Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 38 (1997), No. 1, 145-148.

Correspondence between Representations of the Dual Reductive Pair $O(n)$ and SL_2

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Abstract. We state a connection between the Weil representation of the dual reductive pair $O(n)$ and SL_2, and prove that they have eigenvectors in common.

1. Introduction

In this exposition we continue to study the Weil representation of $Sp(2n, k)$ that we studied in "Commutativity of the Schur algebra". There we were interested in the decomposition of its induced representation whereas here we are interested in the irreducible components of its restriction to the product of the dual reductive pair SL_2 and $O(n)$.

Dual reductive pairs are playing an important role in the study of representation theory. We shall prove here a result on the correspondence between representations of a dual reductive pair, based on Schur theory.

We recall the definition of a dual reductive pair.

Definition. *A dual reductive pair for $Sp(2n, k)$ is a pair G, H of reductive subgroups of $Sp(2n, k)$. Each is the full centralizer of the other.* (See Howe [1].)

2. Construction of the dual reductive pair $O(n), SL_2$

We are interested in the following examples of a dual reductive pair: Let k be a finite field. Let $X = k^n$ with symmetric form $(\ ,\ )$ $n \geq 6$, which the group $O(n) = O(n, k)$ preserves. Let $Y = k^2$ with some antisymmetric form $\langle\ ,\ \rangle$, which $SL_2 = Sp(2, k)$ preserves. Look at $Z = X \otimes Y$. Clearly $\dim Z = 2n$ and Z has an antisymmetric form induced by $(\ ,\ )$ and $\langle\ ,\ \rangle$ as follows: $(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1, x_2)\langle y_1, y_2\rangle$. The group $Sp(2n) = Sp(2n, k)$

0138-4821/93 \$ 2.50

preserves its antisymmetric form. The action of $O(n)$ on X induces an action of $O(n)$ on Z which preserves $\langle\ ,\ \rangle$. The action of SL_2 on Y induces an action of SL_2 on Z which preserves its antisymmetric form. Thus $O(n) \hookrightarrow Sp(2n)$ and $SL_2 \hookrightarrow Sp(2n)$.

Denote the standard basis in X by $e_1, \ldots, e_n$ and in Y by y_1, y_2. Take the following basis in Z : $e_1 \otimes y_1, \ldots, e_n \otimes y_1, e_1 \otimes y_2, \ldots, e_n \otimes y_2$. With respect to this basis $O(n)$ is embedded by $A \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and SL_2 by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix}$. The groups $O(n)$ and SL_2 are both reductive and they are each other's centralizer. So they are a dual reductive pair.

Choose $T = \begin{pmatrix} & & 1 \\ & \ldots & \\ 1 & & \end{pmatrix}$ to induce ( ) and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to induce $\langle , \rangle$. Then with respect to the chosen basis of $Z, \langle , \rangle$ is represented on Z by $\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$.

3. The Weil representation of $O(n) \times SL_2$

Let (W_ψ, V) be a Weil representation of $Sp(2n)$. (See Teicher [3] or Gerardin [1]; Thm. 2.4). To choose a model of W_ψ we have to choose a polarization. Take $Z_1 = X \otimes y_1$, $Z_2 = X \otimes y_2$. Then $Z = Z_1 + Z_2$, Z_1 and Z_2 are maximal isotropic subspaces. In this case $V = \{\phi : Z_1 \to \mathbb{C}\}$, $Z_1 \sim X$. Thus we can take $V = \{\phi : X \to \mathbb{C}\}$.

We shall consider X, when regarded as the domain of the definition for ϕ, as row vectors. We shall need the precise equation of $O(n)$ acting on V :

$$W_\phi \begin{pmatrix} T^t A^{-1} T & 0 \\ 0 & A \end{pmatrix} \phi(x) = \phi(xA), \ x \in k^n, \ A \in GL(n).$$

For $A \in O(n)$, $T^t A^{-1} T = A$ and $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. So then:

$$W_\phi \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \phi(x) = \phi(xA), \ x \in O(n).$$

$W_\phi \mid_{O(n) \times SL_2} = \sum \tau \times \sigma$ where σ are the irreducible representations of SL_2.

4. Construction of the character χ_t

Recall from part II of Teicher [4] the following notations:

$L = \mathrm{Span}(e_2, \ldots, e_{n-1})$, U the subgroup of $O(n)$ characterized by $u \in U \Leftrightarrow \exists a \in L$ such that $ue_1 = e_1$, $u\ell = \ell - (a, \ell)e_1 \ \ \forall \ell \in L$, $ue_n = e_1 + a - \frac{(a,a)}{2} e_n$. Thus $U \stackrel{\sim}{\to} L$ by $u \to a$. For $m \in O(L)$, $m^{-1} U m = U$.

Looking at X as row vectors we then take $e_1 = (1,0,\ldots,0),\ldots,e_n = (0,\ldots,0,1)$. Then $(x,y) = xT^ty$. The group $O(n)$ then acts on X from the right and preserves $(\ ,\ )$: $g \in O(n) \Rightarrow gT^tg = T$. Thus $(xg,yg) = xgT^t(yg) = xgT^tg^ty = xT^ty = (x,y)$. We also have for u in U: $e_nu = e_n,\ e_1u = e_1 - aT - \frac{1}{2}(a,a)e_n$.

Fix ψ, an additive character of k. Let $t \in L$. Let $\chi = \chi_t$ be the following character of U: $\chi(u) = \psi((u,t))$ when considering the isomorphism $U \xrightarrow{\sim} L$.

5. The main result

Theorem. *Let $\tau \times \sigma$ be a component of $(W_\psi, V)\big|_{O(n)\times SL_2}$. Let $\phi_0 \in V$ be an eigenvector with respect to U, i.e., $\exists t \in L$ such that for $\chi = \chi_t$, $\tau(u)\phi_0 = \chi(u)\phi_0$. Assume that ϕ_0 is supported on an orbit W of the right action of U on X, for W that satisfies $(x,e_n) \neq 0$ for $x \in W$. Then ϕ_0 is also an eigenvector with respect to $M_1 = Stab_X(O(L))$.*

Proof. Recall that we consider a function ϕ in V, $\phi : X \to \mathbb{C}$ as a function on row vectors.

Take $x_1, x_2 \in W$. W is an orbit under U, so there is $u \in U$ such that $x_1u = x_2$. Therefore $(x_2,e_n) = (x_1u,e_n) = (x_1u,e_nu) = (x_1,e_n)$ because $e_nu = e_n$. Since $(x_2,x_2) = (x_1u,x_1u) = (x_1,x_1)$, we can write $W \subset \{x \in X \mid (x,e_n) = \alpha, (x,x) = f\}$. We claim that equality actually holds. By the assumption $\alpha \neq 0$. Take x_1, x_2 such that $(x_1,e_n) = (x_2,e_n) = \alpha \neq 0$ and $(x_1,x_1) = (x_2,x_2) = f$. A priori $x_i = a_ie_1 + \ell_i + b_ie_n$ for $\ell_i \in L$, $i = 1,2$ where $a_i = (x_i,e_n)$. Thus $a_1 = a_2 = \alpha \neq 0$. Let $u_i \in U$ such that $e_1u_i = e_1 - \dfrac{\ell_i}{\alpha} + \dfrac{(\ell_1,\ell_1)}{2\alpha\alpha}e_n$, $i = 1,2$. Then $x_iu_i = \alpha e_1 + c_ie_n$ for some $c_i \in k$, $i = 1,2$, and $(x_iu_i, x_iu_i) = 2c_i\alpha$. On the other hand, $(x_iu_1, x_iu_1) = (x_i,x_k) = f$. Therefore $c_1 = c_2$ and $x_1u_1 = x_2u_2$. Thus $x_2 = x_1u_1u_2^{-1}$ and x_1 and x_2 are in one orbit. This orbit must be W. So $W = \{x \in X \mid (x,e_n) = \alpha,\ (x,x) = f\}$.

Take $m \in M_1$ and look at $\tau(m)\phi_0$:
$\tau(m)\phi_0(x) = W_\psi \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \phi_0(x) = \phi_0(xm)$. We claim that $\tau(m)\phi_0$ is also an eigenvector with respect to U :

$$\begin{aligned}(\tau(u)(\tau(m)\phi_0))(x) &= (\tau(um)\phi_0)(x) = W_\psi \begin{pmatrix} um & 0 \\ 0 & um \end{pmatrix} \phi_0(x) = \phi_0(xum) = \phi_0(xmm^{-1}um)\\ &= \chi(m^{-1}um)\phi_0(xm) = \chi(u)\phi_0(xm) = \chi(u)(\tau(m)\phi_0)(x)\end{aligned}$$

because $m^{-1}um \in U\ \forall m \in O(L)$ and $x(m^{-1}um) = x(u)\ \forall m \in M_1$. Therefore $\tau(m)\phi_0$ is also an eigenvector of U with respect to the same eigenvalue.

Take $m \in M_1$ and $x \in X$. We have: $(xm,xm) = (x,x)$ and m^{-1} fixes e_n. Thus $(xm,e_n) = (x,e_nm^{-1}) = (x,e_n)$ and $x \in W \Leftrightarrow xm \in W$. Since ϕ_0 was supported on W, and $(\tau(m)\phi_0)(x) = \phi_0(xm)$ then $\tau(m)\phi_0$ is also supported on W.

Therefore $\tau(m)\phi_0$ is an eigenvector of U with respect to χ and supported on W.

We claim that ϕ_0 and $\tau(m)\phi_0$ differ by a constant: Put $W = \mathrm{Orbit}(x_0)$, $c_0 = \phi_0(x_0)$, $c_1(m) = (\tau(m)\phi_0)(x_0)$. Take $x \in W$, $x = (x_0)u$. Then $(\tau(m)\phi_0)(x) = (\tau(m)\phi_0)((x_0)u) =$

$x(u)\tau(m)\phi_0)(x_0) = \chi(u)c_1(m) = \dfrac{c_1(m)}{c_0}x(u)c_0 = c_m x(u)\phi_0(x_0) = c_m\phi_0(x_0u) = c_m\phi_0(x)$. Thus $\tau(m)\phi_0 = c_m\phi_0$.

Since c_m is a character of M_1 where $M_1 \sim O(n-3)$, $n-3 \geq 3$ then $c_m = (\det m)^\varepsilon$. Thus $\tau(m)\phi_0 = (\det m)^\varepsilon_{\phi_0}$. □

Note that if ϕ is an eigenvector supported on $\{x \mid (x,x) = f\}$ we can look at $\delta_\alpha\phi$ where δ_α is the charcteristic function of $\{x \mid (x,e_n) = \alpha\}$, and get a vector that is supported on an orbit. Furthermore such vectors form a basis for the eigenvectors.

References

[1] Gérardin, P.: *Weil's representations associated to finite fields.* J. of Algebra **46** (1977), 54–101.

[2] Howe, R.: *θ series and invariant theory, part 1.* Proceedings of Symposia in Pure Math. **33** (1979), 275–285.

[3] Teicher, M.: *Commutativity of the Schur algebra.* Beitr. Algebra und Geometrie **38** (1997), 135–143.

[4] Teicher, M.: *Multiplicity one theorem for $GS_p(k,2n)$ and $O(k,n)$.* J. of Algebra **107** (1987), 436–465.

Received January 3, 1996