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A Lower Estimate for the Number of Zero-Torsion Points of a Space Curve

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Abstract. We obtain an inequality regarding the numbers of zero-torsion points, zero curvature points, support triangles and the number of segments of a C^3-closed embedded space curve lying on the boundary of its convex hull. This generalizes the 4-vertex theorem for space curves.

Introduction

According to [5], any smooth generic closed curve in a 3-space satisfies

$$V + 2d = 4 + T, \tag{1}$$

where V represents the number of vertices (zero-torsion points) of the curve, T is the number of support triangles (defined by any 3 points of the curve lying on the same support plane) and d is the number of external segments (connected components of the intersection of the curve with the boundary of its convex hull which are diffeomorphic to a segment). If the curve is convex (lies on the boundary of its convex hull) then $d = 0$ and (1) implies the following 4-vertex theorem: any generic closed convex curve has at least four vertices.

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A more general version of the 4-vertex theorem including the whole set of C^3-embedded closed convex space curves with nonvanishing curvature was obtained in [8,9]. This was further generalized in [7] by proving the inequality

$$V + 2K \geq 4,$$

where K denotes the number of zero-curvature points of the considered curve.

It is natural to ask now whether we can get a formula of the kind of (1) relating the numbers of vertices and support triangles of nonnecessarily generic closed space curves. We give the answer in this paper by showing that any C^3-closed embedded space curve satisfies

$$V + 2K + 2d \geq 4 + T, \tag{2}$$

where T is the number of support triangles in which we can decompose the whole set of support polygons of the curve so that all vertices of these triangles belong to its external segments. In other words

$$T = \Sigma_{i=1}^{P}(n_i - 2),$$

where n_i is the number of points of external segments belonging to the i-th support polygon and P is the number of such polygons. The number V may actually be exchanged by the number of external vertices at which the osculating plane is a support plane of the curve, provided the total number of vertices is finite (for example, for generic curves).

Remark. In [3] it is shown (by different methods relying on the definitions of Maxwell graphs for non-necessarily generic closed space curves), that the inequality, $V + 2d \geq 4 + P$, holds for any C^4-closed space curve with nowhere vanishing curvature, no support bitangent lines and finite total order of contact with any plane. This is implied by (2) since $T \geq P$.

As a corollary of (2) we obtain a lower bound for the number of vertices (local extremum points of the curvature function) of any C^3-embedded closed plane curve whose curvature has only finitely many critical points. Namely, if C is the number of vertices at which the osculating circles and lines are support circles (lines) to the curve then

$$C \geq 4 + \Sigma_{i=1}^{Q}(n_i - 2), \tag{3}$$

where Q is the number of support circles and lines that touch the curve in at least 3 points and n_i is the number of tangency points of the curve with the i-th circle (line). A similar formula has been obtained by B. Wegner [11, Theorem 3.5] as a consequence of his study of the structure of the set of centres of support n-tangent circles, $n \geq 2$, of plane curves. It is also worth mentioning here that R.C. Bose ([2]) obtained, firstly, a similar result for the case of ovals, and this was later extended by O. Haupt ([4]) for a more general case of plane curves.

Remark. Generically $n_i = 3$, for all i. Moreover, the inequality (3) becomes an equality: $C = 4 + Q$. This was independently obtained in [10] and [6].

We must point out that, although working with the usual definitions of zeros of the curvature and torsion of a curve (which, of course, need of the inner product structure) the idea we explore here is that of higher order contacts of the curve with planes and lines in $\mathbb{R}^3$, which does not depend on the euclidean structure of the 3-space, but rather, on its affine structure. So we can say that the results we present here are of affine nature.

1. Statement of the Main Theorem

Consider any C^3-closed embedded space curve γ: $S^1 \to \mathbb{R}^3$. The intersection of all the half-spaces that contain γ is known as its *convex hull.* We say that such a curve is *convex* if it lies on the boundary of its convex hull.

If γ is not convex, the connected components of the intersection of γ with the boundary of its convex hull are diffeomorphic to a segment or to a point. The connected components which are diffeomorphic to a segment will be called *external segments* of γ. When γ is convex we shall understand that the number of its external segments is equal to zero.

Recall that a tangent plane to a curve γ is said to be a *support* plane provided that γ lies entirely in one of the closed half-spaces defined by this plane. A *vertex* (zero-torsion point) of γ is called *external* if the osculating plane to γ at this vertex is a support plane.

The convex hull of the intersection set of a given support plane with γ is said to be a *support polygon* provided it is an n-polygon, with $n \geq 3$. It is easy to see that the interior points of a support polygon cannot belong to any external segment of γ. The vertices of a support polygon must belong to external segments.

Suppose that P is the number of support polygons of a curve γ and T is the number of support triangles in which those polygons can be decomposed (so that all vertices of these triangles belong to external segments of γ).

Main Theorem. *Any C^3-embedded closed space curve with V external vertices, K zero-curvature points and d external segments satisfies the relation*

$$V + 2K + 2d \geq 4 + T,$$

provided the total number of vertices is finite. Moreover, if the curve has infinitely many support triangles, then it has infinitely many vertices, or zero-curvature points or external segments.

The proof of this theorem will be given in section 4. As immediate consequences we get the following

Corollary 1. *If a C^3-embedded closed space curve has finitely many vertices then*

$$V + 2K + 2d \geq 4 + P,$$

where K, V, P, d are the numbers of its zero-curvature points, external vertices, support polygons and external segments, respectively.

Corollary 2. *Any C^3-embedded closed convex space curve having only finitely many vertices satisfies*

$$V + 2K \geq 4 + T,$$

where V, K and T are as above.

Corollary 3 ("Hiding-twice theorem"). *Any C^3-embedded closed space curve with nowhere vanishing curvature and torsion enters the interior of its convex hull at least twice.*

2. Extremal points of a space curve

Consider a C^3-embedded closed curve γ in $\mathbb{R}^3$. Any nonempty open simply-connected subset of γ is called an *arc* of γ. A segment AB that connects two points A and B of γ and contains no other points of the curve is called a *chord* of γ. Both an arc and a chord are said to be *external* if they lie on the boundary Σ of the convex hull of γ.

Suppose that γ is *volumetric*, i.e., it does not lie in any plane (which is always the case when γ has finitely many vertices and zero-curvature points). Then the surface Σ is a topological 2-sphere (see [1]). Consider some fixed orientations of γ and Σ.

An external arc $\widetilde{AB}$ of the curve γ whose end-points A, B define the external chord AB is said to be *complete*. For such arc, the curve $AB \cup \widetilde{AB}$ separates the surface Σ onto two closed domains such that their union is equal to Σ and the intersection to $AB \cup \widetilde{AB}$. These domains are oriented and called *attached* to the arc $\widetilde{AB}$. The attached domain is said to be *left* (*right*) if it lies from the left (right) when we move along the arc $\widetilde{AB}$ according to the orientation of the curve γ.

Definition. *A point A is called left (right) extremal point of γ if there is a contracting system of external complete arcs such that the intersection of the left (right) domains attached to these arcs is equal to A.*

Lemma 1. *An extremal point of the curve γ is an external vertex or a zero-curvature point.*

Lemma 2. *If a given point is at the same time left and right extremal point of γ, then it is a zero-curvature point.*

The proof of these lemmas can be seen in [8]. The underneath idea is that an extremal point lies in the limit (support) plane of a family of support planes, defined by the system of arcs (a plane of the family passes through the end-points of a given arc).

Observe that given any external complete arc l one can define its maximal left (right) *continuation* as the external arc L containing l, belonging to the left (right) domain D attached to l and such that any arc L' strictly containing L does not belong to D.

Lemma 3. *Let l be an external complete arc of the curve γ. Suppose that γ has only finitely many vertices and zero-curvature points. Then either the maximal left (right) continuation of l contains a left (right) extremal point or the interior part of the left (right) domain attached to l contains an external vertex or a zero-curvature point of the curve γ.*

Proof. Denote by L the maximal left (right) continuation of the arc l. If $L = l$ then l contains a left (right) extremal point P according to lemma 2 in [8]. The proof of that lemma is based on the construction of a system of external chords collapsed to the point P and lying in the left (right) domain D attached to l.

One can construct such system of arcs when $L \neq l$ as well. But now, the point P is either a left (right) extremal point belonging to L or an interior point of the domain D (a part of the arc L lies in the interior part of D). In the last case, we can conclude from proposition 4 in [6] that P is a vertex or a zero-curvature point of the curve γ. □

Define

$$\epsilon(d) = \begin{cases} 0, & d \geq 2; \\ 1, & d = 1; \\ 2, & d = 0. \end{cases}$$

Lemma 4. *If a curve γ with finitely many vertices has K zero-curvature points, V external vertices and d external segments then $V + 2K \geq 2\epsilon(d)$.*

Proof. For $d \geq 2$ there is nothing to prove. The case $d = 0$ is considered in [7]. If $d = 1$ then there is at least one external complete arc of the curve γ (the proof of this fact is a slight variation of that of lemma 1 in [8]) and the result follows from lemma 3. □

Definition. *The side AB of a support triangle ABC is called left (right) essential if the arc $\widetilde{AB} \subset \gamma \setminus \{C\}$ with end-points A and B is external and the triangle ABC lies in the right (left) domain attached to $\widetilde{AB}$.*

For example, in figure 1, the side BC of the triangle a) is essential, whereas all the other triangle sides in a) and b) are not essential (dotted parts of the curve represent pieces of internal arcs).

Observe that if one of the essential sides of a support triangle is left (right) essential so must be all its other essential sides.

Definition. *A support triangle having at least one left (right) essential side is called left (right) essential support triangle, otherwise it is called unessential.*

Let E be any subset of the set of all left (right) essential support triangles of γ.

Definition. *We shall say that an essential side AB of some triangle $ABC \in E$ is E-extremal if the left (right) domain attached to the arc $\widetilde{AB} \subset \gamma \setminus \{C\}$ with end-points A and B does not contain any triangles of E.*

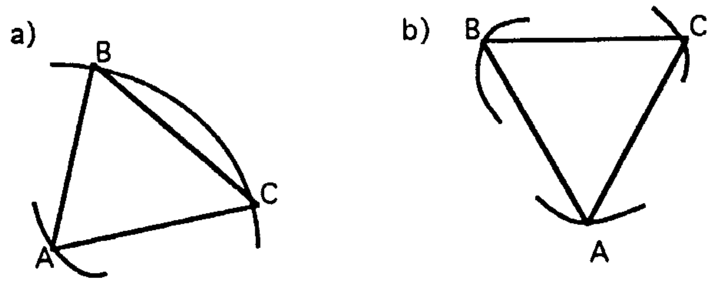

Figure 1. Essential and unessential triangles

Lemma 5. *Let γ have d external segments and E be a nonempty set composed of n left (right) essential support triangles. Then these triangles have at least $\epsilon(d) + n$ left (right) E-extremal sides.*

Proof. We shall prove this by induction on n. Suppose that $n = 1$ and hence E is composed of a single triangle Δ.

If $d = 0$, the curve is convex and thus all the sides of its support triangles are essential. Consequently, the triangle Δ has 3 E-extremal sides. If $d = 1$, it is not difficult to see that Δ has 2 essential sides which must also be E-extremal. Finally, if $d \geq 2$, since by hypothesis Δ is essential, at least one of its sides must be E-extremal.

We shall assume now that the statement is true for $n = k$ and prove for $n = k + 1$.

Suppose thus that E contains $k + 1$ triangles. If all the essential sides of these triangles are E-extremal then the result is immediate. Otherwise there is the essential side AB of some triangle $ABC \in E$ such that the left (right) domain attached to the arc $\widetilde{AB} \subset \gamma \setminus \{C\}$ with end-points A and B contains exactly one triangle $\Delta \in E$.

By the induction assumption the triangles from $E \setminus \{\Delta\}$ have at least $k + \epsilon(d)$ sides which are $E \setminus \{\Delta\}$-extremal. But all the $E \setminus \{\Delta\}$-extremal sides, except AB, are E-extremal. Hence, the triangles in E have at least $k + 1 + \epsilon(d)$ E-extremal sides, for Δ has 2 E-extremal sides. □

Proposition 1. *If γ has V external vertices, K zero-curvature points and d external segments then given any subset of N essential support triangles,*

$$V + 2K \geq 2\epsilon(d) + N.$$

provided the number of all vertices is finite.

The proof follows immediately from the lemmas 1 - 5.

3. Special triangles in spheres with holes

Consider any 2-dimensional sphere S with d holes and denote by σ the standard 2-dimensional simplex. The image of σ by means of a continuous injection to S is called a *special triangle* if the images of the vertices of σ belong to different connected components of the boundary ∂S and the images of the other points of σ are interior points of S. Clearly special triangles on S can exist only if $d \geq 3$.

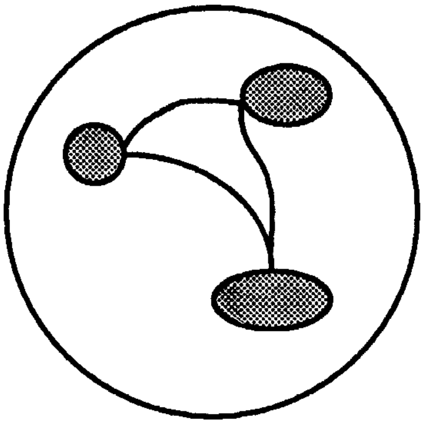

Figure 2. Sphere with 3 holes and a special triangle

Lemma 6. *Given any 2-sphere S with holes, there are at most 2 nonintersecting special triangles having their vertices in the same connected components of ∂S.*

Proof. Let Δ_1 and Δ_2 be 2 nonintersecting special triangles having their vertices in the same connected components $\beta_1, \beta_2, \beta_3$ of ∂S. Then the set $S \setminus (\Delta_1 \cup \Delta_2)$ has exactly 3 connected components every one of which has common points only with two of the curves $\beta_1, \beta_2, \beta_3$. Therefore, S cannot have any other special triangles with vertices in $\beta_1, \beta_2, \beta_3$, that do not intersect Δ_1 or Δ_2. □

The number of ways to choose 3 holes of S is equal to $d(d-1)(d-2)/6$. Hence, by lemma 6, there are at most $d(d-1)(d-2)/3$ nonintersecting special triangles of S.

Lemma 7. *Let N be the maximal number of nonintersecting special triangles of a 2-sphere S with d holes ($N \geq 1$), and $\Delta_1, ..., \Delta_N$ be any set of such triangles. Then the closure of any connected component D of the set $S \setminus (\Delta_1 \cup ... \cup \Delta_N)$ is homeomorphic to a disc, and the set $D \cap \partial S$ has exactly 2 connected components.*

Proof. Since N is the maximal number of nonintersecting special triangles of S, we have that D can have points in at most 2 connected components of ∂S. On the other hand, ∂D contains at least one side of a special triangle and the vertices of this lie each one in a component of ∂S, so we also have that ∂D must contain points of at least these 2 connected components of ∂S. Hence, D has points of exactly 2 connected components, β, β' of ∂S. Moreover, some of these points of β and β' must be contained in the same connected component of ∂D.

Now, all the points of any given connected component of ∂S that belong to ∂D must be in the same connected component of ∂D. Otherwise, if we had points of let's say β in 2 different connected components of ∂D, we would be able to draw a circle contained in the interior of D, and thus in the interior of S, separating them. Therefore, ∂D is connected and its interior is homeomorphic to a disk.

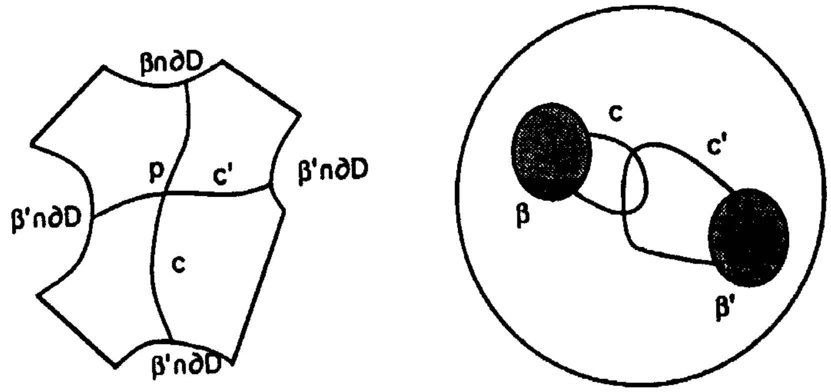

Figure 3. To the proof of lemma 7

The number of connected components of the set $D \cap \partial S$ is at least 2 (one with points of $\partial D \cap \beta$ and another one with points of $\partial D \cap \beta$'). On the other hand, this number is at most 2, i.e. each set $\partial D \cap \beta$ and $\partial D \cap \beta$' can only have one connected component. In fact, suppose that $\partial D \cap \beta$ had points in more than one of the connected components of $D \cap \partial S$. Now, these connected components had to be joined either by sides of essential triangles or by arcs contained in $\partial D \cap \beta$'. It follows that we should have points of $\partial D \cap \beta$' between any two components of $D \cap \partial S$ containing points of $\partial D \cap \beta$. It would be then possible to draw two curves, c and c', joining two different points in $\partial D \cap \beta$ and $\partial D \cap \beta$' respectively (as shown in fig. 3a), having transversal intersection at a unique common point p. But such two curves, considered in the sphere S, must always have an even number of transversal intersections (see fig. 3b)). Hence the set $D \cap \partial S$ has exactly two connected components.

Thus, the set $\partial D \setminus D$ is made by 2 sides of 2 different special triangles and, in particular, the closure of D in the sphere S is homeomorphic to a disc. □

Lemma 8. *Any 2-sphere S with d holes, $d \geq 3$, can have at most $2(d-2)$ nonintersecting special triangles simultaneously.*

Proof. Denote by N the maximal number of nonintersecting special triangles of S and fix the set $\Delta_1, ..., \Delta_N$ of such triangles. According to lemma 7, these triangles define a cell decomposition of S. If M is the number of connected components of the set $S \setminus (\Delta_1 \cup ... \cup \Delta_N)$, then the number of 0-dimensional, 1-dimensional and 2-dimensional cells are respectively equal to $3N, 4M$ and $N + M$. Hence, we get

$$3N - 4M + (N + M) = \chi,$$

where $\chi = 2 - d$ is the Euler number of S. But $3N = 2M$ by lemma 7. Therefore, $N = -2\chi$ and the result is proven. □

Proposition 2. *If a closed space volumetric curve γ has d external segments, then it has at most $2(d - 2 + \epsilon(d))$ unessential support triangles.*

Proof. The surface $\Sigma \setminus \gamma$ is homeomorphic to the interior of a 2-sphere S with d holes. Suppose that h: $\Sigma \setminus \gamma \to S \setminus \partial S$ is the homeomorphism between them. Then the closure of the image $h(ABC \setminus \{A, B, C\})$, for any unessential support triangle ABC of γ, is a special triangle of S. Hence from lemma 8 we conclude that if $d \geq 3$, γ has at most $2(d-2)$ unessential support triangles. On the other hand, if $d \leq 2$, the curve γ cannot have any unessential triangles. Therefore we have that γ must have at most $2(d - 2 + \epsilon(d))$ unessential support triangles $\forall d$. □

4. Proof of the Main Theorem

Let us consider a closed embedded space curve γ with only finitely many vertices. Suppose that γ has V external vertices, K zero-curvature points and d external segments. Then it is volumetric and the boundary Σ of its convex hull is homeomorphic to a 2-sphere.

Fix an orientation of γ and Σ. According to propositions 1 and 2, γ has only finitely many essential and unessential triangles.

Let T_e (T_u) be the number of essential (unessential) support triangles of γ. Then

$$V + 2K \geq 2\epsilon(d) + T_e$$

by proposition 1, and

$$2(d-2) + 2\epsilon(d) \geq T_u$$

by proposition 2.

Now these inequalities lead to

$$V + 2K + 2(d - 2) \geq T,$$

where $T = T_e + T_u$ is the number of all the support triangles of γ, and the theorem is completely proved.

5. Application to plane curves

Given a plane curve α: $S^1 \to \mathbb{R}^2$, stereographic projection s: $\mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ turns α into a spherical curve α^*. A circle (straight line) δ in $\mathbb{R}^2$ becomes, under the same mapping, a circle δ^* on S^2, cut out by a plane π in $\mathbb{R}^3$. If δ is tangent to α at p, then π is tangent to α^* at $s(p) \in S^2$, and the orders of these tangencies are equal. Consequently,

a) if δ is the *osculating* (curvature) circle of α at p, π must be the osculating plane of α^* at $s(p)$,

b) if δ is a tangent line with order of tangency higher than 2 (*osculating line*) to α at p then π must be the osculating plane of α^* at $s(p)$ and it passes through the north pole of the sphere, and

c) if δ is an n-tangent circle (line) to α, then π must be an n-tangent plane to α^*.

We say that a circle (line) δ in $\mathbb{R}^2$ is a *support* circle (line) to a curve α if and only if α is entirely contained in one of the 2 closed regions in which δ decomposes $\mathbb{R}^2$. Clearly, if δ is a support circle (line) to α in $\mathbb{R}^2$ then π is a support plane to α^* in $\mathbb{R}^3$.

A *vertex* of a curve α (local extremum point of its curvature) is said to be *external* if the osculating circle or line of α at this vertex is a support one. It follows from the above considerations that the number C of external vertices of α is the same than the number of external vertices of α^*.

Let Q be the number of support n-tangent circles and lines ($n \geq 3$) to α. This number is the same than the number of support n-polygons of α^*. Since spherical curves in $\mathbb{R}^3$ do not have zero-curvature points, we can easily conclude from corollary 3 above

Corollary 4. *Suppose that the curvature function of a C^3-embedded closed plane curve has finitely many critical points. Then the relation*

$$C \geq 4 + Q$$

holds for the number C of external vertices of the curve.

Moreover, if n_i denotes the number of tangency points of i-th support circle (line) with the curve, then our Main Theorem implies

Corollary 5. *The number C of external vertices of any C^3-embedded closed plane curve satisfies*

$$C \geq 4 + \Sigma_{i=1}^{Q}(n_i - 2),$$

provided the number of critical points of its curvature is finite.

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