mathbb{R}^n$ with centre of symmetry at the origin. A *Minkowskian* $(n-1)$*-area* is defined by a function

$$\alpha : \mathcal{C}^{n-1} \to \mathbb{R}^+$$

satisfying the following properties (M1) – (M4).

(M1) α is invariant under linear transformations of $\mathbb{R}^n$,

(M2) α is continuous,

(M3) $\alpha(E_{n-1}) = \kappa_{n-1}$ if E_{n-1} is an $(n-1)$-dimensional ellipsoid.

The continuity in (M2) may be understood with respect to the Hausdorff metric induced by the auxiliary Euclidean metric (but compare Thompson [15] for the case where Minkowskian areas are defined for all dimensions $1, \ldots, n-1$ simultaneously).

If such a function α and a Minkowski space $(\mathbb{R}^n, \|\cdot\|_B)$ are given, then the induced Minkowski $(n-1)$-area α_B on $(\mathbb{R}^n, \|\cdot\|_B)$ is defined by

$$\alpha_B(M) := \frac{\alpha(B \cap E_0)}{\lambda_{n-1}(B \cap E_0)} \lambda_{n-1}(M) \qquad \text{for } M \in \mathcal{B}(E),\ E \in \mathcal{E}^n_{n-1},\ E_0 = E - E,$$

and more generally by

$$\alpha_B(M) = \int_M \frac{\alpha(B \cap T_x M)}{\lambda_{n-1}(B \cap T_x M)}\, d\lambda_{n-1}(x)$$

for compact C^1 submanifolds M. The result is independent of the choice of the auxiliary Euclidean metric.

It is convenient to define the *homogeneous scaling function* (depending on the Euclidean metric) by

$$\sigma_{\alpha,B}(u) := |u| \frac{\alpha(B \cap u^\perp)}{\lambda_{n-1}(B \cap u^\perp)} \qquad \text{for } u \in \mathbb{R}^n \setminus \{0\}. \tag{9}$$

Here $u^\perp$ denotes the hyperplane through 0 orthogonal to u. One usually imposes a fourth axiom:

(M4) $\sigma_{\alpha,B}$ is a convex function.

If this is satisfied, then $\sigma_{\alpha,B}$ is the support function of a convex body $\mathbf{I}_{\alpha,B}$, called the *isoperimetrix* of the pair (α, B). (See Thompson [15] for the motivation, in particular for the fact that the homothets of the isoperimetrix solve the isoperimetric problem. Usually the isoperimetrix is normalized in a different way, but in our context this is immaterial.)

Among the possible Minkowskian $(n-1)$-areas satisfying these axioms, the following two have attracted particular attention. *Busemann's area* is defined by the constant function

$$\alpha = \kappa_{n-1}.$$

The *Holmes-Thompson area* is defined by the function

$$\alpha(C) = \frac{1}{\kappa_{n-1}} vp(C) \qquad \text{for } C \in \mathcal{C}^{n-1}, \tag{10}$$

where vp denotes the volume product, that is,

$$vp(C) = \lambda_{n-1}(C)\lambda_{n-1}(C^o).$$

Here C^o denotes the polar body of C, taken in the linear hull of C. For the function defined by (10), the $(n-1)$-area α_B coincides with τ_{n-1} as defined in the introduction, since $(B \cap E)^o = B^o|E$. The isoperimetrix for the Busemann area is given by

$$\mathbf{I}_{\alpha,B} = \kappa_{n-1}(\mathrm{I}B)^o,$$

where I denotes the intersection body, and for the Holmes-Thompson area the isoperimetrix is

$$\mathbf{I}_{\alpha,B} = \frac{1}{\kappa_{n-1}} \Pi B^o,$$

where Π denotes the projection body (intersection and projection bodies are explained in [10] and, to a fuller extent, in [5]).

If one tries to extend the simplest integral-geometric formula for k-areas from Euclidean to Minkowski spaces, one is led to the following notion (see [13]). A *Crofton measure* associated with a Minkowskian k-area α_k on $(\mathbb{R}^n, \|\cdot\|_B)$ is a translation invariant, locally finite measure φ_{n-k} on $\mathcal{E}^n_{n-k}$ satisfying

$$\varphi_{n-k}(\{E \in \mathcal{E}^n_{n-k} : E \cap P \neq \emptyset\}) = \alpha_k(P)$$

for all k-dimensional parallelepipeds P. It is known (see [13], Section 3) that the Minkowskian $(n-1)$-area α_B admits an associated Crofton measure if and only if the corresponding isoperimetrix $\mathbf{I}_{\alpha,B}$ is a zonoid. Since projection bodies are zonoids, the Holmes-Thompson $(n-1)$-area always admits an associated Crofton measure. This fact, which was the basis for much of [13], is now complemented by Theorem 1.

Proof of Theorem 1. We shall show that the space ℓ^n_∞ has the property described in Theorem 1. This space can be considered as $(\mathbb{R}^n, \|\cdot\|_C)$, with C the unit cube with vertices $\pm e_1 \pm \ldots \pm e_n$, where $(e_1, \ldots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$ with respect to the auxiliary Euclidean structure. Let α_C be any Minkowskian $(n-1)$-area on ℓ^n_∞, given by a function α, and recall that the support function of the corresponding isoperimetrix $\mathbf{I}_{\alpha,C}$ is given by (9). Let $a > 0$ be such that $a\sigma_{\alpha,C}(e_1) = 1$ and write $a\sigma_{\alpha,C} =: \sigma$ and $a\mathbf{I}_{\alpha,C} =: \mathbf{I}$. Since $\mathbf{I}$ has the same (Euclidean) symmetries as C, the point e_1 is a point of $\mathbf{I}$. Now consider the vector $u := e_1 + e_2$. Since α is invariant under linear transformations and $C \cap u^\perp$ is linearly equivalent to $C \cap e_1^\perp$ and since

$$\lambda_{n-1}(C \cap e_1^\perp) = \lambda_{n-1}(C \cap u^\perp)\frac{\langle u, e_1\rangle}{|u|},$$

we see from (9) that

$$\sigma(u) = \langle u, e_1\rangle.$$

Hence, the support hyperplane to $\mathbf{I}$ with exterior normal vector u contains the point e_1. Similarly, the support hyperplane to $\mathbf{I}$ with normal vector $e_1 - e_2$ contains e_1.

Now suppose that the $(n-1)$-area α_C admits an associated Crofton measure. Then $\mathbf{I}$ is a zonoid, which means that

$$\sigma(u) = \int_{S^{n-1}} |\langle u, v\rangle| \, d\rho(v), \qquad u \in \mathbb{R}^n,$$

with an even measure ρ on the unit sphere S^{n-1}. As seen above, we have $\sigma(u) = \langle u, e_1\rangle$ for $u = e_1$ and for $u = e_1 \pm e_2$, thus

$$\sigma(2e_1) = \sigma(e_1 + e_2) + \sigma(e_1 - e_2). \tag{11}$$

On the other hand,

$$|\langle 2e_1, \cdot\rangle| \le |\langle e_1 + e_2, \cdot\rangle| + |\langle e_1 - e_2, \cdot\rangle|, \tag{12}$$

with strict inequality on the set

$$A_{1,2} := \{v \in S^{n-1} : |\langle e_1, v\rangle| < |\langle e_2, v\rangle|\}.$$

Integrating (12) with respect to ρ, we obtain an inequality, which must coincide with the equality (11), hence $\rho(A_{1,2}) = 0$. Since $\mathbf{I}$ has at least the same symmetries as C and the even measure ρ is uniquely determined, ρ vanishes on the images of $A_{1,2}$ under the symmetry group of C. Each vector $u \in S^{n-1}$ which is not one of the vectors

$$\frac{\pm e_1 \pm \ldots \pm e_n}{\| \pm e_1 \pm \ldots \pm e_n\|}, \tag{13}$$

has two coordinates (with respect to $e_1, \ldots, e_n$) with different absolute values, say $|\langle u, e_1\rangle| < |\langle u, e_2\rangle|$, without loss of generality. Thus $u \in A_{1,2}$, therefore u is not an element of the support of the measure ρ. It follows that the support of ρ is contained in the finite set given by (13). By symmetry, ρ assigns the same mass to each of these points. The vectors (13) are precisely the exterior unit normal vectors of the facets of the crosspolytope C^o. We conclude that $\mathbf{I}$ is a multiple of the projection body ΠC^o. This proves that the $(n-1)$-area α_C is a multiple of the Holmes-Thompson area, which completes the proof of Theorem 1.

It is of interest to observe that also the space ℓ_1^n has the property of Theorem 1. The space ℓ_1^n can be identified with $(\mathbb{R}^n, \|\cdot\|_Q)$, where Q is the crosspolytope with vertices $\pm e_1, \ldots, \pm e_n$. Let α be a function satisfying (M1) – (M4) so that the induced $(n-1)$-area α_Q admits an associated Crofton measure and thus the isoperimetrix $\mathbf{I}_{\alpha,Q}$ is a zonoid. Choose $a > 0$ so that $a\sigma_{\alpha,Q}(e_1) = 1$ and put $a\sigma_{\alpha,Q} = \sigma$ and $a\mathbf{I}_{\alpha,Q} = \mathbf{I}$. Since $\mathbf{I}$ has at least the same symmetries as Q, it must be contained in the unit cube C. Let $v = \alpha e_1 + \beta e_2$ with $\alpha, \beta \ge 0$ and $\alpha + \beta > 0$. All sections $Q \cap v^\perp$ are linearly equivalent, hence

$$\sigma(v) = |v| \frac{\lambda_{n-1}(Q \cap e_1^\perp)}{\lambda_{n-1}(Q \cap v^\perp)} = \langle v, e_1 + e_2\rangle.$$

Since $|\langle e_1 + e_2, v\rangle| < |\langle e_1, v\rangle| + |\langle e_2, v\rangle|$ if $\langle e_1, v\rangle\langle e_2, v\rangle < 0$, but $\sigma(e_1 + e_2) = \sigma(e_1) + \sigma(e_2)$, we conclude as above that the generating measure ρ of the zonoid $\mathbf{I}$ vanishes on the set

$$\{v \in S^{n-1} : \langle e_1, v\rangle\langle e_2, v\rangle < 0\}$$

and on the images of this set under the symmetry group of Q. Therefore, ρ is concentrated in the points $\pm e_1, \ldots, \pm e_n$. By symmetry, $\mathbf{I}$ is a multiple of the cube C and hence a multiple of ΠQ^o. Thus the $(n-1)$-area α_Q is again a multiple of the Holmes-Thompson area.

We conclude this section with a few comments. In the above considerations, besides the cube C and the crosspolytope Q, the following polytopes implicitly played a role. Let $H_{1,2}$ be the supporting halfspace of $\{e_1\}$ with exterior normal vector $e_1 + e_2$, let R be the intersection of all images of $H_{1,2}$ under the symmetries of C. Further, let P be the convex hull of the points $\pm e_i \pm e_j$, $i \neq j$. Then we have shown that a suitable multiple, $\mathbf{I}$, of the isoperimetrix of any Minkowskian $(n-1)$-area satisfies $Q \subset \mathbf{I} \subset R$ on the space $(\mathbb{R}^n, \|\cdot\|_C)$, and $P \subset \mathbf{I} \subset C$ on the space $(\mathbb{R}^n, \|\cdot\|_Q)$. For $n = 3$, where R is a rhombic dodecahedron and P is a cuboctahedron, this was pointed out by Thompson [15], Examples 5.3.5 (2), (3). We have shown in addition that a zonoid Z satisfying $Q \subset Z \subset R$ is uniquely determined, coinciding with R for $n = 3$ (but not for $n > 3$), and that the only zonoid Z with $P \subset Z \subset C$ is C itself. Closer inspection of the first proof shows that it can be extended to some polytopes different from the cube. Besides symmetry properties, the proof used the fact that every boundary point of the polytope different from a vertex lies in a suitable open cylindrical region of the boundary. A three-dimensional example for which the proof carries over is given by the truncated octahedron, whose boundary consists of regular hexagons and squares. Further examples are the zonoids (4,6,8) and (4,6,10) on p. 109 of [4].

The above examples show, in particular, that for the Busemann $(n-1)$-area there does not always exist an associated Crofton measure. This was conjectured in [13], Section 4, in the following equivalent form. There are Minkowski unit balls B for which $(\mathrm{I}B)^o$, the polar of the intersection body, is not a zonoid. As we have seen, the crosspolytope and the cube are such examples. On the other hand, it would be rash to conjecture that $(\mathrm{I}B)^o$ is a zonoid only if B is an ellipsoid. The techniques of [9] should show (we did not check the details) that the polar intersection body remains a zonoid under small and sufficiently smooth (convexity preserving) perturbations of the ellipsoid.

3. Proof of Theorem 3

We assume, first, that in a Minkowski space $(\mathbb{R}^n, \|\cdot\|_B)$ with unit ball B formula (7) holds for $k = 1$, for some convex body K (always with interior points) and for all closed segments M. For $x \in \mathbb{R}^n \setminus \{0\}$ and the segment $[0, x]$ joining 0 and x we thus have

$$I(x) := \int_{\mathbb{R}^n} \mathrm{card}([0,x] \cap (\partial K + t))\, d\lambda_n(t) = c_{n1}(K)\|x\|_B. \tag{14}$$

It is easy to see that

$$I(x) = 2|x|\lambda_{n-1}(K|x^\perp).$$

Now

$$\lambda_{n-1}(K|x^\perp) = \frac{1}{2|x|} \int_{S^{n-1}} |\langle x, v\rangle|\, dS_{n-1}(K, v),$$

where $S_{n-1}(K,\cdot)$ denotes the area measure of K (see, e.g., [10], in particular formula (5.3.32)). Since

$$\|x\|_B = h(B^o, x),$$

where h denotes the support function, equality (14) leads to

$$h(B^o, x) = c_{n1}^{-1}(K) \int_{S^{n-1}} |\langle x, v\rangle| \, dS_{n-1}(K, v) \qquad \text{for } x \in \mathbb{R}^n.$$

In other words, B^o is a multiple of the projection body, ΠK, of K (see, e.g., [10], p. 296). Thus B^o is a zonoid, which means that our Minkowski space is hypermetric. Vice versa, if $B^o = \Pi K$, then we see that

$$\|x\|_B = \frac{1}{2} \int_{\mathbb{R}^n} \operatorname{card}([0,x] \cap (\partial K + t)) \, d\lambda_n(t) \tag{15}$$

for $x \in \mathbb{R}^n$. This equality is due to El-Ekhtiar [3] and appears also as Theorem 6.4.8 in Thompson [15].

Formula (15) is a special case of (7). To prove the general case for a hypermetric Minkowski space, we first establish a more general formula. The proof makes use of some properties of mixed volumes, for which we refer to [10]. We state one of these results (which follows from [10], (5.3.23)) as a lemma. For notation not explained, we also refer to [10].

Lemma. *Let $m \in \{1, \dots, n-1\}$ and $E_m \in \mathcal{L}_m^n$. If $K \subset \mathbb{R}^n$ and $U \subset E_m^\perp$ are convex bodies, then*

$$\binom{n}{m} V(K[m], U[n-m]) = \lambda_m(K|E_m)\lambda_{n-m}(U).$$

By $\mathcal{H}^k$ we denote the k-dimensional Hausdorff measure, based on the Euclidean metric used on $\mathbb{R}^n$. For the notion of surface used in the integral-geometric formulas to be considered, we do not aim here at the greatest possible generality (see Wieacker [16] for more general versions). A subset $M \subset \mathbb{R}^n$ is *k-rectifiable* if either $k = 0$ and M is finite, or $k \in \{1, \dots, n-1\}$ and M is the image of some bounded subset of $\mathbb{R}^k$ under a Lipschitz map. In the following, a k-rectifiable and $\mathcal{H}^k$-measurable subset of $\mathbb{R}^n$ will briefly be called a *k-surface*. Examples are compact C^1 submanifolds of dimension k and, for $k = n-1$, the boundaries of convex bodies. For a k-surface M, the approximate tangent space $\operatorname{Tan}^k(\mathcal{H}^k \llcorner M, x)$ at x is for $\mathcal{H}^k$-almost all $x \in M$ a k-dimensional vector subspace of $\mathbb{R}^n$; we denote it by $T_x M$. (Clearly, these notions do not depend on the choice of the Euclidean metric.)

For an $(n-1)$-surface (or hypersurface) H, the *projection body* ΠH is defined as follows (see [13], §6, for references). Its support function is given by

$$h(\Pi H, x) := \frac{1}{2} \int_{\mathbb{R}^n} \operatorname{card}([0,x] \cap (H + t)) \, d\lambda_n(t), \qquad x \in \mathbb{R}^n;$$

it can also be represented in the form

$$h(\Pi H, x) = \frac{1}{2} \int_H |\langle x, u(y)\rangle| \, d\mathcal{H}^{n-1}(y), \qquad x \in \mathbb{R}^n,$$

where $u(y)$ is a unit normal vector to H at y.

Now we assume that H is an $(n-1)$-surface and M is a k-surface, where $k \in \{1, \dots, n-1\}$. Then

$$\begin{aligned} I : &= \int_{\mathbb{R}^n} \tau_{k-1}(M \cap (H+t))\, d\lambda_n(t) \\ &= \int_{\mathbb{R}^n} \int_{D_t} \sigma_{k-1}(T_x D_t)\, d\mathcal{H}^{k-1}(x)\, d\lambda_n(t) \end{aligned} \tag{16}$$

with

$$D_t := M \cap (H+t),$$

by the definition of τ_{k-1}, namely (3), extended to $(k-1)$-surfaces. According to Lemma 6.1 of [13], this can be written in the form

$$I = \int_M \int_H \sigma_{k-1}(T_x H \cap T_y M)[T_x^\perp H, T_y^\perp M]\, d\mathcal{H}^{n-1}(x)\, d\mathcal{H}^k(y). \tag{17}$$

(The meaning of the bracket $[\cdot,\cdot]$ is explained in [13], and also by (18) below.) For $\mathcal{H}^{n-1}$-almost all $x \in H$, the tangent space $T_x H$ is a unique $(n-1)$-dimensional linear subspace, and we denote by $S(x)$ the segment of unit (Euclidean) length orthogonal to it and with centre at the origin. Similarly, for $\mathcal{H}^k$-almost all $y \in M$, the tangent space $T_y M$ is unique and of dimension k, and we let $P(y)$ be the $(n-k)$-dimensional Euclidean ball in $T_y^\perp M$ with centre 0 and $\lambda_{n-k}(P(y)) = 1$. Then

$$[T_x^\perp H, T_y^\perp M] = \lambda_{n-k+1}(S(x) + P(y)), \tag{18}$$

and from (2) and the Lemma we get

$$\begin{aligned} & \sigma_{k-1}(T_x H \cap T_y M)[T_x^\perp H, T_y^\perp M] \\ = \;& \kappa_{k-1}^{-1} \lambda_{k-1}(B^o | T_x H \cap T_y M)\, \lambda_{n-k+1}(S(x) + P(y)) \\ = \;& \kappa_{k-1}^{-1} \binom{n}{k-1} V(B^o[k-1], (S(x)+P(y))[n-k+1]) \\ = \;& a_{nk} V(S(x), P(y)[n-k], B^o[k-1]) \end{aligned} \tag{19}$$

with

$$a_{nk} := \kappa_{k-1}^{-1} \binom{n}{k-1} (n-k+1).$$

In the last step, we used the multilinearity properties of the mixed volume and the fact that $V(S(x), S(x), \dots) = 0$, since $S(x)$ is a line segment.

We now use the representation of mixed volumes in terms of support functions and mixed area measures ([10], (5.1.18)) together with Fubini's theorem to obtain (denoting the endpoints of $S(x)$ by $\pm\frac{1}{2}u(x)$)

$$\int_H V(S(x), P(y)[n-k], B^o[k-1])\, d\mathcal{H}^{n-1}(x)$$

$$\begin{aligned}
&= \frac{1}{n}\int_H \int_{S^{n-1}} h(S(x),v)\,dS(P(y)[n-k],B^o[k-1],v)\,d\mathcal{H}^{n-1}(x) \\
&= \frac{1}{n}\int_{S^{n-1}} \frac{1}{2}\int_H |\langle u(x),v\rangle|\,d\mathcal{H}^{n-1}(x)\,dS(P(y)[n-k],B^o[k-1],v) \qquad (20)\\
&= \frac{1}{n}\int_{S^{n-1}} h(\Pi H,v)\,dS(P(y)[n-k],B^o[k-1],v) \\
&= V(\Pi H,P(y)[n-k],B^o[k-1]). \qquad (21)
\end{aligned}$$

The mixed volume (21) is a measurable function of y, hence we obtain from (16), (17), (19), (21)

$$\int_{\mathbb{R}^n} \tau_{k-1}(M\cap(H+t))\,d\lambda_n(t) = a_{nk}\int_M V(\Pi H,P(y)[n-k],B^o[k-1])\,d\mathcal{H}^k(y). \qquad (22)$$

This general formula now simplifies if we introduce the special assumption that $\Pi H = B^o$. Since

$$\binom{n}{k} V(B^o[k],P(y)[n-k]) = \lambda_k(B^o|T_yM)$$

by the Lemma, we then arrive at

$$\int_{\mathbb{R}^n} \tau_{k-1}(M\cap(H+t))\,d\lambda_n(t) = \frac{k\kappa_n}{\kappa_{k-1}}\tau_k(M). \qquad (23)$$

The assumption $\Pi H = B^o$ implies, in particular, that B^o is a zonoid and thus $(\mathbb{R}^n,\|\cdot\|_B)$ is hypermetric. If, vice versa, B^o is a zonoid, there exists a centrally symmetric convex body K such that $B^o = \Pi K$, where $\Pi K = \Pi\partial K$ is the ordinary projection body of K. Thus (23) is satisfied with $H = \partial K$. This completes the proof of Theorem 3.

If we assume only that ΠK is homothetic to B^o, we obtain formula (7) with a factor $c_{nk}(K)$ depending on K.

In general, the projection body ΠK does not determine K up to translation, so if B^o is a zonoid, there will also be convex bodies K without a centre of symmetry for which (7) holds. The only exception where the equality $\Pi K = B^o$ determines K up to translation is the case where B^o is a parallelepiped (Gardner and Volčič [6]), that is, the case where $(\mathbb{R}^n,\|\cdot\|_B)$ is isometric to ℓ_1^n.

4. A general translative intersection formula

In this section, we want to obtain a formula for the translative intersection integral appearing in (7), namely

$$\int_{\mathbb{R}^n} \tau_{k-1}(M\cap(\partial K+t))\,d\lambda_n(t) \qquad (24)$$

for a k-surface M and a convex body K, without the assumption that the underlying Minkowski space is hypermetric. It is clear from the Euclidean case treated by Wieacker

[16], Theorem 5, that the best we can hope for is expressing (24) as a mixed volume involving K and an auxiliary zonoid associated with M. In Minkowski space, such an associated zonoid, denoted by $\Pi_k M$, can be defined by its support function given by

$$h(\Pi_k M, u) := \frac{1}{2}\int_{\mathbb{R}} \tau_{k-1}(M \cap (u^\perp + \alpha u))\, d\alpha \qquad \text{for } u \in S^{n-1} \tag{25}$$

and homogeneous extension. Thus $\Pi_k M$ extends Wieacker's body Φ_M^k (see Wieacker [16], p. 236, and [17], p. 77) from Euclidean to Minkowski spaces. The Minkowskian nature of $\Pi_k M$ becomes clear from the following equivalent definition not using a Euclidean metric. We denote $\mathbb{R}^n$ by X and its dual space by X^* and then define the support function of $\Pi_k M$, as a function on X^*, by

$$h(\Pi_k M, x^*) = \frac{1}{2}\int_{\mathbb{R}} \tau_{k-1}(M \cap \{x \in X : x^*(x) = \alpha\})\, d\alpha, \qquad x^* \in X^*.$$

Thus $\Pi_k M$ is a body in X, whereas the projection body ΠH considered in §3 is a body in the dual space X^* (see also [13], Section 9).

To see that (25) yields, in fact, the support function of a zonoid, we use the Euclidean metric again. For given $u \in S^{n-1}$ we choose a compact set $H \subset u^\perp$ with $\lambda_{n-1}(H) = 1$. By Fubini's theorem we can write

$$\int_{\mathbb{R}^n} \tau_{k-1}(M \cap (H + t))\, d\lambda_n(t) = \int_{\mathbb{R}} \int_{u^\perp} \tau_{k-1}(M \cap (H + z + \alpha u))\, d\lambda_{n-1}(z)\, d\lambda_1(\alpha).$$

For λ_1-almost all α, the set $M \cap (u^\perp + \alpha u) := M_\alpha$ is a $(k-1)$-surface, and from [11], Satz 1.2.7 (applied to $\tau_{k-1} \llcorner M_\alpha$, the restriction of τ_{k-1} to the subsets of M_α, and to λ_{n-1}, in $u^\perp + \alpha u$) we have

$$\int_{u^\perp} \tau_{k-1}(M \cap (H + z + \alpha u))\, d\lambda_{n-1}(z) = \tau_{k-1}(M \cap (u^\perp + \alpha u)).$$

Thus from (22) we get

$$\begin{aligned} h(\Pi_k M, u) &= \frac{1}{2}\int_{\mathbb{R}^n} \tau_{k-1}(M \cap (H + t))\, d\lambda_n(t) \\ &= \frac{a_{nk}}{2}\int_M V(\mathrm{conv}\,\{-u/2, u/2\}, P(y)[n-k], B^o[k-1])\, d\mathcal{H}^k(y). \end{aligned} \tag{26}$$

If in (19) we take $H = u^\perp$ (and $x \in H$ arbitrary), we see that

$$\begin{aligned} & a_{nk} V(\mathrm{conv}\,\{-u/2, u/2\}, P(y)[n-k], B^o[k-1]) \\ =\; & \kappa_{k-1}^{-1} \lambda_{k-1}(B^o | u^\perp \cap T_y M)[u^\perp, T_y^\perp M] \\ =\; & \kappa_{k-1}^{-1} h(\Pi'(B^o | T_y M), u). \end{aligned}$$

Here $\Pi'(B^o|T_y M)$ is the ordinary projection body of $B^o|T_y M$, taken in the k-dimensional vector space $T_y M$. Thus we have

$$h(\Pi_k M, u) = \frac{1}{2\kappa_{k-1}} \int_M h(\Pi'(B^o | T_y M), u)\, d\mathcal{H}^k(y). \tag{27}$$

This shows that $\Pi_k M$ is a zonoid, since each body $\Pi'(B^o|T_y M)$ is a zonoid. It also shows how the representation of $h(\Phi^k_M, \cdot)$ given in Wieacker [16], Theorem 2(a), extends to Minkowski spaces. In the Euclidean case, we simply have

$$\frac{1}{\kappa_{k-1}}\Pi'(B^o|T_y M) = B^n \cap T_y M$$

($= B^y$ in Wieacker's notation), where $B^n = \{x \in \mathbb{R}^n : |x| \le 1\}$.

Now we prove Theorem 4 for a convex body $K \subset \mathbb{R}^n$ with interior points and a k-surface M, where $k \in \{1, \ldots, n-1\}$. For $\mathcal{H}^{n-1}$-almost all points $x \in \partial K$ there is a unique outer unit normal vector $u(x)$ to ∂K at x. We shall use the transformation formula

$$\int_{\partial K} f(u(x))\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} f(u)\, dS_{n-1}(K, u) \tag{28}$$

for a nonnegative Borel measurable function f; this follows from [10], (4.2.24). From (16), (17), (19), (26) we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \tau_{k-1}(M \cap (\partial K + t))\, d\lambda_n(t) \\
= \;& a_{nk} \int_M \int_{\partial K} V(S(x), P(y)[n-k], B^o[k-1])\, d\mathcal{H}^{n-1}(x)\, d\mathcal{H}^k(y) \\
= \;& a_{nk} \int_M \int_{S^{n-1}} V(\mathrm{conv}\,\{-u/2, u/2\}, P(y)[n-k], B^o[k-1])\, dS_{n-1}(K, u)\, d\mathcal{H}^k(y) \\
= \;& 2 \int_{S^{n-1}} h(\Pi_k M, u)\, dS_{n-1}(K, u) \\
= \;& 2nV(\Pi_k M, K, \ldots, K).
\end{aligned}$$

This completes the proof of Theorem 4.

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