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Discriminant Algebras and Adjoints of Quadratic Forms

Ottmar Loos
Institut für Mathematik, Universität Innsbruck
A-6020 Innsbruck, Austria

Introduction

The discriminant algebra of a nonsingular quadratic form q is usually defined as C^{C_0}, the centralizer of the even Clifford algebra in the full Clifford algebra. In [5], Haag introduced the discriminant algebra $\mathfrak{D}(q)$ of a possibly singular quadratic form on a finitely generated free module by working with generic forms and then applying specialization. His method depends on a choice of basis, and although the result is up to isomorphism independent of that choice, an extra effort is required to extend the definition to non-free finitely generated projective modules [9, (4.8.11)].

In this paper, we present a definition of $\mathfrak{D}(q)$ in a natural basis-free way which avoids the Clifford algebra entirely. We sketch this definition for the most interesting case of a quadratic module M of even rank $2n$ over a commutative ring k, and assume first that M is free and $q = \sum a_{ij}\xi_i\xi_j$ with a (in general not symmetric) $2n \times 2n$-matrix $A = (a_{ij})$. The construction is based on the elementary observation that there exists a unique polynomial $\operatorname{qdet}(A)$ in the a_{ij} with integral coefficients, homogeneous of degree $2n$, such that

$$4\operatorname{qdet}(A) = \det(A + A^T) - (-1)^n \det(A - A^T).$$

We call this the *quarter determinant* of A. An explicit formula for $\operatorname{qdet} A$ is provided (1.1). Then the discriminant algebra is isomorphic to the free quadratic algebra with generators 1 and z and the relation

$$z^2 - \operatorname{Pf}(A - A^T)z + (-1)^{n+1}\operatorname{qdet}(A).1 = 0,$$

where Pf denotes the Pfaffian. This gives a convenient way of computing the discriminant algebra in the free case.

In general, let L be the invertible module $\bigwedge^{2n} M$ and consider bilinear forms f, g on M representing q in the sense that $q(x) = f(x,x) = g(x,x)$ for all $x \in M$. We introduce an Amitsur 1-cocycle κ_{fg} on such forms with values in the dual of L, defined in terms of Pfaffian powers of $f - f^T$ and $f - g$ (1.5). The module underlying $\mathfrak{D}(q)$ is presented by generators 1 and $s_f(u)$ ($u \in L$), k-linear in u, and satisfying $s_f(u) - s_g(u) = \kappa_{fg}(u).1$ for all bilinear forms f, g representing q. As a k-module, $\mathfrak{D}(q)$ is isomorphic to $k.1 \oplus L$ but not in a canonical way. There is a linear form τ_f and a symmetric bilinear form γ_f on L such that

$$\tau_f(u) = \mathrm{Pf}\,\big(f(x_i,x_j) - f(x_j,x_i)\big), \qquad \gamma_f(u,u) = (-1)^{n+1}\,\mathrm{qdet}\,\big(f(x_i,x_j)\big),$$

for decomposable $u = x_1 \wedge \ldots \wedge x_{2n} \in L$. The algebra structure on $\mathfrak{D}(q)$ is then determined by

$$s_f(u) \cdot s_f(v) - \tau_f(u)s_f(v) + \gamma_f(u,v)1 = 0 \quad (u, v \in L).$$

The τ_f and γ_f interact correctly with the cocycle, ensuring that the multiplication is independent of the choice of f (2.3). As a byproduct, we obtain an explicit formula for the Dickson homomorphism (2.4). The functor $\mathfrak{D}$ from quadratic modules to graded quadratic algebras is multiplicative in the sense that $\mathfrak{D}(q_1 \perp q_2) \cong \mathfrak{D}(q_1) \Box \mathfrak{D}(q_2)$ (2.11) for a suitably defined product $\Box$ which generalizes the well-known product in the separable case [6, 9].

In section 3 we show that there is a natural homomorphism $\psi_q \colon \mathfrak{D}(q) \to C^{C_0}$ which is an isomorphism if q is nonsingular or semiregular, depending on whether M has even or odd rank (3.14). This makes use of Bourbaki's λ-operations between Clifford algebras [2], for which we derive an explicit formula (3.2).

In section 4 we introduce, for a quadratic module (M, q) of odd rank $2n+1$, a unital quadratic form $q^\sharp$ of rank $2n+2$ on a module $\mathfrak{A}(q)$ which is constructed from the second-to-the-highest exterior power of M by the same method as $\mathfrak{D}(q)$ in the even rank case. We call this the *adjoint* of q since it is similar to the adjoint of a matrix; e.g., the discriminants of q and $q^\sharp$ are related by $\delta_{q^\sharp} \cong \delta_q^{\otimes 2n}$ (4.4). In 4.6 we establish a natural homomorphism between the tensor product of q with its discriminant and the space of trace zero elements of $\mathfrak{A}(q)$. In the opposite direction, there is a natural homomorphism from the $\Box$-product of a unital quadratic form Q of even rank with its discriminant algebra to the adjoint $\mathfrak{A}(Q^0)$ where Q^0 denotes the restriction of $-Q$ to the trace zero elements (4.9).

Just as $\mathfrak{D}(q)$ maps naturally into the Clifford algebra so does $\mathfrak{A}(q)$, and its image is contained in a Jordan subalgebra J which we call the *Jordan center* since it is defined by suitable centralizer conditions (5.1). In case q is nonsingular of even rank or semiregular of odd rank, we determine the structure of J completely (5.4, 5.5).

Throughout, k denotes an arbitrary commutative base ring. Tensor symbols without subscripts are understood over k. We mostly follow the notation and terminology of [9]. A quadratic form on a k-module M is always assumed to be of the type $q(x) = f(x,x)$ for some bilinear form f, in which case we say that q is *represented* by f and write $q = [f]$. A quadratic module (M, q) is a finitely generated projective k-module M with quadratic form q. To simplify statements, it is usually assumed that M has constant rank.

1. Exterior and Pfaffian powers of bilinear forms

1.1. The half- and the quarter-determinant. Let A and B be $m \times m$-matrices with coefficients in k. Since $\det A$ is a homogeneous polynomial of degree m in the coefficients of A, there is an expansion

$$\det(A+B) = \sum_{j=0}^{m} \Delta_j(A,B) \tag{1}$$

where $\Delta_j(A,B)$ is homogeneous of bidegree $(m-j,j)$ in (A,B). Explicitly,

$$\Delta_j(A,B) = \sum_{|J|=j} \det C_J(A,B) \tag{2}$$

where, for a subset $J \subset \{1,\ldots,m\}$, the i-th column of the matrix $C_J(A,B)$ equals the i-th column of B if $i \in J$ and the i-th column of A otherwise. Clearly $\Delta_0(A,B) = \det A$ and $\Delta_j(A,B) = \Delta_{m-j}(B,A)$. Since the determinant is invariant under transposition we have $\Delta_j(A^T,B^T) = \Delta_j(A,B)$. Combining these two formulas yields

$$\Delta_j(A,A^T) = \Delta_{m-j}(A,A^T). \tag{3}$$

For odd $m = 2n+1$ we define the *half-determinant*

$$\operatorname{hdet} A = \sum_{i=0}^{n} \Delta_i(A,A^T). \tag{4}$$

This is a homogeneous polynomial of degree m with integer coefficients in the entries of A which satisfies

$$2\operatorname{hdet} A = \det(A+A^T), \tag{5}$$

an easy consequence of (1) and (3). Note also that $\operatorname{hdet} A = P_m(a_{ii}, a_{ij}+a_{ji})$, where P_m is the polynomial defined in [9, IV, (3.1.2)] or [11, (1.6.1)].

Now let $m = 2n$ be even and define the *quarter-determinant* by

$$\operatorname{qdet} A = \sum_{i<n,\, i\not\equiv n(2)} \Delta_i(A,A^T). \tag{6}$$

This is again a homogeneous polynomial of degree m with integer coefficients in the entries of A, and it satisfies

$$4\operatorname{qdet} A = \det(A+A^T) - (-1)^n \det(A-A^T). \tag{7}$$

Indeed, by (1) and (3),

$$
\begin{aligned}
\det(A + A^T) - (-1)^n \det(A - A^T) &= \sum_{i=0}^{2n} \Delta_i(A, A^T) - (-1)^n \Delta_i(A, -A^T) \\
&= \sum_{i=0}^{2n} (1 - (-1)^{n+i}) \Delta_i(A, A^T) \\
&= 2 \sum_{i \not\equiv n(2)} \Delta_i(A, A^T) \\
&= 4 \sum_{i<n,\, i \not\equiv n(2)} \Delta_i(A, A^T).
\end{aligned}
$$

1.2. Formulas for the half- and quarter-determinant. The half-determinant of a 1×1-matrix (a) is obviously a, and $\operatorname{qdet} A = \det A$ for a 2×2-matrix follows immediately from the definition. Next, let $A_i \in \operatorname{Mat}_{r_i}(k)$. Then for $r_i = 2n_i$ even,

$$
\begin{aligned}
\operatorname{qdet} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} &= 4 \operatorname{qdet}(A_1) \operatorname{qdet}(A_2) + (-1)^{n_2} \operatorname{qdet}(A_1) \det(A_2 - A_2^T) \\
&\quad + (-1)^{n_1} \operatorname{qdet}(A_2) \det(A_1 - A_1^T),
\end{aligned} \tag{1}
$$

while for r_i both odd,

$$
\operatorname{qdet} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \operatorname{hdet}(A_1) \operatorname{hdet}(A_2). \tag{2}
$$

If r_1 is odd and r_2 is even, we have

$$
\operatorname{hdet} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \operatorname{hdet}(A_1) \det(A_2 + A_2^T). \tag{3}
$$

Since the half- and quarter-determinant is a polynomial with integral coefficients it suffices to prove these formulas for the torsion-free polynomial ring in the indeterminate entries of the A_i with integer coefficients. Then they follow by straightforward computations from 1.1.5 and 1.1.7. The details are left to the reader. By induction, we obtain

$$
\operatorname{qdet} \operatorname{diag}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = 0, \tag{4}
$$

$$
\operatorname{hdet} \operatorname{diag}(1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = (-1)^n. \tag{5}
$$

We also note

$$
\operatorname{qdet} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = 0 \tag{6}
$$

for a $2n \times 2n$-matrix subdivided into 4 blocks of size $n \times n$, since

$$\begin{aligned}
4\,\mathrm{qdet}\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix} &= \det\begin{pmatrix}0 & B\\ B^T & 0\end{pmatrix} - (-1)^n \det\begin{pmatrix}0 & B\\ -B^T & 0\end{pmatrix}\\
&= \det\begin{pmatrix}0 & B\\ B^T & 0\end{pmatrix} - (-1)^n \det\begin{pmatrix}0 & B\\ B^T & 0\end{pmatrix}\det\begin{pmatrix}-1_n & 0\\ 0 & 1_n\end{pmatrix}\\
&= \bigl(1-(-1)^{2n}\bigr)\det\begin{pmatrix}0 & B\\ B^T & 0\end{pmatrix} = 0.
\end{aligned}$$

1.3. Exterior powers of bilinear forms. Let M be a k-module and $f\colon M\times M\to k$ a bilinear form, with transposed form $f^T(x,y)=f(y,x)$. For m-tuples $\mathbf{x}=(x_1,\dots,x_m)$ and $\mathbf{y}=(y_1,\dots,y_m)$ of elements of M let $f(\mathbf{x},\mathbf{y})$ be the $m\times m$-matrix with entries $f(x_i,y_j)$. Then we have $f(\mathbf{x},\mathbf{y})^T=f^T(\mathbf{y},\mathbf{x})$. Recall that the m-th exterior power of f is the unique bilinear form $\bigwedge^m f$ on $\bigwedge^m M$ satisfying

$$\bigwedge^m f(x_1\wedge\ldots\wedge x_m, y_1\wedge\ldots\wedge y_m)=\det f(\mathbf{x},\mathbf{y}).$$

Let g be a second bilinear form. With Δ_j as in 1.1, we claim that the expression

$$\Delta_j(f(\mathbf{x},\mathbf{y}),g(\mathbf{x},\mathbf{y})) \tag{1}$$

is a multilinear and alternating function of $\mathbf{x}$ and also of $\mathbf{y}$. Indeed, let t be an indeterminate. Then $h=f+tg$ is a bilinear form on M with values in $k[t]$. Therefore, $\det h(\mathbf{x},\mathbf{y})=\det(f(\mathbf{x},\mathbf{y})+tg(\mathbf{x},\mathbf{y}))$ is a multilinear and alternating function of $\mathbf{x}$ and $\mathbf{y}$, and so are the coefficients of t^j which are given by (1). By the universal property of the m-th exterior power, there exists a unique bilinear form $\bigwedge^{(m-j,j)}(f,g)$ on $\bigwedge^m M$ such that

$$\bigwedge^{(m-j,j)}(f,g)(x_1\wedge\ldots\wedge x_m, y_1\wedge\ldots\wedge y_m)=\Delta_j(f(\mathbf{x},\mathbf{y}),g(\mathbf{x},\mathbf{y})), \tag{2}$$

for all $x_i,y_j\in M$. Now 1.1.1 yields the expansion

$$\bigwedge^m(f+g)=\sum_{j=0}^m\bigwedge^{(m-j,j)}(f,g). \tag{3}$$

Clearly $\bigwedge^{(m,0)}(f,g)=\bigwedge^m(f)$ and $\bigwedge^{(m-j,j)}(f,g)$ is homogeneous of degree $m-j$ in f and of degree j in g. From the corresponding properties of the Δ_j we obtain

$$\bigwedge^{(m-j,j)}(f,g)=\bigwedge^{(j,m-j)}(g,f),\qquad \Bigl(\bigwedge^{(m-j,j)}(f,g)\Bigr)^T=\bigwedge^{(m-j,j)}(f^T,g^T),$$

which implies

$$\Bigl(\bigwedge^{(m-j,j)}(f,f^T)\Bigr)^T=\bigwedge^{(j,m-j)}(f,f^T). \tag{4}$$

Define the bilinear form $\delta_f=\delta_f^{(m)}$ on $\bigwedge^m M$ by

$$\delta_f^{(m)} = \begin{cases} (-1)^n \sum_{i=0}^n \bigwedge^{(2n+1-i,i)}(f, f^T) & \text{if } m = 2n+1, \\ (-1)^n \bigwedge^{2n}(f + f^T) & \text{if } m = 2n. \end{cases} \tag{5}$$

Then for decomposable $u = x_1 \wedge \ldots \wedge x_m$ we have, by (2) and 1.1.4,

$$\delta_f^{(m)}(u, u) = \begin{cases} (-1)^n \operatorname{hdet} f(\mathbf{x}, \mathbf{x}) & \text{if } m = 2n+1, \\ (-1)^n \det\big(f(\mathbf{x}, \mathbf{x}) + f(\mathbf{x}, \mathbf{x})^T\big) & \text{if } m = 2n. \end{cases} \tag{6}$$

Also, (3) and (4) imply

$$(-1)^n \bigwedge^{2n+1}(f + f^T) = \delta_f^{(2n+1)} + (\delta_f^{(2n+1)})^T. \tag{7}$$

Now let $m = 2n$ be even and define the bilinear form $\gamma_f = \gamma_f^{(2n)}$ on $\bigwedge^{2n} M$ by

$$\gamma_f^{(2n)} = (-1)^{n+1} \sum_{i<n,\, i \not\equiv n(2)} \bigwedge^{(2n-i,i)}(f, f^T). \tag{8}$$

For decomposable $u = x_1 \wedge \ldots \wedge x_{2n}$ we have

$$\gamma_f(u, u) = (-1)^{n+1} \operatorname{qdet} f(\mathbf{x}, \mathbf{x}). \tag{9}$$

Moreover, (4) shows

$$\gamma_f + \gamma_f^T = (-1)^{n+1} \sum_{i \not\equiv n(2)} \bigwedge^{(2n-i,i)}(f, f^T), \tag{10}$$

and from (3) and (10) we obtain by a similar calculation as for 1.1.7,

$$\delta_f^{(2n)} = \bigwedge^{2n}(f - f^T) - 2(\gamma_f + \gamma_f^T). \tag{11}$$

1.4. Pfaffian powers of alternating bilinear forms. Let a be an alternating bilinear form on M and $\mathbf{x} = (x_1, \ldots, x_{2n}) \in M^{2n}$. Then the matrix $a(\mathbf{x}, \mathbf{x})$ is alternating, and from the definition of the Pfaffian of an alternating matrix [1, §5.2] it follows easily that $\operatorname{Pf}(a(\mathbf{x}, \mathbf{x}))$ is an alternating and multilinear function of $\mathbf{x}$. Hence there exists a unique linear form $\pi_n(a)$ on $\bigwedge^{2n} M$, called the n-th *Pfaffian power* of a, such that

$$\pi_n(a)(x_1 \wedge \ldots \wedge x_{2n}) = \operatorname{Pf}(a(\mathbf{x}, \mathbf{x})), \tag{1}$$

for all $\mathbf{x} \in M^{2n}$. Since the square of the Pfaffian is the determinant, (1) implies

$$\big(\pi_n(a)(u)\big)^2 = (\bigwedge^{2n} a)(u, u) \tag{2}$$

for all decomposable $u = x_1 \wedge \ldots \wedge x_{2n}$. If M is finitely generated projective of rank $2n$ then (2) holds for all u. Indeed, after localizing we may assume M is free and then every element of $\bigwedge^{2n} M$ is decomposable.

The Pfaffian of an alternating $2n \times 2n$-matrix is a homogeneous polynomial of degree n in the matrix coefficients. Hence, $\pi_n(a)$ is homogeneous of degree n in a. For the same reason, if A and A' are alternating $2n \times 2n$-matrices, there is an expansion

$$\mathrm{Pf}(A + A') = \sum_{j=0}^{n} P_j(A, A'), \tag{3}$$

where $P_j(A, A')$ is homogeneous of bidegree $(n - j, j)$ in (A, A'). Now let a and a' be alternating bilinear forms on M. Similar arguments as in 1.3 show that there exist unique linear forms $\pi_{n-j,j}(a, a')$ on $\bigwedge^{2n} M$, homogeneous of bidegree $(n - j, j)$, called the *mixed Pfaffian powers* of a and a', such that

$$\pi_{n-j,j}(a, a')(x_1 \wedge \ldots \wedge x_{2n}) = P_j(a(\mathbf{x}, \mathbf{x}), a'(\mathbf{x}, \mathbf{x})), \tag{4}$$

for all $\mathbf{x} \in M^{2n}$. Let t be an indeterminate and define the "difference quotient" $\mathit{\Pi}_n(t, a, a')$ by

$$\pi_n(a + ta') = \pi_n(a) + t\mathit{\Pi}_n(t, a, a'). \tag{5}$$

Then (3) and (4) show

$$\mathit{\Pi}_n(t, a, a') = \sum_{j=1}^{n} t^{j-1}\pi_{n-j,j}(a, a'). \tag{6}$$

1.5. Definition. Let M be a k-module. Two bilinear forms f and g are said to be *equivalent*, $f \sim g$, if they represent the same quadratic form $[f] = [g]$, i.e., if $f - g$ is alternating. In this case, define linear forms τ_f and κ_{fg} on any even exterior power $\bigwedge^{2n} M$ by

$$\tau_f = \tau_f^{(n)} = \pi_n(f - f^T), \tag{1}$$

$$\kappa_{fg} = \kappa_{fg}^{(n)} = \mathit{\Pi}_n(-2, f - f^T, f - g). \tag{2}$$

We will prove below that κ_{fg} is an Amitsur cocycle on bilinear forms with respect to the equivalence relation $\sim$.

A similitude $\varphi\colon (\tilde{M}, \tilde{q}) \to (M, q)$ with multiplier μ between modules with quadratic forms is a linear map $\varphi\colon \tilde{M} \to M$ such that $\tilde{q}(x) = \mu q(\varphi(x))$, for all $x \in \tilde{M}$. (This way of writing the multiplier μ on the "wrong" side allows for non-units as multipliers.) If q is represented by f then it is clear that $\tilde{q}$ is represented by

$$\varphi^* f = \mu f \circ (\varphi \times \varphi). \tag{3}$$

We extend φ to a linear map $\mathit{\Phi}$: $\bigwedge^m \tilde{M} \to \bigwedge^m M$ by

$$\mathit{\Phi}(u) = \mu^{[\frac{m}{2}]} \bigwedge^{m} \varphi(u). \tag{4}$$

1.6. Proposition. *Let M be a k-module with quadratic form q, represented by bilinear forms f, g, h.*
(a) *κ satisfies the relations*

$$\kappa_{fg} + \kappa_{gh} = \kappa_{fh}, \tag{1}$$
$$\tau_f - \tau_g = 2\kappa_{fg}. \tag{2}$$

(b) *Let $u = x_1 \wedge \ldots \wedge x_m \in \bigwedge^m M$ be decomposable. Then*

$$\gamma_g(u,u) = \gamma_f(u,u) - \kappa_{fg}(u)\tau_f(u) + \kappa_{fg}(u)^2 \quad (m = 2n), \tag{3}$$
$$\delta_g^{(m)}(u,u) = \delta_f^{(m)}(u,u). \tag{4}$$

If M is finitely generated projective of rank m then (3) and (4) hold for all $u \in \bigwedge^m M$.
(c) *Let $\varphi\colon (\tilde{M}, \tilde{q}) \to (M, q)$ be a similitude with multiplier μ. Then with the notations of 1.5,*

$$\kappa_{\varphi^* f, \varphi^* g} = \kappa_{fg} \circ \Phi, \tag{5}$$
$$\tau_{\varphi^* f} = \tau_f \circ \Phi, \qquad \gamma_{\varphi^* f} = \gamma_f \circ (\Phi \times \Phi) \quad (m = 2n), \tag{6}$$
$$\delta_{\varphi * f}^{(m)} = \begin{cases} \delta_f^{(m)} \circ (\Phi \times \Phi) & (m = 2n), \\ \mu\, \delta_f^{(m)} \circ (\Phi \times \Phi) & (m = 2n+1). \end{cases} \tag{7}$$

Proof. (a) Let a, a', a'' be alternating bilinear forms on M. Then the polynomials Π_n of 1.4 satisfy

$$\Pi_n(t, a, a') + \Pi_n(t, a + ta', a'') = \Pi_n(t, a, a' + a''). \tag{8}$$

Indeed, we replace a' by $a' + a''$ in 1.4.5 and obtain

$$\begin{aligned} \pi_n(a) + t\Pi_n(t, a, a' + a'') &= \pi_n((a + ta') + ta'') \\ &= \pi_n(a + ta') + t\Pi_n(t, a + ta', a'') \\ &= \pi_n(a) + t\big(\Pi_n(t, a, a') + \Pi_n(t, a + ta', a'')\big) \end{aligned}$$

which implies (8). Now let $a = f - f^T$, $a' = f - g$, $a'' = g - h$ and $t = -2$. Then $a - 2a' = f - f^T - 2(f - g) = 2g - (f + f^T) = 2g - (g + g^T) = g - g^T$, and therefore (8) implies (1) by definition of $\kappa_{f,g}$. Formula (2) follows in the same way from 1.4.5 and 1.5.1.

(b) Formulas (3) and (4) are polynomial identities with integer coefficients between the $f_{ij} = f(x_i, x_j)$ and $g_{ij} = g(x_i, x_j)$. Hence, it suffices to prove them when the f_{ij} and g_{ij} are indeterminates, the base ring is $R = \mathbb{Z}[f_{ij}, g_{ij}]$ and $M = R^m$ is free with basis $x_1, \ldots, x_n$, and then apply specialization. Let m be even. Then $\delta_f(u,u) = \delta_g(u,u)$ by 1.3.5, and 1.3.11 and 1.4.2 yield $4\gamma_f(u,u) = \tau_f(u)^2 - \delta_f(u,u)$. Hence by (2), $4\big(\gamma_g(u,u) - \gamma_f(u,u)\big) = \tau_g(u)^2 - \tau_f(u)^2 = \big(\tau_f(u) - 2\kappa_{fg}(u)\big)^2 - \tau_f(u)^2 = 4\big(\kappa_{fg}(u)^2 - \kappa_{fg}(u)\tau_f(u)\big)$, and since R has no 2-torsion, we have (3). Similarly, 1.3.7 shows for $m = 2n+1$ that $2\delta_f(u,u) = (-1)^n \bigwedge^{2n+1}(f + f^T)(u,u) = 2\delta_g(u,u)$, which implies (4). To prove the last statement of (b), let M be finitely generated projective of rank m. After a faithfully flat base extension we may assume M free. Then every element of $\bigwedge^m M$ is decomposable, so that (3) and (4) hold for all $u \in \bigwedge^m M$.

Finally, (5) – (7) are easily seen to hold for decomposable elements and therefore by linearity in general. This completes the proof.

1.7. Discriminants. Let M be a finitely generated projective module of rank r over k. Then $L = \bigwedge^r M$ has rank 1 which implies $\bigwedge^2 L = 0$. Hence there are no nonzero alternating bilinear forms on L. As a consequence, every bilinear form on L is symmetric, and bilinear and quadratic forms on L are in bijection, since every quadratic form on L is obtained from some bilinear form, unique modulo alternating bilinear forms, by setting the arguments equal.

Now let $q = [f]$ be a quadratic form on M, with polar form $b = f + f^T$. The (signed) discriminant of q is the bilinear form $\delta_q = \delta_f^{(r)}$ on L, which by the above remarks and 1.6.4 is symmetric and independent of the choice of f representing q. For even $r = 2n$ we note the formula

$$\delta_q = \tau_f^{\langle 2\rangle} - 4\gamma_f \tag{1}$$

which follows from 1.3.11 and 1.4.2 for $a = f - f^T$ due to the symmetry of γ_f. Here $\tau_f^{\langle 2\rangle}$ denotes the bilinear form $(u, v) \mapsto \tau_f(u)\tau_f(v)$. In case of odd rank $r = 2n + 1$, we have

$$2\delta_q = (-1)^n \bigwedge^r b \tag{2}$$

by 1.3.7 and the symmetry of $\delta_f^{(r)}$.

1.8. Proposition. *Let M_i be finitely generated projective k-modules of rank r_i with quadratic forms $q_i = [f_i] = [g_i]$, and let $q = q_1 \perp q_2$ on $M = M_1 \oplus M_2$, represented by $f = f_1 \perp f_2$ and $g = g_1 \perp g_2$. Also, let $L_i = \bigwedge^{r_i} M_i$ and identify $L = \bigwedge^{r_1+r_2} M$ with $L_1 \otimes L_2$ in the canonical way.*

(a) *Let r_1 and r_2 be even. Then*

$$\tau_f = \tau_{f_1} \otimes \tau_{f_2}, \quad \gamma_f = \gamma_{f_1} \otimes \tau_{f_2}^{\langle 2\rangle} + \tau_{f_1}^{\langle 2\rangle} \otimes \gamma_{f_2} - 4\gamma_{f_1} \otimes \gamma_{f_2}, \tag{1}$$

$$\kappa_{fg} = \kappa_{f_1 g_1} \otimes \tau_{f_2} + \tau_{f_1} \otimes \kappa_{f_2 g_2} - 2\kappa_{f_1 g_1} \otimes \kappa_{f_2 g_2}. \tag{2}$$

(b) *If $r_1 + r_2$ is odd,*

$$\delta_q = \delta_{q_1} \otimes \delta_{q_2}. \tag{3}$$

(c) *Let r_1 and r_2 be odd. Then*

$$\tau_f = 0, \quad \gamma_f = \delta_{q_1} \otimes \delta_{q_2}, \tag{4}$$

$$\kappa_{fg} = 0. \tag{5}$$

Proof. By localization, we may assume M_i free. Suppose $r_i = 2n_i$ is even and let $n = n_1 + n_2$. Then the well-known formula

$$\operatorname{Pf}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \operatorname{Pf}(A_1)\operatorname{Pf}(A_2), \tag{6}$$

for alternating matrices A_i of order $2n_i$ implies

$$\pi_n(a_1 \perp a_2) = \pi_{n_1}(a_1) \otimes \pi_{n_2}(a_2) \tag{7}$$

for alternating bilinear forms a_i on M_i. Setting $a_i = f_i - f_i^T$ yields the first formula of (1), while the second formula follows from 1.3.9 and 1.2.1. Now replace a_i by $a_i + ta_i'$ in (7). Then 1.4.5 implies by a simple calculation

$$\begin{aligned}\Pi_n(t, a_1 \perp a_2, a_1' \perp a_2') &= \pi_{n_1}(a_1) \otimes \Pi_{n_2}(t, a_2, a_2') \\ &\quad + \Pi_{n_1}(t, a_1, a_1') \otimes \pi_{n_2}(a_2) + t \cdot \Pi_{n_1}(t, a_1, a_1') \otimes \Pi_{n_2}(t, a_2, a_2'),\end{aligned}$$

and (2) follows by specializing $a_i = f_i - f_i^T$, $a_i' = f_i - g_i$, $t = -2$.

Formula (3) follows from 1.2.3. Now suppose r_1 and r_2 odd and let again A_i be alternating matrices of order r_i. Then

$$\operatorname{Pf}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = 0. \tag{8}$$

Indeed, it suffices to prove this when the base ring is the polynomial ring in the indeterminate coefficients of the A_i over the integers. Then $\det(A_i) = 0$, and therefore $\operatorname{Pf}\operatorname{diag}(A_1, A_2)^2 = \det(A_1)\det(A_2) = 0$. Since the base ring is now an integral domain, we have (8). Let $r_1 + r_2 = 2m$. Then (8) implies $\pi_m(a_1 \perp a_2) = 0$ for alternating bilinear forms a_i on M_i. Applied to $a_i = f_i - f_i^T$ we obtain the first formula of (4). Also, (8) yields, by a similar argument as above, that $\Pi_m(t, a_1 \perp a_2, a_1' \perp a_2') = 0$. For $a_i' = f_i - g_i$ and $t = -2$ we have (5). Finally, the second formula of (4) is a consequence of 1.2.2 and the definition of the discriminants in 1.7.

2. Discriminant algebras

2.1. Quadratic algebras. Recall [6, 9] that a quadratic (i.e., rank 2) algebra S over k has a unique standard involution σ such that all norms $x\sigma(x)$ and traces $x + \sigma(x)$ are scalar multiples of 1. Conversely, if S is a rank 2 module with a quadratic form N and a distinguished element 1 such that $N(1) = 1$, then S carries a unique structure of a quadratic algebra such that 1 is the unit element and N is the norm. It is given by

$$xy = F(x, 1)y + xF(1, y) - F(x, y)1 \tag{1}$$

where F is any bilinear form representing N [10, 1.6]. Denote the polar form of N by $N(x, y)$ and let $T(x) = N(x, 1)$ be the trace form. Then $x + \sigma(x) = T(x)1$ and every $x \in S$ satisfies $x^2 - T(x)x + N(x)1 = 0$. A graded quadratic algebra is a quadratic algebra S with a specified $\mathbb{Z}/2\mathbb{Z}$-grading $S = S^0 \oplus S^1$ such that $S^i S^j \subset S^{i+j}$. Then $1 \in S^0$ and $S^0 \perp S^1$ with respect to the polar form of N. Conversely, it is easy to see that these conditions suffice to make the quadratic algebra constructed from N by (1) graded. We say S is even or odd if S^1 has (constant) rank 0 or 1, respectively.

2.2. Let M be a k-module with quadratic form $q = [f]$ and $L = \bigwedge^{2n} M$. We define $E_{2n}(q)$ as the k-module generated by symbols 1 and $s_f(u)$, subject to the relations

$$s_f(u) \quad \text{is } k\text{-linear in } u, \tag{1}$$
$$s_f(u) - s_g(u) = \kappa_{fg}(u)1, \tag{2}$$

for all $u \in L$ and all bilinear forms f, g representing q. Clearly $s_f: L \to E_{2n}(q)$ is a module homomorphism, and there is a well defined homomorphism $p: E_{2n}(q) \to L$ such that $p(1) = 0$ and $p(s_f(u)) = u$. Moreover, the cocycle relation 1.6.1 guarantees that there is a well-defined linear form $\rho_f: E_{2n}(q) \to k$ such that

$$\rho_f(1) = 1, \quad \rho_f(s_g(u)) = -\kappa_{fg}(u). \tag{3}$$

Hence, the sequence

$$0 \longrightarrow k \longrightarrow E_{2n}(q) \xrightarrow{\ p\ } \textstyle\bigwedge^{2n} M \longrightarrow 0 \tag{4}$$

is exact and split, every f representing q giving rise to a splitting determined by s_f and ρ_f.

2.3. Theorem. *Let (M, q) be a quadratic module M of rank $2n$.*
(a) *There is a unique structure of a quadratic algebra on $\mathfrak{D}(q) = E_{2n}(q)$ having 1 as unit element and satisfying*

$$s_f(u) \cdot s_f(v) - \tau_f(u)s_f(v) + \gamma_f(u, v)1 = 0, \tag{1}$$

for all bilinear forms f representing q and all $u, v \in \bigwedge^{2n} M$. Norm and trace of $\mathfrak{D}(q)$ are given by

$$N(\lambda 1 + s_f(u)) = \lambda^2 + \lambda\tau_f(u) + \gamma_f(u, u), \tag{2}$$
$$T(\lambda 1 + s_f(u)) = 2\lambda + \tau_f(u). \tag{3}$$

(b) *$\mathfrak{D}(q)$ depends functorially on q with respect to similitudes: If $\varphi: (\tilde{M}, \tilde{q}) \to (M, q)$ is a similitude of quadratic modules of rank $2n$ with multiplier μ, then there exists a unique algebra homomorphism $\mathfrak{D}(\varphi): \mathfrak{D}(\tilde{q}) \to \mathfrak{D}(q)$ such that, with the notations of 1.5, $s_{\varphi^* f}(u) \mapsto s_f(\Phi(u))$ for all $u \in \bigwedge^{2n} \tilde{M}$.*
(c) *$\mathfrak{D}(q)$ commutes with change of base ring: If $\tilde{k}$ is a commutative k-algebra then there is a natural isomorphism $\mathfrak{D}(q) \otimes \tilde{k} \cong \mathfrak{D}(q \otimes \tilde{k})$.*
(d) *The discriminants of q and N are related by the formula*

$$\delta_q(u, v) = \delta_N(1 \wedge s_f(u), 1 \wedge s_f(v)). \tag{4}$$

Proof. (a) Fix a bilinear form f representing q. By 2.2, the map $(\lambda, u) \mapsto \lambda 1 + s_f(u)$ is a module isomorphism between $k \times \bigwedge^{2n} M$ and $\mathfrak{D}(q)$. Hence the right hand side of (2) defines a quadratic form N_f on $\mathfrak{D}(q)$. If $g \sim f$ then by (2) and (3) of 1.6,

$$\begin{aligned}N_f(\lambda 1 + s_g(u)) &= N_f\big((\lambda - \kappa_{fg}(u))1 + s_f(u)\big)\\ &= \lambda^2 - 2\lambda\kappa_{fg}(u) + \kappa_{fg}(u)^2 + (\lambda - \kappa_{fg}(u))\tau_f(u) + \gamma_f(u,u)\\ &= \lambda^2 + \lambda\tau_g(u) + \gamma_g(u,u) = N_g\big(\lambda 1 + s_g(u)\big),\end{aligned} \tag{5}$$

whence $N = N_f$ is independent of the choice of f. The bilinear form F defined by $F(1,1) = 1$, $F(s_f(u), 1) = \tau_f(u)$, $F(1, s_f(u)) = 0$, $F(s_f(u), s_f(v)) = \gamma_f(u,v)$ represents N. Now it follows from 2.1.1 that the quadratic algebra determined by N is given by (1), and (3) is an immediate consequence of (2).

(b) By 2.2 it is clear that there exists a unique module homomorphism $h\colon \mathfrak{D}(\tilde{q}) \to \mathfrak{D}(q)$ as indicated. We show that h is independent of the choice of f. Indeed, if g also represents q then by 1.6.5 and 2.2.2,

$$\begin{aligned}h(s_{\varphi^* g}(u)) &= h\big(s_{\varphi^* f}(u) - \kappa_{\varphi^* f, \varphi^* g}(u)1\big) = s_f(\Phi(u)) - \kappa_{\varphi^* f, \varphi^* g}(u)1\\ &= s_f(\Phi(u)) - \kappa_{fg}(\Phi(u))1 = s_g(\Phi(u)),\end{aligned}$$

proving our assertion. To show that h is an algebra homomorphism, it suffices to verify that h preserves norms, which is an easy consequence of 1.6.6.

(c) This follows from the fact that the exterior powers and the linear and bilinear forms τ_f, γ_f, and κ_{fg} commute in the obvious sense with change of base ring.

(d) By (2) and the remarks at the beginning of 1.7 the polar form $N(x,y)$ of N satisfies $N(s_f(u), s_f(v)) = \gamma_f(u,v) + \gamma_f(v,u) = 2\gamma_f(u,v)$. Now we use 1.7.1:

$$\begin{aligned}\delta_N(1 \wedge s_f(u), 1 \wedge s_f(v)) &= -\det\begin{pmatrix} N(1,1) & N(1, s_f(v)) \\ N(1, s_f(u)) & N(s_f(u), s_f(v))\end{pmatrix}\\ &= -\det\begin{pmatrix} 2 & \tau_f(v) \\ \tau_f(u) & 2\gamma_f(u,v)\end{pmatrix} = \delta_q(u,v).\end{aligned}$$

2.4. Corollary. *Let (M,q) be a quadratic module of rank $2n$.*
(a) *Let $\varphi \in \mathrm{GO}(q)$, the group of similitudes of $q = [f]$, with multiplier $\mu \in k^\times$. Then with the notation of* 1.5,

$$\mathfrak{D}(\varphi)(s_f(u)) = \kappa_{f,\varphi^* f}(u)1 + \mu^n \det(\varphi) s_f(u). \tag{1}$$

(b) *Let q be nonsingular and suppose that q admits a representative f such that $f - f^T$ is nonsingular. Then $\tau_f\colon \bigwedge^{2n} M \to k$ is an isomorphism, and the Dickson homomorphism* $\mathrm{dick}\colon \mathrm{GO}(q) \to \mathbf{Z}_2(k)$ *is given by the formula*

$$\mathrm{dick}(\varphi) = \kappa_{f,\varphi^* f}(e) \tag{2}$$

where $\tau_f(e) = 1$.

Proof. (a) We have $\Phi(u) = \mu^n \det(\varphi) u$, for $u \in \bigwedge^{2n} M$. Hence, by 2.2.2 and 2.3(c),

$$\begin{aligned}\mathfrak{D}(\varphi)(s_f(u)) &= \mathfrak{D}(\varphi)(\kappa_{f,\varphi^* f}(u)1 + s_{\varphi^* f}(u))\\ &= \kappa_{f,\varphi^* f}(u)1 + \mu^n \det(\varphi) s_f(u).\end{aligned}$$

(b) When M is free and $f - f^T = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$, it is clear that $\tau_f = \pi_n(f - f^T)$ is an isomorphism between $\bigwedge^{2n} M$ and k. The general case follows by descent and [9, I, §4]. By 2.3.4, q is nonsingular if and only if $\mathfrak{D}(q)$ is separable. It is well-known that the automorphism group of a separable quadratic algebra with conjugation σ is isomorphic to $\mathbf{Z}_2(k)$, the group of idempotents of k with multiplication $\varepsilon \bullet \varepsilon' = \varepsilon + \varepsilon' - 2\varepsilon\varepsilon'$, under the map $\varepsilon \mapsto \alpha(\varepsilon) = (1 - \varepsilon)\mathrm{Id} + \varepsilon\sigma$. The Dickson homomorphism is $\mathrm{dick}(\varphi) = \alpha^{-1}(\mathfrak{D}(\varphi))$ (this is consistent with the usual definition in terms of the center of the even Clifford algebra by Theorem 3.14(c)). By 2.3.3, $\sigma(s_f(u)) = \tau_f(u)1 - s_f(u)$. Hence $\alpha(\varepsilon)(s_f(e)) = \varepsilon\tau_f(e)1 + (1 - 2\varepsilon)s_f(e)$. Now (2) follows from this and (1) by comparison.

We remark that representatives of the type required in (b) obviously exist for the split form on k^{2n} and also if k is local or $2 = 0$ in k, but not in general since $\bigwedge^{2n} M$ need not be free.

2.5. Definition. The *discriminant algebra* of a quadratic module (M, q) of even rank $r = 2n$ is the even (trivially graded) quadratic algebra $\mathfrak{D}(q)$ of Theorem 2.3. If M has odd rank $r = 2n+1$, the discriminant algebra is the odd quadratic algebra $\mathfrak{D}(q) = \mathfrak{D}^0(q) \oplus \mathfrak{D}^1(q)$ defined by

$$\mathfrak{D}^0(q) = k.1, \quad \mathfrak{D}^1(q) = \bigwedge^r M, \quad uv = \delta_q(u, v)1 \tag{1}$$

for $u, v \in \mathfrak{D}^1(q)$, where δ_q is the discriminant of q as in 1.7. Hence norm, trace and standard involution are given by

$$N(u) = -\delta_q(u, u), \quad T(u) = 0, \quad \sigma(u) = -u \quad (u \in \mathfrak{D}^1(q)). \tag{2}$$

If M does not have constant rank, then $k = k_1 \times \ldots \times k_s$ decomposes as a direct product of rings such that $(M_i, q_i) = (M, q) \otimes k_i$ has constant rank r_i. Then $\mathfrak{D}(q)$ is the graded quadratic k-algebra $\mathfrak{D}(q_1) \times \ldots \times \mathfrak{D}(q_s)$. From 2.3(b) and 1.6(c) it follows that $\mathfrak{D}(q)$ is a functor from quadratic modules of the same rank to quadratic graded algebras, and it is easy to see that it commutes with change of base ring.

2.6. Example. Let us compute $\mathfrak{D}(q)$ for a hyperbolic module $M = P \oplus P^*$ where P is finitely generated and projective of rank n, with the bilinear form $f(x \oplus \alpha, y \oplus \beta) = \beta(x)$, and the quadratic form $q = [f]$. We claim that $\gamma_f = \gamma_f^{(2n)} = 0$ and $\tau_f = \tau_f^{(2n)}\colon L = \bigwedge^{2n} M \to k$ is an isomorphism. Indeed, by localization we may assume P free, say with basis $x_1, \ldots, x_n$. Let $\alpha_i = x_{n+i}$ $(i = 1, \ldots, n)$ be the dual basis of P^* and $\mathbf{x} = (x_1, \ldots, x_{2n})$. Then $f(\mathbf{x}, \mathbf{x}) = \begin{pmatrix} 0 & 1_n \\ 0 & 0 \end{pmatrix}$. Since $u = x_1 \wedge \ldots \wedge x_{2n}$ is a basis of L, it follows from 1.2.6 and 1.3.9 that $\gamma_f(u, u) = 0$ and hence $\gamma_f = 0$, and by 1.4.1 and 1.5.1, $\tau_f(u) = \mathrm{Pf}\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} = 1$. Thus if we identify k with L via τ_f and L with $s_f(L)$, then 2.3.1 shows that $\mathfrak{D}(q) \cong k \times k$ is a split quadratic algebra.

2.7. Free quadratic modules. Let M be free with basis $e_1, \ldots, e_r$ and let $A = (f(e_i, e_j))$ be the matrix of a bilinear form f representing q. Then $\mathfrak{D}(q)$ is isomorphic to the free quadratic algebra with generators 1 and z and the relation

$$z^2 = \begin{cases} \mathrm{Pf}(A - A^T)z + (-1)^n \,\mathrm{qdet}(A) & \text{if } r = 2n \text{ is even,} \\ (-1)^n \,\mathrm{hdet}(A) & \text{if } r = 2n+1 \text{ is odd.} \end{cases} \tag{1}$$

Indeed, $\bigwedge^r M$ is free with basis $e = e_1 \wedge \ldots \wedge e_r$. In the even rank case, set $z = s_f(e)$. Then the assertion follows from 2.3.1 and the fact that $\tau_f(e) = \mathrm{Pf}(A - A^T)$ and $\gamma_f(e,e) = (-1)^{n+1} \mathrm{qdet}(A)$ by 1.4.1, 1.5.1 and 1.3.9. For odd r, we let $z = e$ and use 1.3.6 and 2.5.1 above.

We now show that $\mathfrak{D}(q)$ is isomorphic to Haag's discriminant algebra. Let $R = \mathbb{Z}[a_i, b_{ij}]$ where the a_i and b_{ij} $(1 \le i < j \le r)$ are indeterminates. The generic quadratic form of rank r is $q_r = \sum a_i \xi_i^2 + \sum_{i<j} b_{ij}\xi_i\xi_j$. Define the matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} b_{ij} & \text{if } i < j, \\ a_i & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Haag's discriminant algebra $D(q_r)$ [5, Satz 1] is the free quadratic algebra $R \oplus Rz_r$ with the relation $z_r^2 = bz_r + c$, where $b = \mathrm{Pf}(A - A^T)$ and $b^2 + 4c = (-1)^n \det(A + A^T)$ if $r = 2n$, while $b = 0$ and $c = (-1)^n \,\mathrm{hdet}(A)$ in case $r = 2n+1$. Let f be the bilinear form on R^r determined by $f(e_i, e_j) = a_{ij}$. Then f represents q_r and thus $D(q_{2n+1}) \cong \mathfrak{D}(q_{2n+1})$ is clear from (1). In the even rank case, we have $4c = 4(-1)^n \,\mathrm{qdet}(A)$ by 1.1.7, and since R has no 2-torsion, $c = (-1)^n \,\mathrm{qdet}(A)$, whence $D(q_{2n}) \cong \mathfrak{D}(q_{2n})$ by (1). For an arbitrary free quadratic module, Haag defines $D(q)$ by specialization. Since our discriminant algebra commutes with change of base ring, it follows again that the definitions are consistent.

2.8. The product of quadratic algebras. Let S_i be graded quadratic algebras with standard involutions σ_i and $S_1 \hat{\otimes} S_2$ their graded tensor product. If the algebras are separable, their product $S_1 \,\square\, S_2$ is the fixed point set of $\sigma_1 \hat{\otimes} \sigma_2$ [9, III, (6.2)]. The discriminant algebras of quadratic modules are in general not separable, so we need to extend this definition to the nonseparable case. Trivial examples show that in general the fixed point set of $\sigma_1 \hat{\otimes} \sigma_2$ is not even a rank 2 module; a different definition is therefore required. By decomposing the spectrum of the base ring, it suffices to consider the case where the odd part S_i^1 has rank 0 or 1, i.e., where S_i is even or odd.

Let first both S_1 and S_2 be even, and let N_i and T_i be their norm and trace forms. Then the product $S_1 \,\square\, S_2$ is the even quadratic algebra determined as in 2.1 from the quadratic form $N_1 \,\square\, N_2$ constructed in [10, Theorem 2.4]. We quickly recall this construction and refer to [10, section 2] for details and proofs. Consider the exact sequences

$$0 \longrightarrow k \longrightarrow S_i \xrightarrow{\text{can}} \dot{S}_i \longrightarrow 0 \tag{1}$$

where $\dot{S}_i = S_i/k.1$. A linear form α_i on S_i is called unital if $\alpha_i(1) = 1$, i.e., if it determines a splitting of (1). We denote the corresponding section of can by s_{α_i}, so that $\mathrm{Ker}\,\alpha_i = \mathrm{Im}\,s_{\alpha_i}$. For unital linear forms α_i and β_i the formula $c_{\alpha_i\beta_i}(\dot{x}) = -\alpha_i(x) + \beta_i(x)$ defines a linear form on $\dot{S}_i$. Consider the linear and quadratic forms t_{α_i} and q_{α_i} on $\dot{S}_i$ given by

$$t_{\alpha_i} = T_i \circ s_{\alpha_i}, \quad q_{\alpha_i} = N_i \circ s_{\alpha_i}. \tag{2}$$

For pairs $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ of unital linear forms, the linear form $c_{\alpha\beta}$ on $\dot{S}_1 \otimes \dot{S}_2$ defined by

$$c_{\alpha\beta} = c_{\alpha_1\beta_1} \otimes t_{\alpha_2} + t_{\alpha_1} \otimes c_{\alpha_2\beta_2} - 2c_{\alpha_1\beta_1} \otimes c_{\alpha_2\beta_2} \tag{3}$$

satisfies the cocycle relation $c_{\alpha\beta} + c_{\beta\gamma} = c_{\alpha\gamma}$. The module underlying $S_1 \,\Box\, S_2$ is presented by generators 1 and $u_1 \,\Box_\alpha\, u_2$ and the relations saying that $u_1 \,\Box_\alpha\, u_2$ is bilinear in u_1 and u_2, and

$$u_1 \,\Box_\alpha\, u_2 - u_1 \,\Box_\beta\, u_2 = c_{\alpha\beta}(u_1 \otimes u_2)1, \tag{4}$$

for all $u_i \in \dot{S}_i$ and all pairs of unital linear forms α, β. Similar to the situation in 2.2, there is a split exact sequence

$$0 \longrightarrow k \longrightarrow S_1 \,\Box\, S_2 \xrightarrow{p} \dot{S}_1 \otimes \dot{S}_2 \longrightarrow 0 \tag{5}$$

where $p(1) = 0$ and $p(u_1 \,\Box_\alpha\, u_2) = u_1 \otimes u_2$, and any α yields a section s_α of p by $u_1 \otimes u_2 \mapsto u_1 \,\Box_\alpha\, u_2$. Now define linear and quadratic forms t_α and q_α on $\dot{S}_1 \otimes \dot{S}_2$ by

$$t_\alpha = t_{\alpha_1} \otimes t_{\alpha_2}, \quad q_\alpha = q_{\alpha_1} \otimes t_{\alpha_2}^{\langle 2 \rangle} + t_{\alpha_1}^{\langle 2 \rangle} \otimes q_{\alpha_2} - 2q_{\alpha_1} \otimes q_{\alpha_2}. \tag{6}$$

Then the norm form $N = N_1 \,\Box\, N_2$ and its trace T are given by

$$\begin{aligned} N(\lambda 1 + s_\alpha(u)) &= \lambda^2 + \lambda t_\alpha(u) + q_\alpha(u), \\ T(\lambda 1 + s_\alpha(u)) &= 2\lambda + t_\alpha(u). \end{aligned} \tag{7}$$

In view of 2.1.1 this suffices to determine the algebra $S_1 \,\Box\, S_2$.

If S is any quadratic algebra, we denote by d_S the bilinear form on $\dot{S} = S/k.1$ given by

$$d_S(\dot{x}, \dot{y}) = \begin{cases} T(x)T(y) - 2N(x,y) & \text{if } S \text{ is even,} \\ \dot{x}\dot{y} & \text{if } S \text{ is odd.} \end{cases} \tag{8}$$

Here $\dot{x} = x + k.1 \in \dot{S}$ and we identify $\dot{S}$ canonically with S^1 if S is odd. Note that in the even case d_S corresponds to the discriminant of N under the isomorphism $\dot{S} \cong \bigwedge^2 S$, $\dot{x} \mapsto 1 \wedge x$. On the other hand, an odd algebra S is entirely determined by d_S since its norm form is $\langle 1 \rangle \perp [-d_S]$.

Now we define the product of quadratic algebras S_1 and S_2 in case one of the factors is odd by $S_1 \,\Box\, S_2 = k.1 \oplus (\dot{S}_1 \otimes \dot{S}_2)$, with the norm

$$N = N_1 \,\Box\, N_2 = \langle 1 \rangle \perp (-1)^{1+\deg S_1 \deg S_2}[d_{S_1} \otimes d_{S_2}], \tag{9}$$

and let the grading be even or odd depending on whether both or only one of the factors is odd. By 2.1.1, the multiplication in $S_1 \,\Box\, S_2$ is

$$(\dot{x}_1 \otimes \dot{x}_2)(\dot{y}_1 \otimes \dot{y}_2) = \begin{cases} (\dot{x}_1\dot{y}_1)d_{S_2}(\dot{x}_2, \dot{y}_2) & \text{if } S_1 \text{ is odd and } S_2 \text{ is even,} \\ -(\dot{x}_1\dot{y}_1)(\dot{x}_2\dot{y}_2) & \text{if both } S_1 \text{ and } S_2 \text{ are odd.} \end{cases} \tag{10}$$

2.9. Proposition. *Let S_1 and S_2 be graded quadratic algebras. There is a natural homomorphism $\eta\colon S_1 \,\square\, S_2 \to \mathrm{Fix}(\sigma_1 \hat{\otimes} \sigma_2) \subset S_1 \hat{\otimes} S_2$ of unital graded algebras, given as follows.*
(a) *If S_1 and S_2 are both even then, with the notation of* 2.8,

$$\eta(u_1 \,\square_\alpha\, u_2) = s_{\alpha_1}(u_1) \hat{\otimes} t_{\alpha_2}(u_2) + t_{\alpha_1}(u_1) \hat{\otimes} s_{\alpha_2}(u_2) - 2 s_{\alpha_1}(u_1) \hat{\otimes} s_{\alpha_2}(u_2). \tag{1}$$

(b) *If S_1 is odd and S_2 is even,*

$$\eta(\dot{x}_1 \otimes \dot{x}_2) = \dot{x}_1 \hat{\otimes} \big(t_{\alpha_2}(\dot{x}_2) - 2 s_{\alpha_2}(\dot{x}_2)\big). \tag{2}$$

(c) *If both S_1 and S_2 are odd,*

$$\eta(\dot{x}_1 \otimes \dot{x}_2) = \dot{x}_1 \hat{\otimes} \dot{x}_2. \tag{3}$$

Proof. (a) By [10, Prop. 2.7(a)], there is a well-defined k-linear map η, independent of the choice of the α_i, satisfying $\eta(1) = 1 \hat{\otimes} 1$ and (1). A tedious but straightforward computation shows that η is an algebra homomorphism. Noting that $\eta(u_1 \,\square_\alpha\, u_2) = \big(\mathrm{Id} \hat{\otimes} \sigma_2 + \sigma_1 \hat{\otimes} \mathrm{Id}\big)\big(s_{\alpha_1}(u_1) \hat{\otimes} s_{\alpha_2}(u_2)\big)$, we see that η takes values in the fixed point set of $\sigma_1 \hat{\otimes} \sigma_2$.

(b) If β_2 is another unital linear form on S_2 then by [10, 1.4], $s_{\alpha_2} - s_{\beta_2} = c_{\alpha_2\beta_2}$ and $t_{\alpha_2} - t_{\beta_2} = 2c_{\alpha_2\beta_2}$, whence the right hand side of (2) is independent of the choice of α_2 and therefore η is well-defined. A computation shows that η is an algebra homomorphism and similarly as before, $\eta(\dot{x}_1 \otimes \dot{x}_2) = \big(\mathrm{Id} \hat{\otimes} \sigma_2 + \sigma_1 \hat{\otimes} \mathrm{Id}\big)\big(\dot{x}_1 \hat{\otimes} s_{\alpha_2}(\dot{x}_2)\big)$, so that η takes values in $\mathrm{Fix}(\sigma_1 \hat{\otimes} \sigma_2)$.

(c) This is immediate from the definitions.

We now show that the product $\square$ is naturally associative and commutative and admits the even algebra $E = k \times k$ as neutral element, more precisely:

2.10. Proposition. *There are natural isomorphisms*

$$(S_1 \,\square\, S_2) \,\square\, S_3 \cong S_1 \,\square\, (S_2 \,\square\, S_3), \tag{1}$$
$$S_1 \,\square\, S_2 \cong S_2 \,\square\, S_1, \tag{2}$$
$$S \,\square\, E \cong S \cong E \,\square\, S, \tag{3}$$

satisfying the compatibility axioms of [8] *and thus making the category of graded quadratic algebras a symmetric tensor category.*

Proof. By the usual decomposition of the base ring, it suffices to assume that the S_i are either even or odd. When all S_i are even the proposition is a special case of [10, Theorem 2.6(a)], so we assume that at least one factor is odd. For odd S we have $S \square E = k.1 \oplus (\dot{S} \otimes \dot{E})$ with norm $\langle 1 \rangle \perp [d_S \otimes d_E]$. Let $e = (1, 0) \in E$. Then $(\dot{E}, d_E) \cong (k, \langle 1 \rangle)$ by mapping $\dot{e} \mapsto 1$ [10, 1.2], and this induces the first of the isomorphisms (3), the second being analogous. The symmetries (2) are defined by $u_1 \,\square_{(\alpha_1,\alpha_2)}\, u_2 \mapsto u_2 \,\square_{(\alpha_2,\alpha_1)}\, u_1$ in case both S_1 and S_2 are even [10, 2.6.10], and by

$$u_1 \otimes u_2 \mapsto (-1)^{\deg u_1 \deg u_2} u_2 \otimes u_1 \tag{4}$$

when one factor is odd.

Next, we establish associativity. Setting $d_i = d_{S_i}$, we have the formula

$$d_{S_i \Box S_j} \cong \begin{cases} d_i \otimes d_j & \text{if } S_i \text{ or } S_j \text{ is even,} \\ -4d_i \otimes d_j & \text{if both } S_i \text{ and } S_j \text{ are odd.} \end{cases} \tag{5}$$

When both factors are even this is [10, Theorem 2.6(b)], otherwise it follows from 2.8.8 and 2.8.9. By listing all possibilities, one sees that the required isomorphism (1) is a consequence of (5) and the natural associativity of the tensor product of bilinear forms, $(d_1 \otimes d_2) \otimes d_3 \cong d_1 \otimes (d_2 \otimes d_3)$, except when two adjacent factors are odd and the third is even. Let, say, S_1 and S_2 be odd and S_3 be even. By definition, $S = S_1 \Box S_2$ is the even algebra determined by the quadratic form $\langle 1 \rangle \perp [d_1 \otimes d_2]$ on $k.1 \oplus (\dot{S}_1 \otimes \dot{S}_2)$. Let α_{12} be the unital linear form on S with kernel $\dot{S}_1 \otimes \dot{S}_2$. Then with the notations of 2.8, we have

$$t_{\alpha_{12}} = 0, \quad q_{\alpha_{12}} = [d_1 \otimes d_2]. \tag{6}$$

Now let α_3 be any unital linear form on S_3, and set $\alpha = (\alpha_{12}, \alpha_3)$. Define $\varphi\colon S \Box S_3 \to S_1 \Box (S_2 \Box S_3) = k.1 \oplus \ \dot{S}_1 \otimes (\dot{S}_2 \otimes \dot{S}_3)$ by $1 \mapsto 1$ and $(u_1 \otimes u_2) \Box_\alpha u_3 \mapsto u_1 \otimes (u_2 \otimes u_3)$. Then φ is well-defined, independent of the choice of α_3. (Note that there is no ambiguity in the choice of α_{12} since by construction $S_1 \Box S_2$ comes equipped with the canonical unital linear form α_{12}). Indeed, if β_3 is another unital linear form on S_3 and $\beta = (\alpha_{12}, \beta_3)$ then by 2.8.3, $c_{\alpha\beta} = 0$ because $t_{\alpha_{12}} = c_{\alpha_{12}\alpha_{12}} = 0$. Now 2.8.4 shows that φ is well-defined, and it is clearly an isomorphism of k-modules. It remains to show that it is a homomorphism of algebras, equivalently, that it preserves norms and traces. By 2.8.6 and 2.8.7 and (6) above, $T((u_1 \otimes u_2) \Box_\alpha u_3) = t_{\alpha_{12}}(u_1 \otimes u_2) t_{\alpha_3}(u_3) = 0$, while $T(u_1 \otimes (u_2 \otimes u_3)) = 0$ by 2.8.9. Similarly,

$$\begin{aligned} N((u_1 \otimes u_2) \Box_\alpha u_3) &= q_{\alpha_{12}}(u_1 \otimes u_2) t_{\alpha_3}(u_3)^2 - 4q_{\alpha_{12}}(u_1 \otimes u_2) q_{\alpha_3}(u_3) \\ &= d_1(u_1, u_1) d_2(u_2, u_2) \big(t_{\alpha_3}(u_3)^2 - 4q_{\alpha_3}(u_3)\big) \end{aligned}$$

while $N(u_1 \otimes (u_2 \otimes u_3)) = d_1(u_1, u_1) d_2(u_2, u_2) d_3(u_3, u_3)$. Since

$$d_3(u_3, u_3) = T_3(s_{\alpha_3}(u_3))^2 - 4N_3(s_{\alpha_3}(u_3)) = t_{\alpha_3}(u_3)^2 - 4q_{\alpha_3}(u_3)$$

by 2.8.2, we have equality. This establishes the isomorphisms (1), and a tedious but straightforward verification shows that they satisfy the compatibility conditions of [8] defining a symmetric tensor category.

2.11. Theorem. *The discriminant algebra functor $\mathfrak{D}$ is multiplicative, more precisely: There are natural isomorphisms*

$$\vartheta : \mathfrak{D}(q_1) \Box \mathfrak{D}(q_2) \to \mathfrak{D}(q_1 \perp q_2), \tag{1}$$

$$\vartheta_0 : E \to \mathfrak{D}(0), \tag{2}$$

which make $\mathfrak{D}$ a symmetric tensor functor from quadratic modules (with $\perp$) to graded quadratic algebras (with $\Box$).

Proof. By the usual localization argument, it suffices to establish (1) when the (M_i, q_i) are quadratic modules of constant rank r_i. We use the notations of 1.8 and distinguish the following cases.

(a) *r_1 and r_2 are even.* Let $\alpha_i = \rho_{f_i}: \mathfrak{D}(q_i) \to k$ be the unital linear forms defined in 2.2.3, set $\alpha = (\alpha_1, \alpha_2)$ and define ϑ by

$$\vartheta(1) = 1, \quad \vartheta(u_1 \,\Box_\alpha\, u_2) = s_f(u_1 \otimes u_2) \quad (u_i \in L_i). \tag{3}$$

Then ϑ is independent of the choice of representatives f_i of q_i. Indeed, let β_i and β be analogously constructed from g_i. Since $s_{f_i} = s_{\alpha_i}: L_i \to \mathfrak{D}(q_i)$ and therefore

$$t_{\alpha_i} = \tau_{f_i}, \quad q_{\alpha_i} = [\gamma_{f_i}], \tag{4}$$

we have $c_{\alpha_i\beta_i}(u_i) = -\alpha_i(s_{f_i}(u_i)) + \beta_i(s_{f_i}(u_i)) = \kappa_{f_i g_i}(u_i)$ by 2.8 and 2.2.3. Hence 2.8.3 and 1.8.2 imply $\kappa_{fg} = c_{\alpha\beta}$. Now 2.8.4 and 2.2.2 show that ϑ is independent of the choice of f_i, and it is clearly an isomorphism of k-modules. Further, (4), 1.8.1 and 2.8.6 show that it preserves norms and traces and is therefore an algebra isomorphism. Observe here that $(q_{\alpha_1} \otimes q_{\alpha_2})(u_1 \otimes u_2) = 2q_{\alpha_1}(u_1)q_{\alpha_2}(u_2)$ by definition of the tensor product of quadratic forms and therefore $2q_{\alpha_1} \otimes q_{\alpha_2} = 4[\gamma_{f_1} \otimes \gamma_{f_2}]$.

(b) *$r_1 + r_2$ is odd.* First note that

$$d_{\mathfrak{D}(q_i)} = \delta_{q_i} \tag{5}$$

which follows from 1.7.1 and 2.5.1. Now 1.8.3 shows that $\vartheta = \mathrm{Id}$ is an isomorphism.

(c) *r_1 and r_2 are odd.* Then (5) shows $\mathfrak{D}(q_1) \,\Box\, \mathfrak{D}(q_2) = k.1 \oplus (L_1 \otimes L_2)$ with norm $\langle 1 \rangle \perp [\delta_{q_1} \otimes \delta_{q_2}]$. Define ϑ by

$$\vartheta(1) = 1, \quad \vartheta(u_1 \otimes u_2) = s_f(u_1 \otimes u_2).$$

Then 1.8.5 shows ϑ is independent of the choice of f_i, and by 1.8.4 it preserves norms and traces. In all three cases it is easy to see that ϑ depends functorially on q_1 and q_2.

For $M = \{0\}$, $q = 0$ we have $\bigwedge^0 M = k$ and $f = 0$ is the unique bilinear form representing q. Define ϑ_0 by $\vartheta_0((1,0)) = s_f(1)$, and $\vartheta_0((1,1)) = 1$. Then ϑ_0 is a module isomorphism. Since the Pfaffian and the quarter-determinant of an empty matrix are 1 and 0, respectively, we have $\tau_f(1) = 1$ and $\gamma_f = 0$, whence ϑ_0 is an algebra isomorphism.

To complete the proof, we must show that $\mathfrak{D}$ is compatible with the associativity, commutativity and unity constraints of 2.10. Let us check this for commutativity, leaving the other cases to the persistent reader. We must show that the diagram

$$\begin{tikzcd}
\mathfrak{D}(q_1) \,\Box\, \mathfrak{D}(q_2) \arrow[r, "\vartheta"] \arrow[d] & \mathfrak{D}(q_1 \perp q_2) \arrow[d, "\mathfrak{D}(\omega)"] \\
\mathfrak{D}(q_2) \,\Box\, \mathfrak{D}(q_1) \arrow[r, "\vartheta"'] & \mathfrak{D}(q_2 \perp q_1)
\end{tikzcd}$$

commutes, where the left vertical arrow is as in 2.10.2 and the right one is induced from the switch $\omega: M_1 \oplus M_2 \to M_2 \oplus M_1$. Let $x_1^{(i)}, \ldots, \ldots x_{r_i}^{(i)} \in M_i$ and $u_i = x_1^{(i)} \wedge \ldots \wedge x_{r_i}^{(i)} \in$

$L_i = \bigwedge^{r_i} M_i$. Let $r = r_1 + r_2$ and identify $L_1 \otimes L_2$ with $\bigwedge^r(M_1 \oplus M_2)$ via $u_1 \otimes u_2 \mapsto x_1^{(1)} \wedge \ldots \wedge x_{r_1}^{(1)} \wedge x_1^{(2)} \wedge \ldots \wedge x_{r_2}^{(2)}$ and similarly $L_2 \otimes L_1$ with $\bigwedge^r(M_2 \oplus M_1)$ via $u_2 \otimes u_1 \mapsto x_1^{(2)} \wedge \ldots \wedge x_{r_2}^{(2)} \wedge x_1^{(1)} \wedge \ldots \wedge x_{r_1}^{(1)}$. Then the map $\bigwedge^r \omega$ is given by $u_1 \otimes u_2 \mapsto (-1)^{r_1 r_2} u_2 \otimes u_1$. Now it is easy to check the commutativity of the diagram from the definitions. Note that the sign occurring in 2.10.4 is essential for this.

3. Mapping the discriminant algebra into the Clifford algebra

3.1. Bourbaki's λ-operation. Let M be a k-module, q a quadratic form on M and $C(q)$ the Clifford algebra of q. Since we assume that q is represented by some bilinear form (cf. 1.5), the canonical map $M \to C(q)$ is injective and we may identify M with its image in $C(q)$. Now let f be a bilinear form on M and consider the quadratic form $q' = q + [f]$. By [1, §9.3, Prop. 3] there exists a unique isomorphism

$$\lambda_f : C(q') \to C(q) \tag{1}$$

of k-modules, compatible with the $\mathbb{Z}/2\mathbb{Z}$-gradings and the filtrations, such that

$$\lambda_f(1) = 1, \qquad \lambda_f(xu) = x\lambda_f(u) + i_x^f(\lambda_f(u)), \tag{2}$$

for all $x \in M$, $u \in C(q')$. Here i_x^f denotes the graded derivation of odd degree of $C(q)$ determined by

$$i_x^f(1) = 0, \qquad i_x^f(y) = f(x, y), \tag{3}$$

for all $x, y \in M$ [1, §9.2, Lemma 1]. We also recall that λ_f depends exponentially on f in the sense that

$$\lambda_0 = \mathrm{Id}, \qquad \lambda_{f+g} = \lambda_f \circ \lambda_g : C(q'') \to C(q), \tag{4}$$

for a second bilinear form g and $q''(x) = q'(x) + g(x, x)$. Furthermore, λ_f is compatible with orthogonal sums: If $q_i' = q_i + [f_i]$ then the diagram

$$\begin{tikzcd}
C(q_1') \hat{\otimes} C(q_2') \arrow[r, "\lambda_{f_1} \hat{\otimes} \lambda_{f_2}"] \arrow[d] & C(q_1) \hat{\otimes} C(q_2) \arrow[d] \\
C(q_1' \perp q_2') \arrow[r, "\lambda_{f_1 \perp f_2}"'] & C(q_1 \perp q_2)
\end{tikzcd} \tag{5}$$

is commutative, where the vertical arrows are the well-known isomorphisms [9, IV, (1.3.1)]. This is easily proved from the definitions.

We derive next an explicit formula for λ_f. Let $I = \{1, \ldots, m\}$, let $x_1, \ldots, x_m \in M$ and let $A = (a_{ij})$ be the alternating $m \times m$ matrix satisfying $a_{ij} = f(x_i, x_j)$ for $i < j$. For any subset $J = \{j_1, \ldots, j_r\}$ of I, $j_1 < \ldots < j_r$, let $x_J = x_{j_1} \ldots x_{j_r}$ and

$$A_J = (a_{ij})_{(i,j) \in J \times J}.$$

Finally, if J and L are subsets of I we set $\varepsilon(J, L) = 0$ if J and L are not disjoint, and $(-1)^\nu$ otherwise, where ν is the number of pairs $(j, l) \in J \times L$ such that $j > l$.

3.2. Proposition. *With the above notations, λ_f is given by the formula*

$$\lambda_f(x_I) = \sum_{K\subset I,\ |K| \text{ even}} \varepsilon(K, I\text{-}K)\,\mathrm{Pf}(A_K)\,x_{I\text{-}K}\,.$$

The proof is by induction on the cardinality $|I|$ of I, the case $I = \emptyset$ being trivial. For the induction step, suppose the formula holds for $I = \{1, \ldots, m\}$ and let $I' = \{0\} \cup I$. Also set $w = \lambda_f(x_I)$ and write $i_x = i^f_x$ for simplicity. Then

$$\begin{aligned}\lambda_f(x_{I'}) &= \lambda_f(x_0x_I) = x_0w + i_{x_0}w\\ &= \sum_{K\subset I,\ |K| \text{ even}} \varepsilon(K, I\text{-}K)\,\mathrm{Pf}(A_K)\,x_0x_{I\text{-}K} \ + \ i_{x_0}w.\end{aligned} \tag{1}$$

Here $\varepsilon(K, I\text{-}K) = \varepsilon(K, I'\text{-}K)$. Indeed, the $\varepsilon(J, L)$ are the structure constants of the exterior algebra of the free $\mathbb{Z}$-module with basis $e_0, \ldots, e_m$, i.e., $e_J \wedge e_L = \varepsilon(J, L)e_{J\cup L}$ [2, III, §7.8, (20)]. Since $|K|$ is even, e_K is central in this exterior algebra. Hence $\varepsilon(K, I\text{-}K)e_{I'} = \varepsilon(K, I\text{-}K)e_0 \wedge e_I = e_0 \wedge e_K \wedge e_{I\text{-}K} = e_K \wedge e_0 \wedge e_{I\text{-}K} = e_K \wedge e_{I'\text{-}K} = \varepsilon(K, I'\text{-}K)e_{I'}$. Now (1) becomes

$$\lambda_f(x_{I'}) = \sum_{0\notin K\subset I',\ |K| \text{ even}} \varepsilon(K, I'\text{-}K)\,\mathrm{Pf}(A_K)\,x_{I'\text{-}K} \ + \ i_{x_0}w,$$

and it remains to show

$$i_{x_0}w = \sum_{0\in K\subset I',\ |K| \text{ even}} \varepsilon(K, I'\text{-}K)\,\mathrm{Pf}(A_K)x_{I'\text{-}K}\,. \tag{2}$$

From the definition of i_{x_0} it is easy to see that

$$i_{x_0}x_J = \sum_{j\in J} \varepsilon(\{j\}, J\text{-}\{j\})\,a_{0j}\,x_{J\text{-}\{j\}}\,.$$

Hence the induction hypothesis yields

$$i_{x_0}w = \sum_{K\subset I,\ |K| \text{ even}} \varepsilon(K, I\text{-}K)\,\mathrm{Pf}(A_K) \sum_{j\in I\text{-}K} \varepsilon(\{j\}, I\text{-}(K \cup \{j\}))\,a_{0j}\,x_{I\text{-}(K\cup\{j\})}\,.$$

When K runs over all even subsets of I and $j \in I\text{-}K$ then $J = K \cup \{j\}$ runs over all odd subsets of I. Hence by rearranging the double sum, we obtain

$$i_{x_0}w = \sum_{J\subset I,\ |J| \text{ odd}} \alpha_J x_{I\text{-}J}, \tag{3}$$

where

$$\alpha_J = \sum_{j\in J} \varepsilon(J\text{-}\{j\}, I\text{-}(J\text{-}\{j\}))\varepsilon(\{j\}, I\text{-}J)a_{0j}\,\mathrm{Pf}(A_{J\text{-}\{j\}}). \tag{4}$$

We claim that

$$\varepsilon(J\text{-}\{j\}, I\text{-}(J\text{-}\{j\}))\varepsilon(\{j\}, I\text{-}J) = \varepsilon(J, I\text{-}J)\varepsilon(\{j\}, J\text{-}\{j\}). \tag{5}$$

Indeed, interpreting the $\varepsilon(J, L)$ as before, we have

$$e_j \wedge e_{J\text{-}\{j\}} \wedge e_{I\text{-}J} = \varepsilon(\{j\}, J\text{-}\{j\})e_J \wedge e_{I\text{-}J} = \varepsilon(J, I\text{-}J)\varepsilon(\{j\}, J\text{-}\{j\})e_I.$$

On the other hand, $e_{J\text{-}\{j\}}$ is an element of even degree and therefore

$$\begin{aligned} e_j \wedge e_{J\text{-}\{j\}} \wedge e_{I\text{-}J} = e_{J\text{-}\{j\}} \wedge e_j \wedge e_{I\text{-}J} &= \varepsilon(\{j\}, I\text{-}J)e_{J\text{-}\{j\}} \wedge e_{I\text{-}(J\text{-}\{j\})} \\ &= \varepsilon(J\text{-}\{j\}, I\text{-}(J\text{-}\{j\}))\varepsilon(\{j\}, I\text{-}J)e_I, \end{aligned}$$

which proves (5). From (4) and (5) and the well-known expansion formula for the Pfaffian we obtain

$$\alpha_J = \varepsilon(J, I\text{-}J)\sum_{j\in J}\varepsilon(\{j\}, J\text{-}\{j\})\, a_{0j}\ \mathrm{Pf}(A_{J\text{-}\{j\}}) = \varepsilon(J, I\text{-}J)\,\mathrm{Pf}(A_{\{0\}\cup J}). \tag{6}$$

Let $J' = \{0\} \cup J$. Then $I'\text{-}J' = I\text{-}J$ and one sees easily that $\varepsilon(J', I'\text{-}J') = \varepsilon(J, I\text{-}J)$. By substituting (6) in (3) we obtain (2) and the proof is complete.

3.3. Corollary. *Let a be an alternating bilinear form on M. Then the map $\lambda_a\colon \bigwedge M \to \bigwedge M$ satisfies*

$$\lambda_a(x_I) = \sum_{K\subset I,\ |K|\ \text{even}} \varepsilon(K, I\text{-}K)\pi_{|K|/2}(a)(x_K)\, x_{I\text{-}K}.$$

Indeed, since a is alternating, the matrix A introduced in 3.1 equals $a(\mathbf{x}, \mathbf{x})$, and $\mathrm{Pf}(A_K) = \pi_{|K|/2}(a)(x_K)$ by 1.4.1.

3.4. Let q be a quadratic form on a k-module M, represented by a bilinear form f, let t be an indeterminate and $R = k[t, t^{-1}]$ the Laurent polynomial ring. We denote the scalar extensions of f and q to $M \otimes R$ by the same letters. Then $f[t] = f - (1+t)f^T = (f - f^T) - tf^T$ is a bilinear form on $M \otimes R$, and

$$f\{t\} = t^{-1}f[t] = -f^T + t^{-1}(f - f^T)$$

represents $-q$. Hence by 3.1, $\lambda_{f\{t\}}\colon \bigwedge M \otimes R \to C(q) \otimes R$ is an isomorphism of R-modules. Now define k-linear maps $\zeta_{2n}(t, f)\colon \bigwedge^{2n} M \to C_0(q) \otimes R$ and $\zeta_{2n+1}(t, f)\colon \bigwedge^{2n+1} M \to C_1(q) \otimes R$ by the formulas

$$\tau_f(u)1 + t\zeta_{2n}(t, f)(u) = \lambda_{f\{t\}}(t^n u), \tag{1}$$

$$\zeta_{2n+1}(t, f)(v) = \lambda_{f\{t\}}(t^n v), \tag{2}$$

where τ_f is as in 1.5.1. To derive explicit formulas, let $\mathbf{x} \in M^m$ and let $A(t) = (a_{ij}(t))$ be the alternating $m \times m$-matrix such that $a_{ij}(t) = f(x_i, x_j) - (1+t)f(x_j, x_i)$ for $i < j$. Then $A(0) = (f - f^T)(\mathbf{x}, \mathbf{x})$ and $A(t) = A(0) + t\dot{A}(0)$, where $\dot{A}(0) = (\dot{a}_{ij}(0))$ is the alternating matrix satisfying $\dot{a}_{ij}(0) = f(x_i, x_j)$ for $i > j$. With the notations and conventions introduced in 3.1 and 1.4, we then have

3.5. Lemma. *Let $I = \{1, \ldots, 2n\}$ and $I' = \{0, \ldots, 2n\}$. Then*

$$\begin{aligned}
&\zeta_{2n}(t, f)(x_1 \wedge \ldots \wedge x_{2n}) = \\
&\quad \sum_{K \subsetneqq I,\ |K| \text{ even}} t^{n - \frac{|K|}{2} - 1}\, \varepsilon(K, I\text{-}K)\, \mathrm{Pf}(A(t)_K) x_{I\text{-}K} \\
&\qquad + \sum_{i=1}^{n} P_i(A(0), \dot{A}(0)) t^{i-1},
\end{aligned} \tag{1}$$

$$\begin{aligned}
&\zeta_{2n+1}(t, f)(x_0 \wedge \ldots \wedge x_{2n}) = \\
&\quad \sum_{K \subset I',\ |K| \text{ even}} t^{n - \frac{|K|}{2}}\, \varepsilon(K, I'\text{-}K)\, \mathrm{Pf}(A(t)_K) x_{I'\text{-}K}.
\end{aligned} \tag{2}$$

In particular, $\zeta_m(t, f)(w) \in C(q) \otimes k[t]$ is a polynomial in t, for all $w \in \bigwedge^m M$.

Proof. Since the Pfaffian of an alternating $2r \times 2r$-matrix is a homogeneous polynomial of degree r, we have $\mathrm{Pf}(t^{-1} A(t)_K) = t^{-\frac{|K|}{2}} \mathrm{Pf}(A(t)_K)$. Proposition 3.2 implies

$$\begin{aligned}
\lambda_{f\{t\}}(t^n x_1 \wedge \ldots \wedge x_{2n}) &= \sum_{K \subset I,\ |K| \text{ even}} \varepsilon(K, I\text{-}K) t^{n - \frac{|K|}{2}}\ \mathrm{Pf}(A(t)_K) x_{I\text{-}K} \\
&= t \left(\sum_{K \subsetneqq I,\ |K| \text{ even}} t^{n - \frac{|K|}{2} - 1}\, \varepsilon(K, I\text{-}K)\, \mathrm{Pf}(A(t)_K) x_{I\text{-}K} \right) + \mathrm{Pf}(A(t)).
\end{aligned}$$

By 1.4,

$$\begin{aligned}
\mathrm{Pf}\, A(t) &= \mathrm{Pf}\left(A(0) + t\dot{A}(0)\right) \\
&= \mathrm{Pf}\, A(0) + t \sum_{i=1}^{n} P_i(A(0), \dot{A}(0)) t^{i-1} \\
&= \tau_f(x_1 \wedge \ldots \wedge x_{2n}) + t \sum_{i=1}^{n} P_i(A(0), \dot{A}(0)) t^{i-1}.
\end{aligned}$$

Now (1) follows from the definition of ζ_{2n}. The proof of (2) is similar, and the last statement is a consequence of (1) and (2).

Remark. It should be noted that the two terms on the right hand side of (1) are not alternating functions of $x_1, \ldots, x_{2n}$ but their sum is.

3.6. Definition. By the lemma, we can specialize $t \to -2$ in $\zeta_m(t, f)$ and thus obtain a well-defined map

$$z_f = z_f^{(m)} = \zeta_m(-2, f) \colon \bigwedge^m M \to C(q),$$

with values in $C_0(q)$ or $C_1(q)$, depending on whether m is even or odd. Note that $a_{ij}(-2) = -a_{ji}(-2) = b(x_i, x_j)$ for $i < j$ where $b = f + f^T = f[-2]$ is the polar form of q. Hence 3.5.2 shows that $z_f^{(m)}$ is independent of the choice of f representing q for m odd (but not so in the even case). Therefore, we will sometimes write z_q instead of $z_f^{(2n+1)}\colon \bigwedge^{2n+1} M \to C_1(q)$. In particular, the map $z_q\colon M \to C_1(q)$ is simply the canonical embedding of M into $C(q)$, whereas $z_f\colon \bigwedge^2 M \to C_0(q)$ is given by

$$z_f(x_1 \wedge x_2) = x_1x_2 - f(x_2, x_1)1. \tag{1}$$

3.7. Proposition. *Let $M = M_1 \oplus M_2$ be an orthogonal sum of submodules and suppose the representative $f = f_1 \perp f_2$ of q decomposes accordingly. Let $u_i \in L_i = \bigwedge^{m_i} M_i$ and identify L_i with a submodule of $\bigwedge^{m_i} M$. Then for $m_i = 2n_i$ even,*

$$z_f(u_1 \wedge u_2) = z_f(u_1)\tau_f(u_2) + \tau_f(u_1)z_f(u_2) - 2z_f(u_1)z_f(u_2). \tag{1}$$

If m_1 is odd and m_2 is even,

$$z_f(u_1 \wedge u_2) = z_f(u_1)\big(\tau_f(u_2)1 - 2z_f(u_2)\big). \tag{2}$$

If m_1 and m_2 are both odd,

$$z_f(u_1 \wedge u_2) = z_f(u_1)z_f(u_2). \tag{3}$$

Proof. We only show (1) since the proof of the other formulas is similar. Clearly $f\{t\} = f_1\{t\} \perp f_2\{t\}$ decomposes orthogonally. By 3.1.5 we have, identifying $C(q)$ with $C([f_1]) \hat{\otimes} C([f_2])$ and setting $n = n_1 + n_2$,

$$\lambda_{f\{t\}}(t^n u_1 \wedge u_2) = \lambda_{f_1\{t\}}(t^{n_1}u_1) \hat{\otimes} \lambda_{f_2\{t\}}(t^{n_2}u_2) = \lambda_{f\{t\}}(t^{n_1}u_1)\lambda_{f\{t\}}(t^{n_2}u_2).$$

By substituting 3.4.1, expanding and comparing the coefficient of t, we obtain

$$\begin{aligned}\zeta_{2n}(t,f)(u_1 \wedge u_2) = \tau_f(u_1)\zeta_{2n_2}(t,f)(u_2) &+ \zeta_{2n_1}(t,f)(u_1)\tau_f(u_2)\\ &+ t\zeta_{2n_1}(t,f)(u_1)\zeta_{2n_2}(t,f)(u_2).\end{aligned}$$

Now (1) follows for $t \to -2$.

3.8. Example. Let $q = \xi_1\xi_2 + \ldots + \xi_{2n-1}\xi_{2n}$ be the standard form on k^{2n} with basis $e_1, \ldots, e_{2n}$. Then $f = f_1 \perp \ldots \perp f_n$ represents q where f_i is the bilinear form with matrix $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ on k^2. The elements $c_j = e_{2j-1}e_{2j}$ are pairwise commuting idempotents of $C(q)$ and 3.6.1. shows $z_f(e_{2j-1}\wedge e_{2j}) = c_j$. By 3.7.1 we have $z_f(e_1\wedge\ldots\wedge e_4) = c_1+c_2-2c_1c_2 = c_1\bullet c_2$, the well-known product of commuting idempotents which is again an idempotent. Hence induction yields

$$z_f(e_1 \wedge \ldots \wedge e_{2n}) = c_1 \bullet \ldots \bullet c_n = c, \qquad c^2 = c. \tag{1}$$

Now let $q' = \xi_0^2 \perp q$ on k^{2n+1} with basis $e_0, \ldots, e_{2n}$, represented by $f' = \langle 1 \rangle \perp f$, and set $e = e_1 \wedge \ldots \wedge e_{2n}$ and $e' = e_0 \wedge e$. By 1.5.1, $\tau_{f'}(e) = \operatorname{Pf}\operatorname{diag}\big(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\big) = 1$, so that 3.7.2 implies

$$z_{f'}(e') = z_{f'}(e_0)(1 - 2z_{f'}(e)) = e_0(1 - 2c). \tag{2}$$

As $e_0^2 = 1$ and e_0 commutes with the c_j and hence with c, we obtain

$$z_{f'}(e')^2 = e_0^2(1 - 4c + 4c^2) = 1. \tag{3}$$

3.9. Proposition. *Let q be a quadratic form on M, represented by bilinear forms f and g. Then for $x \in M$, $u \in \bigwedge^{2n} M$, $v \in \bigwedge^{2n+1} M$ the following relations hold:*

$$z_f(u) - z_g(u) = \kappa_{fg}(u)1, \tag{1}$$
$$z_q(x \wedge u) = \tau_f(u)x - xz_f(u) - z_f(u)x, \tag{2}$$
$$2z_f(x \wedge v) = \tau_f(x \wedge v)1 + [x, z_q(v)]. \tag{3}$$

Proof. Let again $R = k[t, t^{-1}]$ be the Laurent polynomial ring and define, for an alternating bilinear form a on $M \otimes R$, the linear map $\eta_{2n}(t, a)$: $\bigwedge^{2n} M \otimes R \to \bigwedge M \otimes R$ by

$$\pi_n(a)(u)1 + t\eta_{2n}(t, a)(u) = \lambda_{a/t}(t^n u). \tag{4}$$

From 3.3 it follows easily that $\eta_{2n}(t, a)$ is a polynomial in t whenever a is. If a' is another alternating bilinear form then

$$\lambda_{a'} \circ \eta_{2n}(t, a) = \eta_{2n}(t, a + ta') + \Pi_n(t, a, a')1. \tag{5}$$

For the proof, one replaces a by $a+ta'$ in (4) and uses 1.4.5 and 3.1.4. Now let in particular $a = f - f^T$. Then $f\{t\} = -f^T + a/t$ and $\pi_n(a) = \tau_f$. Hence by applying λ_{-f^T} to (4) we obtain

$$\zeta_{2n}(t, f) = \lambda_{-f^T} \circ \eta_{2n}(t, f - f^T). \tag{6}$$

Next, substitute $t = -2$ and $a' = f - g$ in (5). Since $g - g^T = f - a' - (f - a')^T = a - 2a'$, the result is, by 1.5.2,

$$\lambda_{a'} \circ \eta_{2n}(-2, f - f^T) = \eta_{2n}(-2, g - g^T) + \kappa_{fg}1. \tag{7}$$

Finally, we apply λ_{-g^T} to (7) and use (6) and the relation $-g^T + a' = -f^T$:

$$\begin{aligned}\zeta_{2n}(-2, f) &= \lambda_{-g^T} \circ \lambda_{a'} \circ \eta_{2n}(t, f - f^T)\\ &= \lambda_{-g^T} \circ \eta_{2n}(-2, g - g^T) + \kappa_{fg}1\\ &= \zeta_{2n}(-2, g) + \kappa_{fg}1.\end{aligned}$$

This proves (1).

Let $b = f+f^T$ be the polar form of q and let $w \in C(q)$ be a homogeneous element. Then it is easy to see that the graded derivation i_x^b of $C(q)$ is just the inner graded derivation determined by x, i.e.,

$$i_x^b(w) = xw - (-1)^{\deg w} wx. \tag{8}$$

In $C(q) \otimes R$ we have, since i_x^f is linear in f and annihilates 1,

$$\begin{aligned}\zeta_{2n+1}(t, f)(x \wedge u) &= \lambda_{f\{t\}}(t^n x \wedge u)\\ &= (x + i_x^{f\{t\}})\lambda_{f\{t\}}(t^n u)\\ &= (x + i_x^{f\{t\}})(\tau_f(u)1 + t\zeta_{2n}(t, f)(u))\\ &= \tau_f(u)x + tx\zeta_{2n}(t, f)(u) + i_x^{f[t]}\zeta_{2n}(t, f)(u).\end{aligned} \tag{9}$$

If we specialize $t \to -2$ in this formula, observe $f[-2] = f - f^T + 2f^T = b$ and use (8) we obtain (2). Similarly,

$$\begin{aligned}\tau_f(x \wedge v)1 + t\zeta_{2n+2}(t,f)(x \wedge v) &= \lambda_{f\{t\}}(t^{n+1}x \wedge v)\\ &= (x + i_x^{f\{t\}})\lambda_{f\{t\}}(t.t^n v)\\ &= (tx + i_x^{f[t]})\zeta_{2n+1}(t,f)(v) \qquad (10)\end{aligned}$$

yields (3) for $t \to -2$.

We remark that (9) and (10) give an inductive definition of $\zeta_m(t,f)$.

3.10. Corollary. (a) *For any n, there is a well-defined homomorphism of k-modules $\psi_q\colon E_{2n}(q) \to C_0(q)$ such that $\psi_q(1) = 1$ and $\psi_q(s_f(u)) = z_f(u)$ for all $u \in \bigwedge^{2n} M$.*
(b) *If M is finitely generated and projective of rank $2n$ and $u \in \bigwedge^{2n} M$ then $z_f(u)$ is central in $C_0(q)$ and satisfies*

$$xz_f(u) = \big(\tau_f(u)1 - z_f(u)\big)x, \qquad (1)$$

for all $x \in M$.
(c) *If M is finitely generated and projective of rank $2n+1$ and $v \in \bigwedge^{2n+1} M$ then $z_q(v)$ is central in $C(q)$.*

Indeed, (a) follows immediately from 3.9.1 and 2.2, while (b) is a consequence of 3.9.2 and implies, by a simple computation, that $z_f(u)$ is central in $C_0(q)$ since the latter is generated by all xy, $x, y \in M$. Similarly, (c) follows from 3.9.3.

3.11. We will need the behavior of $\zeta_m(t,f)$ under a similitude $(\varphi,\mu)\colon(\tilde{M},\tilde{q}) \to (M,q)$, cf. 1.5. It is easy to see that (φ,μ) induces a k-linear map $\Phi = C(\varphi,\mu)\colon C(\tilde{q}) \to C(q)$ of the Clifford algebras which generalizes 1.5.4 and is induced from the map $x_1 \otimes \ldots \otimes x_m \mapsto \mu^{[\frac{m}{2}]}\varphi(x_1) \otimes \ldots \otimes \varphi(x_m)$ of the tensor algebras [9, IV, (7.1.1)]. The restriction of Φ to the even Clifford algebras is an algebra homomorphism, and so is Φ provided $\mu = 1$. In any case, Φ satisfies

$$\Phi(yw) = \mu^{\deg w}\varphi(y)\Phi(w) \qquad (1)$$

for all $y \in \tilde{M}$ and homogeneous elements $w \in C(\tilde{q})$.

Let h be a bilinear form on M, let $q' = q + [h]$ and define $\tilde{h} = \mu h \circ (\varphi \times \varphi)$ and $\tilde{q}' = \tilde{q} + [\tilde{h}]$. Then (φ,μ) is a similitude between $\tilde{q}'$ and q' as well, and the diagram

$$\begin{tikzcd} C(\tilde{q}') \arrow[r, "\Phi"] \arrow[d, "\lambda_{\tilde{h}}"'] & C(q') \arrow[d, "\lambda_h"] \\ C(\tilde{q}) \arrow[r, "\Phi"'] & C(q) \end{tikzcd} \qquad (2)$$

commutes. The proof is based on the inductive definition of λ_h and makes use of the formula

$$\Phi(i_y^{\tilde{h}}w) = \mu^{\deg w}\, i^h_{\varphi(y)}(\Phi(w)). \qquad (3)$$

The details are left to the reader.

3.12. Lemma. *Let $(\varphi,\mu)\colon(\tilde M,\tilde q)\to(M,q)$ be a similitude, $q=[f]$ and $\tilde f=\mu f\circ(\varphi\times\varphi)$. Then $\Phi\circ\zeta_m(t,\tilde f)=\zeta_m(t,f)\circ\Phi$, in particular, $\Phi\circ z_{\tilde f}=z_f\circ\Phi$.*

Proof. We specialize 3.11.2 to the case $h=f\{t\}$ and $q'=0$. Then $\tilde h=\tilde f\{t\}$, and the diagram

$$\begin{tikzcd} \bigwedge\tilde M \arrow[r, "\Phi"] \arrow[d, "\lambda_{\tilde f\{t\}}"'] & \bigwedge M \arrow[d, "\lambda_{f\{t\}}"] \\ C(\tilde q) \arrow[r, "\Phi"'] & C(q) \end{tikzcd}$$

commutes, with Φ acting on the exterior algebra by $x_1\wedge\ldots\wedge x_m\mapsto\mu^{[\frac{m}{2}]}\varphi(x_1)\wedge\ldots\wedge\varphi(x_m)$. Now the first formula follows easily from the definition of $\zeta_m(t,f)$, together with 1.6(c) in case m is even, and the second is obtained by specializing $t\to-2$.

3.13. Proposition. *Let q be a quadratic form represented by a bilinear form f on M. Then for decomposable multivectors $u\in\bigwedge^m M$ the following relations hold in $C(q)$:*

$$z_f(u)^2-\tau_f(u)z_f(u)+\gamma_f(u,u)1=0\quad(u\in\textstyle\bigwedge^{2n}M),\tag{1}$$

$$z_q(u)^2=\delta_f(u,u)1\quad(u\in\textstyle\bigwedge^{2n+1}M).\tag{2}$$

Proof. We first reduce to the case where $M=k^m$ is free of rank m and u is a basis of $\bigwedge^m M$. Indeed, if $u=x_1\wedge\ldots\wedge x_m$, let $\tilde M=k^m$ with standard basis e_i, and let $\varphi\colon\tilde M\to M$ be the homomorphism mapping e_i to x_i. Set $\tilde q=q\circ\varphi$, $\tilde f=f\circ(\varphi\times\varphi)$ and $e=e_1\wedge\ldots\wedge e_m$. Then φ is a similitude with multiplier 1, and $\Phi(e)=u$. Therefore, if the asserted formulas hold for $\tilde f$, $\tilde q$ and e in place of f, q and u, they follow in general by applying Φ, because of Lemma 3.12 and (6) and (7) of 1.6.

Let $M=k^m$ and consider the k-scheme $\mathbf{B}$ of all bilinear forms on M which, via the standard basis, may be identified with the affine scheme of $m\times m$-matrices. As in [3] we consider schemes as set-valued functors on commutative k-algebras. First suppose $m=2n$ is even and define an open subscheme $\mathbf{U}\subset\mathbf{B}$ by $\mathbf{U}(R)=\{f\in\mathbf{B}(R):\det(f+f^T)\in R^\times\}$, for all k-algebras R. Since $\mathbf{B}$ is smooth with connected fibers and $\mathbf{U}(k)\neq\emptyset$, as for instance $\begin{pmatrix}0&1_n\\0&0\end{pmatrix}\in\mathbf{U}(k)$, $\mathbf{U}$ is universally schematically dense in $\mathbf{B}$ [4, Exp. XVIII, sec. 1].

Let $f\in\mathbf{B}(R)$. Via the standard basis, we identify the R-module underlying $C([f])$ with the exterior algebra over R^m. Then the structure constants of $C([f])$ are polynomials in (the entries of) f and, therefore, so is the expression

$$p(f)=z_f(e)^2-\tau_f(e)z_f(e)+\gamma_f(e,e)1.\tag{3}$$

Since all constructions are compatible with arbitrary base ring extension, (3) defines a morphism p from $\mathbf{B}$ to the affine k-scheme of the exterior algebra of k^m (which may be identified with affine 2^m-space). Formula (1) is equivalent to the vanishing of p on $\mathbf{B}$, and since $\mathbf{U}$ is universally schematically dense, it suffices that p vanish on $\mathbf{U}$.

Thus let now $f \in \mathbf{U}(R)$. Then $[f]$ is nonsingular and therefore splits over some faithfully flat extension S of R [9, IV, (2.2.1)]. By faithfully flat descent, it suffices to prove $p(f) \otimes 1_S = 0$. After extending the base ring from k to S we may assume $k = S$ to simplify notation, and after a change of basis in k^m, which affects e by a unit and therefore $p(f)$ by the square of that unit, we may assume that $[f]$ is the standard form $\xi_1\xi_2 + \ldots + \xi_{2n-1}\xi_{2n}$. This means that $f \sim g = \mathrm{diag}(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right))$. A straightforward computation with 3.9.1 and 1.6(c) shows that $p(f) = p(g)$ for $f \sim g$. We may therefore even assume $f = g$. We compute $\tau_f(e) = \mathrm{Pf}(f - f^T) = 1$, and $\gamma_f(e,e) = (-1)^{n+1}\,\mathrm{qdet}(f) = 0$ by 1.2.4 and 1.3.9. Now 3.8.1 shows $p(f) = 0$, and we have established (1).

Finally, let $m = 2n+1$ be odd. With $\mathbf{B}$ as above, consider the open subscheme $\mathbf{U}' \subset \mathbf{B}$ of all f' for which $\mathrm{hdet}\, f'$ is invertible. Again $\mathbf{U}'$ is universally schematically dense. For $f' \in \mathbf{U}'(R)$ the quadratic form $[f']$ is semiregular and therefore splits over a faithfully flat extension [9, IV, (3.2.1)]. Arguing as before, the validity of (2) needs to be shown only for $u = e' = e_0 \wedge \ldots \wedge e_{2n}$ and the standard form $q' = \xi_0^2 + \xi_1\xi_2 + \ldots + \xi_{2n-1}\xi_{2n}$, represented by $f' = \mathrm{diag}(1, \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right))$. Since $\delta_{f'}(e', e') = 1$ by 1.2.5 and 1.3.6 this follows from 3.8.3.

3.14. Theorem. *Let (M, q) be a quadratic module of rank r.*
(a) *The map $\psi_q: \mathfrak{D}(q) \to C(q)$ defined as in* 3.10(a) *if $r = 2n$ and by $\psi_q(1) = 1$ and $\psi_q(u) = z_q(u)$ for all $u \in \bigwedge^r M$ if $r = 2n+1$, is a homomorphism of unital graded algebras.*
(b) *$\psi: \mathfrak{D} \to C$ is a natural transformation of tensor functors: If $\varphi: (\tilde{M}, \tilde{q}) \to (M, q)$ is a similitude between quadratic modules of the same rank and the multiplier μ is* 1 *in case of odd rank, then the diagram*

$$\begin{tikzcd} \mathfrak{D}(\tilde q) \arrow[r, "\psi_{\tilde q}"] \arrow[d, "\mathfrak{D}(\varphi,\mu)"'] & C(\tilde q) \arrow[d, "C(\varphi,\mu)"] \\ \mathfrak{D}(q) \arrow[r, "\psi_q"'] & C(q) \end{tikzcd} \qquad (1)$$

commutes. Moreover, if $q = q_1 \perp q_2$ is an orthogonal sum then the diagram

$$\begin{tikzcd}[column sep=small] & \mathfrak{D}(q_1)\,\hat\otimes\,\mathfrak{D}(q_2) \arrow[dr, "\psi_{q_1}\hat\otimes\psi_{q_2}"] & \\ \mathfrak{D}(q_1)\,\square\,\mathfrak{D}(q_2) \arrow[ur, "\eta"] \arrow[d, "\vartheta"'] & & C(q_1)\,\hat\otimes\, C(q_2) \arrow[d, "\cong"] \\ \mathfrak{D}(q_1 \perp q_2) \arrow[rr, "\psi_{q_1\perp q_2}"'] & & C(q_1 \perp q_2) \end{tikzcd} \qquad (2)$$

is commutative, where η and ϑ are as in 2.9 *and* 2.11.
(c) *ψ_q takes values in the center of $C_0(q)$ in case of even rank, and in the center of $C(q)$ in case of odd rank. If M has rank at most* 2*, or if q is nonsingular of even rank or semiregular of odd rank, then ψ_q is an isomorphism onto $Z(C_0(q))$ or $Z(C(q))$, respectively.*

Proof. (a) Let $r = 2n$. By a localization argument, 3.13.1 holds for all $u \in L = \bigwedge^r M$. Now 2.3.1 and 3.13.1 show that the bilinear map $\psi_q(s_f(u)s_f(v)) - \psi_q(s_f(u))\psi_q(s_f(v))$ on the rank 1 module L is alternating and therefore vanishes. Hence ψ_q is a homomorphism. The case of odd rank follows similarly from 3.13.2 and 2.5.1.

(b) The commutativity of (1) follows easily from 3.12 and 2.3(b) in the even rank case, and 1.6.7 for odd rank. For (2) one uses 2.9, 2.11, and 3.7.

(c) The first statement is clear from 3.10. If M has rank 1 or 2 then ψ_q is an isomorphism onto $C(q)$ or $C_0(q)$, respectively. This is clear when the rank is 1. In the rank 2 case, $C_0(q) = k.1 + MM$, and hence 3.6.1 shows that $\psi_q: \mathfrak{D}(q) \to C_0(q)$ is surjective. Since both modules have rank 2 the assertion follows. Now assume q is nonsingular or semiregular. After a faithfully flat base ring extension, we may assume that q is the standard form. Then the assertion follows from 3.8 and [9, IV, (2.3), (3.2.4)].

4. Adjoints of quadratic forms of odd rank

4.1. Lemma. *Let M be a finitely generated projective module of rank $m+1$ and p: $\bigwedge^m M \to N$ a quadratic map into some k-module N. If p vanishes on decomposable elements in all base extensions then $p = 0$. In particular, this applies to the formulas in* 1.4.2, 1.6(b) *and* 3.13.

Proof. By localization we may assume M free, say with basis $x_0, \ldots, x_m$. Then $\bigwedge^m M$ is free with basis $u_i = x_0 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_m$. For $i < j$,

$$u_i + u_j = \big((-1)^i x_i + (-1)^{j-1} x_j\big) \wedge x_0 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_m$$

is decomposable. Hence $p(u_i) = p(u_i + u_j) = 0$ which shows that p and its polar form vanishes on the basis u_i.

4.2. Definition. Let (M, q) be a quadratic module of rank $2n+1$ and $E_{2n}(q)$ as in 2.2. Since $\bigwedge^{2n} M$ has rank $2n+1$, 2.2.4 implies that $E_{2n}(q)$ has rank $2n+2$. Except when $n = 0$, it is not possible to define an algebra structure on $E_{2n}(q)$ as in 2.3.1. However, there is a well-defined quadratic form $q^\sharp$ on $E_{2n}(q)$, such that

$$q^\sharp\big(\lambda 1 + s_f(u)\big) = \lambda^2 + \lambda\tau_f(u) + \gamma_f(u, u) \tag{1}$$

for all $\lambda \in k$, $u \in \bigwedge^{2n} M$ and f representing q. Indeed, the proof given in 2.3.5 carries over verbatim because it only uses 1.6(b) which continues to hold in the present situation due to 4.1. Since $q^\sharp(1) = 1$ and $k.1$ is a free direct summand of $E_{2n}(q)$, the triple $\mathfrak{A}(q) = (E_{2n}(q), q^\sharp, 1)$ is a unital quadratic form in the sense of [10], called the *adjoint* of q. By abuse of notation, we will sometimes write $x \in \mathfrak{A}(q)$ instead of $x \in E_{2n}(q)$.

Denote by $B(x, y)$ the polar form of $q^\sharp$, by $T(x) = B(1, x)$ the trace form and by $\sigma(x) = \bar{x} = T(x)1 - x$ the conjugation. Clearly,

$$T\big(\lambda 1 + s_f(u)\big) = 2\lambda + \tau_f(u). \tag{2}$$

Also, σ is an orthogonal transformation of period two and by [7] $\mathfrak{A}(q)$ becomes a quadratic Jordan algebra with unit element 1 and quadratic operators

$$U_x y = B(x, \bar{y})x - q^\sharp(x)\bar{y}, \tag{3}$$

called the *adjoint Jordan algebra* of q.

Just as the discriminant algebra of a quadratic form of even rank, $\mathfrak{A}(q)$ is functorial with respect to similitudes: If $\varphi\colon (\tilde{M}, \tilde{q}) \to (M, q)$ is a similitude with multiplier μ between quadratic modules of rank $2n+1$, then there is a unique morphism $\mathfrak{A}(\varphi, \mu) : \mathfrak{A}(\tilde{q}) \to \mathfrak{A}(q)$ of unital quadratic forms and hence of Jordan algebras, given as in 2.3(b). The proof follows that of 2.3(b) and is therefore omitted. As in 2.3(c), it is easy to see that $\mathfrak{A}(q)$ commutes with change of base ring.

If $n = 0$, i.e., M has rank 1, then $\bigwedge^{2n} M = k$ and $\mathfrak{A}(q) = k \times k$ with $q^\sharp$ the hyperbolic form, since $\tau_f(1) = 1$ as the Pfaffian of an empty matrix, and $\gamma_f = 0$. Hence in this case, $\mathfrak{A}(q)$ carries no information about q.

4.3. Example. Let us compute the adjoint of the standard form $q = \xi_0^2 + \sum \xi_{2i-1}\xi_{2i}$ on $M = k^{2n+1}$ with respect to the standard basis $e_0, \dots, e_{2n}$, represented by $f = \operatorname{diag}(1, \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \dots, \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right))$ as in 3.8. The elements $v_i = e_0 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_{2n}$ $(i = 0, \dots, 2n)$ form a basis of $\bigwedge^{2n} M$. Direct computation yields

$$\tau_f(v_i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > 0, \end{cases} \tag{1}$$

$$\gamma_f(v_i, v_i) = 0 \quad (i = 0, \dots, 2n), \tag{2}$$

$$\gamma_f(v_i, v_j) + \gamma_f(v_j, v_i) = \begin{cases} 1 & \text{if } \{i, j\} = \{2l-1, 2l\} \\ 0 & \text{otherwise} \end{cases} \quad (i \neq j). \tag{3}$$

Let $x_{-1} = 1 - s_f(v_0)$ and $x_i = s_f(v_i)$, for $i = 0, \dots, 2n$. Then these formulas and 4.2.1 show that $\mathfrak{A}(q)$ is the orthogonal sum of $n+1$ hyperbolic planes, spanned by the hyperbolic pairs (x_{2i-1}, x_{2i}), for $i = 0, \dots, n$, and $1 = x_{-1} + x_0$.

Next, we determine the relation between the discriminants of q and its adjoint, which is more complicated than 2.3.5:

4.4. Proposition. *Let (M, q) be a quadratic module of rank $2n+1$ with adjoint $\mathfrak{A}(q) = (E_{2n}(q), q^\sharp, 1)$, and let $L = \bigwedge^{2n+1} M$. Then*

$$\delta_{q^\sharp} \cong (\delta_q)^{\otimes 2n}, \tag{1}$$

more precisely, there is a natural isomorphism $\varphi\colon L^{\otimes 2n} \cong \bigwedge^{2n+2} E_{2n}(q)$, under which $(\delta_q)^{\otimes 2n}$ corresponds to $\delta_{q^\sharp}$. Hence, if $n > 0$, q is semiregular if and only if $q^\sharp$ is nonsingular.

Proof. There is a natural isomorphism $r\colon W = \bigwedge^{2n} M \to \operatorname{Hom}(M, L)$ assigning to an element $u \in W$ the map $r_u\colon M \to L$, $x \mapsto x \wedge u$, and $\operatorname{Hom}(M, L) \cong L \otimes M^*$ since M is finitely generated projective. Moreover, L being of rank 1, we have $\bigwedge^{2n+1}(L \otimes M^*) \cong L^{\otimes 2n+1} \otimes (\bigwedge^{2n+1} M^*) \cong L^{\otimes 2n}$, because $\bigwedge^{2n+1} M^* \cong (\bigwedge^{2n+1} M)^* = L^*$. Altogether, this yields a natural isomorphism $\chi\colon \bigwedge^{2n+1} W \to L^{\otimes 2n}$.

Further, the map $\eta\colon \bigwedge^{2n+1} W \to \bigwedge^{2n+2} E_{2n}(q)$, $w_0 \wedge \dots \wedge w_{2n} \mapsto 1 \wedge s_f(w_0) \wedge \dots \wedge s_f(w_{2n})$, is an isomorphism independent of the choice of representative f of q, by 2.2.2. Let $\varphi = \eta \circ \chi^{-1}$. We must show $(\delta_q)^{\otimes 2n} = \delta_{q^\sharp} \circ (\varphi \times \varphi)$, equivalently,

$$(\delta_q)^{\otimes 2n} \circ (\chi \times \chi) = \delta_{q^\sharp} \circ (\eta \times \eta). \tag{2}$$

By localization we may assume M free, say with basis $e_0, \dots, e_{2n}$. Let $w_i = e_0 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_{2n}$, $w = w_0 \wedge \dots \wedge w_{2n}$ and $e = e_0 \wedge \dots \wedge e_{2n}$. Then w and e are bases of $\bigwedge^{2n+1} W$ and L, respectively, and we claim that $\chi(w) = (-1)^n e^{\otimes 2n}$. Indeed, $r_{w_j}(e_i) = e_i \wedge w_j = (-1)^i \delta_{ij} e$. Thus if e_i^* is the basis dual to the e_i and $e^* = e_0^* \wedge \dots \wedge e_{2n}^*$ then $e \otimes e_i^*$ corresponds to $(-1)^i u_i$ under the isomorphism $L \otimes M^* \cong W$, and therefore $(e \otimes e_0^*) \wedge \dots \wedge (e \otimes e_{2n}^*) = e^{\otimes 2n+1} \otimes e^* = e^{\otimes 2n}$ corresponds to $(-1)^{1+\dots+2n} w_0 \wedge \dots \wedge w_{2n} = (-1)^n w$. Hence (2) is equivalent to $p(f) = 0$ where

$$p(f) = \delta_q(e,e)^{2n} - \delta_{q^\sharp}(\eta(w), \eta(w)), \tag{3}$$

and of course $q = [f]$. As in the proof of 3.13 let $\mathbf{B}$ be the scheme of bilinear forms on M and $\mathbf{U}'$ the open universally schematically dense subscheme of all f such that $\operatorname{hdet} f$ is invertible. Then (3) defines a morphism $p\colon \mathbf{B} \to k_{\mathbf{a}}$ since all constructions are compatible with arbitrary base ring extension. Also $p(f) = p(g)$ for $f \sim g$, because the right hand side of (3) depends only on $[f]$. By density, it suffices to show that p vanishes on $\mathbf{U}'$. For $f \in \mathbf{U}'(R)$ the quadratic form $[f]$ is semiregular and thus splits over a faithfully flat extension. Therefore, after a suitable base extension and a change of basis in M, which affects $p(f)$ by a unit, we are reduced to proving $p(f) = 0$ in case $f = \operatorname{diag}(1, \binom{0\,1}{0\,0}, \dots, \binom{0\,1}{0\,0})$ is the standard form. Then $\delta_q(e,e) = (-1)^n \operatorname{hdet} f = 1$ by 1.2.5. Defining $x_{-1}, \dots, x_{2n}$ as in 4.3, we have $\eta(w) = x_{-1} \wedge \dots \wedge x_{2n}$, and the (x_{2i-1}, x_{2i}) are orthogonal hyperbolic pairs in $\mathfrak{A}(q)$. Hence $\delta_{q^\sharp}(\eta(w), \eta(w)) = 1$ and the proof is complete.

Remark. If q is semiregular then (L, δ_q) is a discriminant module in the sense that $\delta_q\colon L \otimes L \to k$ is an isomorphism, and therefore $(L^{\otimes 2}, \delta_q^{\otimes 2}) \cong (k, \langle 1 \rangle)$. It follows that $q^\sharp$ has discriminant $\langle 1 \rangle$ as well (and also trivial Arf invariant, see 5.6). Thus in contrast to 2.3(d), the discriminant of q cannot be recovered from $\mathfrak{A}(q)$.

4.5. Lemma. *Let $q = [f] = [g]$ be a quadratic form on a k-module M and $b = f + f^T = g + g^T$ the polar form of q. For any bilinear form h on M and $x \in M$ let i_x^h be the graded derivation of degree -1 of $\bigwedge M$ given on M by $i_x^h y = h(x,y)$. Then for all $u \in \bigwedge^{2n+1} M$,*

$$\tau_f(x \wedge u) = 2\tau_f(i_x^f u) - \tau_f(i_x^b u), \tag{1}$$

$$\kappa_{fg}(x \wedge u) = \tau_f(i_x^f u) - \tau_g(i_x^g u) - \kappa_{fg}(i_x^b u). \tag{2}$$

If u is decomposable then

$$\gamma_f(x \wedge u, x \wedge u) = q(x)\delta_q(u,u) - \tau_f(i_x^f u)^2 + \tau_f(i_x^f u)\tau_f(x \wedge u) + \gamma_f(i_x^b u, i_x^b u). \tag{3}$$

If M is finitely generated projective of rank $\le 2n+2$ then (3) holds for all $u \in \bigwedge^{2n+1} M$.

Proof. Since (1) and (2) are linear in u it suffices to assume $u = x_2 \wedge \dots \wedge x_{2n+2}$ decomposable. Let $x_1 = x$ and $\mathbf{x} = (x_1, \dots, x_{2n+2})$. If a is an alternating bilinear form on M, the expansion formula for $\operatorname{Pf}(a(\mathbf{x}, \mathbf{x})) = \pi_{n+1}(a)(x \wedge u)$ implies

$$\pi_{n+1}(a)(x \wedge u) = \pi_n(a)(i_x^a u), \tag{4}$$

and setting $a = f - f^T = 2f - b$, we obtain (1) in view of 1.5.1.

Let a and a' be alternating bilinear forms and t an indeterminate. A simple calculation with (4) and 1.4.5 shows

$$\Pi_{n+1}(t,a,a')(x \wedge u) = \pi_n(a)(i_x^{a'} u) + \Pi_n(t,a,a')(i_x^{a+ta'} u).$$

Now (2) follows by specializing $a = f - f^T$, $a' = f - g$, $t = -2$ in this formula and observing $a - 2a' = 2g - b$:

$$\begin{aligned}\kappa_{fg}(x \wedge u) &= \tau_f(i_x^{f-g} u) + \kappa_{fg}(i_x^{2g-b} u)\\ &= \tau_f(i_x^f u) + (2\kappa_{fg} - \tau_f)(i_x^g u) - \kappa_{fg}(i_x^b u)\\ &= \tau_f(i_x^f u) - \tau_g(i_x^g u) - \kappa_{fg}(i_x^b u).\end{aligned}$$

Here we used 1.6.2 for the last equality.

Formula (3) is a polynomial identity with integer coefficients in the $f_{ij} = f(x_i, x_j)$. Hence it suffices to prove it when the f_{ij} are indeterminates and M is free with basis $x_1, \ldots, x_{2n+2}$ over the torsion-free ring $\mathbb{Z}[f_{ij}]$ and then apply specialization. By expanding the determinant $\det b(\mathbf{x},\mathbf{x}) = \bigwedge^{2n+2} b(x \wedge u, x \wedge u)$ with respect to the first row and column, we obtain

$$\bigwedge^{2n+2} b(x \wedge u, x \wedge u) = b(x,x) \bigwedge^{2n+1} b(u,u) - \bigwedge^{2n} b(i_x^b u, i_x^b u). \tag{5}$$

By 1.3.5, 1.3.11 and 1.4.2,

$$\begin{aligned}\bigwedge^{2n+2} b(x \wedge u, x \wedge u) &= (-1)^{n+1} \delta_f(x \wedge u, x \wedge u)\\ &= (-1)^{n+1} \big(\tau_f(x \wedge u)^2 - 4\gamma_f(x \wedge u, x \wedge u)\big).\end{aligned} \tag{6}$$

Further, 1.3.7 implies

$$b(x,x) \bigwedge^{2n+1} b(u,u) = 4(-1)^n q(x) \delta_f(u,u). \tag{7}$$

Clearly

$$v = i_x^b u = \sum_{j=2}^{2n+2} (-1)^j b(x_1, x_j) x_2 \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_{2n+2}$$

belongs to $\bigwedge^{2n} N$ where N is the submodule of rank $2n+1$ of M spanned by $x_2, \ldots, x_{2n+2}$. By 1.4.2 and 4.1, $\bigwedge^{2n}(f - f^T)(v,v) = \pi_n(f - f^T)(v)^2 = \tau_f(v)^2$, so that, in analogy to (6),

$$\bigwedge^{2n} b(v,v) = (-1)^n \delta_f(v,v) = (-1)^n \big(\tau_f(v)^2 - 4\gamma_f(v,v)\big). \tag{8}$$

By (1), $\tau_f(v) = 2\tau_f(i_x^f u) - \tau_f(x \wedge u)$. By substituting this into (8) and then substituting (6), (7) and (8) into (5) we obtain (3) after canceling a factor 4.

Finally, suppose M has rank $r \le 2n+2$. If $r \le 2n$ we have $u = 0$ and there is nothing to prove. If $r = 2n+1$ we may localize and thus assume M free. Then every $u \in \bigwedge^{2n+1} M$ is decomposable. If $r = 2n+2$, the assertion follows from Lemma 4.1, since the difference of the left and right hand side of (3) is, for fixed x, a quadratic form in u which vanishes on decomposable elements.

4.6. Theorem. *Let (M, q) be a quadratic module of rank $2n+1$ and $L = \bigwedge^{2n+1} M$ with the discriminant form δ_q. Let $\mathfrak{A}(q) = (E_{2n}(q), q^\sharp, 1)$ be the adjoint of q and $E_{2n}(q)^0 = \operatorname{Ker} T$ the elements of trace 0, with the quadratic form $q^{\sharp 0} = -q^\sharp \big| E_{2n}(q)^0$. Then there is a natural morphism of quadratic forms*

$$\varphi : (M \otimes L,\, q \otimes \delta_q) \to \mathfrak{A}(q)^0 = (E_{2n}(q)^0,\, q^{\sharp 0}) \tag{1}$$

such that, for all $x \in M$, $u \in L$ and f representing q,

$$\varphi(x \otimes u) = -\tau_f(i_x^f u)1 + s_f(i_x^b u). \tag{2}$$

If q is semiregular then φ is an isomorphism.

Proof. Since $x \wedge u \in \bigwedge^{2n+2} M = 0$, 4.5.2 and 2.2.2 show that φ is well-defined, independent of the choice of f. An element $v = \lambda 1 + s_f(w)$ of $E_{2n}(q)^0$ has $2\lambda + \tau_f(w) = 0$ by 4.2.2 and therefore $q^{\sharp 0}(v) = -\lambda^2 - \lambda\tau_f(w) - \gamma_f(w, w) = \lambda^2 - \gamma_f(w, w)$. Now $T(\varphi(x \otimes u)) = 0$ follows from 4.5.1, and 4.5.3 yields $q^{\sharp 0}(\varphi(x \otimes u)) = q(x)\delta_q(u, u) = (q \otimes \delta_q)(x \otimes u)$. Since L is an invertible module, this implies by localization that φ is a morphism of quadratic forms. Finally, suppose that q is semiregular. By faithfully flat descent, we may assume q is the standard form as in 4.3. With the notations introduced there, $e' = e_0 \wedge \ldots \wedge e_{2n}$ is a basis of L, and a straightforward computation shows $\varphi(e_0 \otimes e') = x_0 - x_{-1}$, $\varphi(e_{2i-1} \otimes e') = x_{2i}$, $\varphi(e_{2i} \otimes e') = -x_{2i-1}$. Now the assertion follows from the description of $\mathfrak{A}(q)$ given in 4.3.

A semiregular q of odd rank is determined by its adjoint and discriminant:

4.7. Corollary. *Let (M, q) be a semiregular quadratic module of rank $2n + 1$ and let* $\operatorname{dis}(q) = (\bigwedge^{2n+1} M, \delta_q)$ *be its discriminant module. Then* $(M, q) \cong \mathfrak{A}(q)^0 \otimes \operatorname{dis}(q)$.

Indeed, $(M, q) \otimes \operatorname{dis}(q) \cong \mathfrak{A}(q)^0$ by 4.6. Since q is semiregular, $\operatorname{dis}(q) \otimes \operatorname{dis}(q) \cong \langle 1 \rangle$ so the assertion follows by tensoring with $\operatorname{dis}(q)$.

As another corollary, we obtain an easy proof of the following result which was shown in [10, 6.7] by a cohomological argument.

4.8. Corollary. *For a semiregular quadratic module (M, q) of rank $2n + 1$ over k the following conditions are equivalent.*

(i) *There exists a nonsingular unital quadratic form $\mathfrak{X} = (X, Q, 1)$ of rank $2n + 2$ such that $(M, q) \cong \mathfrak{X}^0 = (X^0, Q^0)$;*

(ii) *there exists a separable quadratic algebra R such that* $\operatorname{dis}(R) \cong \operatorname{dis}(q)$;

(iii) $\operatorname{dis}(q) \otimes (k/4k) \cong \langle 1 \rangle$.

Proof. (i) $\Longrightarrow$ (ii): Let $R = \mathfrak{D}(q)$. Then $\operatorname{dis}(q) \cong \operatorname{dis}(Q^0) \cong \operatorname{dis}(Q)$ (by [10, 3.10(b)]) $\cong \operatorname{dis}(R)$ (by 2.3(d)).

(ii) $\Longrightarrow$ (i): $(M, q) \cong \mathfrak{A}(q)^0 \otimes \operatorname{dis}(q)$ (by 4.7) $\cong \mathfrak{A}(q)^0 \otimes \operatorname{dis}(R) \cong \big(\mathfrak{A}(q) \,\square\, R\big)^0$ (by [10, 2.12]).

(ii) $\Longleftrightarrow$ (iii): See [6, (12.6)].

4.9. Theorem. *Let $\mathfrak{X} = (X, Q, 1)$ be a unital quadratic form of rank $2n+2$ with surjective trace form T, and let (X^0, Q^0) be the quadratic module of rank $2n+1$ defined by $X^0 = \operatorname{Ker} T$ and $Q^0 = -Q\big|X^0$. Then there is a natural morphism of unital quadratic forms*

$$\eta : \mathfrak{X} \,\square\, \mathfrak{D}(Q) \to \mathfrak{A}(Q^0), \tag{1}$$

given as follows: Let α_1 be any unital linear form on X, let F be a bilinear form representing Q, and set $\alpha_2 = \rho_F$ as in 2.2 and $\alpha = (\alpha_1, \alpha_2)$. Also let B be the polar form of Q and $F^0 = -F\big|X^0 \times X^0$. Then $\eta(1) = 1$ and

$$\eta(\dot{x} \,\square_\alpha\, w) = \big\{\tau_F(i_x^F i_1^B w) - \alpha_1(x)\tau_F(w)\big\}1 + (-1)^{n+1} s_{F^0}(i_x^B i_1^B w), \tag{2}$$

for all $\dot{x} \in X/k.1$ and $w \in \bigwedge^{2n+2} X \cong \mathfrak{D}(Q)/k.1$. If Q is nonsingular then η is an isomorphism.

Proof. Choose an element $e \in X$ with $T(e) = 1$. Then $X = k.e \oplus X^0$, so every $w \in \bigwedge^{2n+2} X$ can be written uniquely

$$w = e \wedge u, \qquad u = i_1^B w \in \bigwedge^{2n+1} X^0.$$

Also, $x - T(x)e \in X^0$ whence $0 = (x - T(x)e) \wedge u$ or

$$x \wedge u = T(x)w. \tag{3}$$

We begin by showing that the right hand side of (2) vanishes for $x = 1$ and therefore depends only on the class $\dot{x}$ of x in $X/k.1$. This holds for the third term since $(i_1^B)^2 = 0$. Writing $w = e \wedge u$ as above, let $\lambda_F = \tau_F(i_1^F u) - \tau_F(w)$, and suppose G is a second bilinear form representing Q. Then by 4.5.2 and 1.6.2, $\lambda_F - \lambda_G = \kappa_{FG}(1 \wedge u) + \kappa_{FG}(i_1^B u) - 2\kappa_{FG}(w) = \kappa_{FG}(1 \wedge u - 2w) = 0$ since (3) shows, for $x = 1$, that $1 \wedge u = 2w$. Thus $\lambda_F = \lambda_G$ is independent of the choice of F. To prove $\lambda_F = 0$ we may assume k local. Then there are two cases: If $2 \in k^\times$, choose $F = \frac{1}{2}B$ symmetric. Then $i_1^F u = \frac{1}{2} i_1^B u = 0$ and $\tau_F(w) = \pi_{2n+2}(F - F^T)(w) = 0$. If 2 is in the maximal ideal of k, let $y = 1 - 2e \in X^0$. Then by [10, 3.5], $1 - 4Q(e) = -Q(y) = Q^0(y)$ is a unit, so that Q^0 is primitive. By [10, 3.6] we may assume, after a faithfully flat base extension, that X contains a hyperbolic pair (e^+, e^-) such that $1 = e^+ + e^-$. Decompose $X = (k.e^+ + k.e^-) \perp Y$ orthogonally and choose F such that $F(e^+, e^-) = 1$ and $F(e^-, X) = F(e^+, e^+ + Y) = F(Y, e^\pm) = 0$. Also let $e = e^+$. Then $X^0 = k(e^+ - e^-) \oplus Y$ so that $u = (e^+ - e^-) \wedge v$ with $v \in \bigwedge^{2n} Y$. It follows $i_1^F u = -v = i_e^F u = i_e^B u$. Now 4.5.1 implies, for $x = e$, that $\lambda_F = \tau_F(i_1^F u) - 2\tau_F(i_e^F u) + \tau_F(i_e^B u) = 0$.

Next, we show that η is well-defined, independent of the choice of F and α_1. Let η_α be the map defined by $1 \mapsto 1$ and the right hand side of (2). We must verify $\eta_\beta(\dot{x} \,\square_\alpha\, w) = \eta_\alpha(\dot{x} \,\square_\alpha\, w)$, where $\beta = (\beta_1, \beta_2)$ for a second unital linear form β_2 on X and $\beta_2 = \rho_G$ with $Q = [G]$. By definition of the product $\mathfrak{X} \,\square\, \mathfrak{D}(Q)$ of unital quadratic forms [10, 2.1], $\dot{x} \,\square_\alpha\, w = \dot{x} \,\square_\beta\, w + c_{\alpha\beta}(\dot{x} \otimes w)1$, where

$$c_{\alpha\beta}(\dot{x} \otimes w) = c_{\alpha_1\beta_1}(\dot{x})\tau_F(w) + t_{\alpha_F}(\dot{x})\kappa_{FG}(w) - 2c_{\alpha_1\beta_1}(\dot{x})\kappa_{FG}(w). \tag{4}$$

Here $c_{\alpha_1\beta_1}(\dot{x}) = -\alpha_1(x) + \beta_1(x)$ and $t_{\alpha_1}(\dot{x}) = T(x - \alpha_1(x)1) = T(x) - 2\alpha_1(x)$, by [10, 1.4]. Substituting this into (4) and observing 1.6.2 and (3), we obtain

$$c_{\alpha\beta}(\dot{x} \otimes w) = -\alpha_1(x)\tau_F(w) + \beta_1(x)\tau_G(w) + \kappa_{FG}(x \wedge u). \tag{5}$$

Also, $\kappa_{F^0G^0}(i_x^B u) = (-1)^n \kappa_{FG}(i_x^B u)$ since F^0 and G^0 are the restrictions of $-F$ and $-G$ to X^0 and $\kappa_{FG}^{(n)}$ is homogeneous of degree n in (F, G) by 1.5.2 and 1.4. Hence by 2.2,

$$(-1)^{n+1} s_{G^0}(i_x^B u) = (-1)^{n+1} s_{F^0}(i_x^B u) + \kappa_{FG}(i_x^B u)1, \tag{6}$$

and it follows

$$\begin{aligned}
\eta_\beta(\dot{x} \,\square_\alpha\, w) &= \eta_\beta(\dot{x} \,\square_\beta\, w) + c_{\alpha\beta}(\dot{x} \otimes w)1 \\
&= \big\{\tau_G(i_x^G u) - \beta_1(x)\tau_G(w) + c_{\alpha\beta}(\dot{x} \otimes w)\big\}1 + (-1)^{n+1} s_{G^0}(i_x^B u) && \text{(by (2))} \\
&= \big\{\tau_G(i_x^G u) - \alpha_1(x)\tau_F(w) + \kappa_{FG}(x \wedge u) + \kappa_{FG}(i_x^B u)\big\}1 + (-1)^{n+1} s_{F^0}(i_x^B u) \\
& && \text{(by (5) and (6))} \\
&= \big\{\tau_F(i_x^F u) - \alpha_1(x)\tau_F(w)\big\}1 + (-1)^{n+1} s_{F^0}(i_x^B u) && \text{(by 4.5.2)} \\
&= \eta_\alpha(\dot{x} \,\square_\alpha\, w),
\end{aligned}$$

as desired.

Now we prove that η is a morphism of quadratic forms. By localization we may assume the rank 1 module $\bigwedge^{2n+2} X$ free. Then every element of $\dot{X} \,\square_\alpha \bigwedge^{2n+2} X$ is of the form $\dot{x} \,\square_\alpha\, w$ (instead of being a sum of such terms). Since $\eta(1) = 1$ it suffices to show that $T(\dot{x} \,\square_\alpha\, w) = T(\eta(\dot{x} \,\square_\alpha\, w))$ and the same for Q in place of T. By [10, 2.3.1] and (3), $T(\dot{x} \,\square_\alpha\, w) = t_{\alpha_1}(\dot{x}) t_{\rho_F}(w) = \big(T(x) - 2\alpha_1(x)\big)\tau_F(w) = \tau_F(x \wedge u) - 2\alpha_1(x)\tau_F(w)$, while $T(\eta(\dot{x} \,\square_\alpha\, w)) = 2\tau_F(i_x^F u) - 2\alpha_1(x)\tau_F(w) + (-1)^{n+1}\tau_{F^0}(i_x^B u) = 2\tau_F(i_x^F u) - \tau_F(i_x^B u) - 2\alpha_1(x)\tau_F(w)$. Here we used $\tau_{F^0}(v) = \tau_{-F}(v) = (-1)^n \tau_F(v)$ for $v \in \bigwedge^{2n} X^0$, which follows from the fact that $\tau_F = \tau_F^{(n)}$ is homogeneous of degree n in F. Now 4.5.1 shows $T(\dot{x} \,\square_\alpha\, w) = T(\eta(\dot{x} \,\square_\alpha\, w))$. For the quadratic forms we have, by [10, 2.3.2],

$$\begin{aligned}
Q(\dot{x} \,\square_\alpha\, w) &= q_{\alpha_1}(\dot{x}) t_{\rho_F}(w)^2 + \big\{t_{\alpha_1}(\dot{x})^2 - 4q_{\alpha_1}(\dot{x})\big\} q_{\rho_F}(w) \\
&= Q(x - \alpha_1(x)1)\tau_F(w)^2 + \Big\{\big(T(x) - 2\alpha_1(x)\big)^2 - 4Q(x - \alpha_1(x)1)\Big\}\gamma_F(w, w) \\
&= Q(x - \alpha_1(x)1)\tau_F(w)^2 + \big\{T(x)^2 - 4Q(x)\big\}\gamma_F(w, w) \\
&= Q(x)\big\{\tau_F(w)^2 - 4\gamma_F(w, w)\big\} + T(x)^2\gamma_F(w, w) \\
&\qquad + \big(\alpha_1(x)^2 - T(x)\alpha_1(x)\big)\tau_F(w)^2 \\
&= Q(x)\delta_Q(w, w) + \gamma_F(x \wedge u, x \wedge u) \\
&\qquad + \alpha_1(x)^2\tau_F(w)^2 - \alpha_1(x)\tau_F(x \wedge u)\tau_F(w).
\end{aligned} \tag{7}$$

Here we used (3) and 1.7.1 in the last step. On the other hand, $\gamma_{F^0}(v, v) = \gamma_{-F}(v, v) = \gamma_F(v, v)$ for $v \in \bigwedge^{2n} X^0$ since $\gamma_F = \gamma_F^{(2n)}$ is homogeneous of degree $2n$ in F. Hence,

$$
\begin{aligned}
Q(\eta(\dot{x}\,\Box_\alpha\, w)) &= \big(\tau_F(i_x^F u) - \alpha_1(x)\tau_F(w)\big)^2 \\
&\quad - \big(\tau_F(i_x^F u) - \alpha_1(x)\tau_F(w)\big)\tau_F(i_x^B u) + \gamma_F(i_x^B u, i_x^B u) \\
&= \tau_F(i_x^F u)\big(\tau_F(i_x^F u) - \tau_F(i_x^B u)\big) - \alpha_1(x)\tau_F(w)\big(2\tau_F(i_x^F u) - \tau_F(i_x^B u)\big) \\
&\qquad + \alpha_1(x)^2\tau_F(w)^2 + \gamma_F(i_x^B u, i_x^B u) \\
&= \tau_F(i_x^F u)\big(\tau_F(x\wedge u) - \tau_F(i_x^F u)\big) - \alpha_1(x)\tau_f(w)\tau_F(x\wedge u) \\
&\qquad + \alpha_1(x)^2\tau_F(w)^2 + \gamma_F(i_x^B u, i_x^B u) && \text{(by 4.5.1)} \\
&= -Q(x)\delta_F(u,u) + \gamma_F(x\wedge u, x\wedge u) \\
&\qquad\qquad + \alpha_1(x)^2\tau_F(w)^2 - \alpha_1(x)\tau_F(x\wedge u)\tau_F(w) && \text{(by 4.5.3).}
\end{aligned}
$$

Comparing this with (7) we see that it remains to show $\delta_Q(w,w) = -\delta_F(u,u)$. By [10, 3.10(b)], $\delta_Q(w,w) = \delta_{Q^0}(u,u)$, and by 1.7, $\delta_{Q^0}(u,u) = \delta_{F^0}(u,u) = \delta_{-F}(u,u) = -\delta_F(u,u)$ since $\delta_F = \delta_F^{(2n+1)}$ is homogeneous of degree $2n+1$ in F by 1.3.5. This completes the proof that η is a morphism of quadratic forms.

Finally, we show η is an isomorphism if Q is nonsingular. By faithfully flat descent and [10, 3.7(a)] we may assume X free with basis $e_{-1},\dots,e_{2n}$, Q the standard form represented by $F = \operatorname{diag}\big(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\dots,\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\big)$, and $1 = e_{-1}+e_0$. Then X^0 has basis $e_0' = e_{-1} - e_0$ and $e_i' = (-1)^i e_i$ $(i = 1,\dots,2n)$, and the matrix of F^0 with respect to this basis is $\operatorname{diag}\big(1, \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\dots,\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\big)$. Define $v_i = e_0'\wedge\ldots\wedge\hat{e}_i'\wedge\ldots\wedge e_{2n}'$ and let $x_i = s_{F^0}(v_i)$ for $i\ge 0$ and $x_{-1} = 1 - x_0$. Then by 4.3, $x_{-1},\dots,x_{2n}$ is a hyperbolic basis for $\mathfrak{A}(Q^0)$. Finally, let $w = e_{-1}\wedge\ldots\wedge e_{2n}$ and choose the unital linear form $\alpha_1(x) = F(1,x)$ on X. Then a computation shows that $\eta(1) = 1$, $\eta(\dot{e}_{-1}\Box_\alpha w) = 1 - x_0 = -\eta(\dot{e}_0\Box_\alpha w)$, $\eta(\dot{e}_{2j-1}\Box_\alpha w) = x_{2j}$, $\eta(\dot{e}_{2j}\,\Box_\alpha\, w) = x_{2j-1}$. Hence η is an isomorphism and the proof is complete.

4.10. Corollary. *A nonsingular unital quadratic form $\mathfrak{X} = (X,Q,1)$ of even rank is naturally isomorphic to $\mathfrak{A}(Q^0)\,\Box\,\mathfrak{D}(Q)$.*

Proof. Since $\mathfrak{D}(Q)$ is separable, there is a natural isomorphism $\mathfrak{D}(Q)\,\Box\,\mathfrak{D}(Q) \cong \mathfrak{E} = k\times k$ [10, 2.9], and $\mathfrak{E}$ acts like a unit with respect to the product $\Box$ [10, 2.6]. Hence $\Box$-multiplying 4.9.1 with $\mathfrak{D}(Q)$ yields a natural isomorphism $\mathfrak{X}\cong\mathfrak{A}(Q^0)\,\Box\,\mathfrak{D}(Q)$.

5. The Jordan center of the Clifford algebra

5.1. Definition. Let $C = C(q) = C_0\oplus C_1$ be the Clifford algebra of a quadratic form q on M and $Z = Z_0\oplus Z_1$ its center. Define $J = J(q)\subset C\times C$ by

$$(u,v)\in J \quad\Longleftrightarrow\quad ux - xv = xu - vx \in Z \quad \text{for all } x\in M. \tag{1}$$

In view of the properties proved below, we call J the *Jordan center* of C. Clearly J is a Z-submodule of $C\times C$ containing $1_J = (1,1)$, and there is a Z-linear map

$$\chi\colon J\to \operatorname{Hom}_k(M,Z), \qquad \chi(u,v)(x) = ux - xv. \tag{2}$$

The kernel of χ is

$$K = \operatorname{Ker}\chi = \{(u,v)\in C\times C : ux = xv,\ vx = xu \text{ for all } x\in M\}. \tag{3}$$

It is easy to see that K is a subalgebra of $C \times C$ and J and K are stable under the exchange automorphism $(u,v) \mapsto \overline{(u,v)} = (v,u)$. Let $u = u_0 + u_1$, $v = v_0 + v_1$ be the decomposition into homogeneous parts of $(u,v) \in C \times C$. Since $M \subset C_1$ it follows easily that (u,v) belongs to J or K if and only if this holds for (u_i, v_i). Hence $J = J_0 \oplus J_1$ and $K = K_0 \oplus K_1$ are $\mathbb{Z}/2\mathbb{Z}$-graded, and χ maps J_i to $\mathrm{Hom}_k(M, Z_{1-i})$. We finally note that there is a homomorphism $\Delta\colon M \to J_1$ sending $y \in M$ to $(y, -y)$ and making the diagram

$$\begin{tikzcd} M \arrow[r, "b"] \arrow[d, "\Delta"'] & \mathrm{Hom}(M,k) \arrow[d, "i"] \\ J_1 \arrow[r, "\chi"'] & \mathrm{Hom}(M, Z_0) \end{tikzcd} \tag{4}$$

commutative, where b is given by the polar form of q and i is induced from $k \to Z_0$. Indeed, $yx - x(-y) = xy - (-y)x = b(x,y)1$ for all $x \in M$.

5.2. Proposition. (a) *If $(u,v) \in K$ and $w_i \in C_i$ then u and v commute with w_0 and satisfy $uw_1 = w_1 v$ and $vw_1 = w_1 u$; in particular, $\mathrm{pr}_1(K) \subset C^{C_0}$. If q is primitive, $\mathrm{pr}_1 \colon K \to C^{C_0}$ is injective.*
(b) *If $a = (u,v) \in J$ then $Q(a) = uv = vu \in Z$. The map $Q\colon J \to Z$ is a quadratic form on J with $Q(1_J) = 1$, polar form $B(a,a') = uv' + u'v$, trace form $T(a) = B(a, 1_J) = u + v$ and conjugation $\sigma(a) = T(a)1_J - a = \bar{a}$.*
(c) *J is a (quadratic) Jordan subalgebra of $C \times C$ and the Jordan algebra structure induced form $C \times C$ is the one determined by the quadratic form Q with base point 1_J, i.e.,*

$$aba = B(a, \bar{b})a - Q(a)\bar{b} \in J \tag{1}$$

for all $a, b \in J$.

Proof. The first statement of (a) follows from 5.1.3. Let q be primitive and $(0,v) \in K$. Then $q(M)$ generates k as an ideal which implies $1 \in MM$, and $vM = 0$ by 5.1.3. Hence $v = v1 = 0$.

Let $a = (u,v) \in J$. By 5.1.1, $(u+v)x = x(u+v)$ for all $x \in M$. This implies $u + v = t \in Z$ and therefore $uv = u(t-u) = (t-u)u = vu$. Further, let $c = ux - xv = xu - vx \in Z$. Then $uvx = u(xu - c) = (ux - c)u = xvu = xuv$, and since $x \in M$ was arbitrary, $uv = vu \in Z$. The remaining assertions of (b) are now clear. Finally, $a\bar{a} = (uv, vu) = Q(a)1_J = (vu, uv) = \bar{a}a \in Z1_J$. Hence $B(a,\bar{b})1_J = (a + \bar{b})(\bar{a} + b) - a\bar{a} - \bar{b}b = ab + \bar{b}\bar{a}$ which yields $B(a,\bar{b})a - Q(a)\bar{b} = (ab + \bar{b}\bar{a})a - (a\bar{a})\bar{b} = aba$.

5.3. Remark. By specializing $b = 1_J$ in 5.2.1, we see that every $a \in J$ satisfies the quadratic polynomial $a^2 - T(a)a + Q(a)1 = 0$. Also, Q is multiplicative with respect to the Jordan product, $Q(aba)1_J = aba\bar{a}\bar{b}\bar{a} = (a\bar{a})^2 b\bar{b} = Q(a)^2 Q(b)1_J$. Since K_0 is commutative by 5.2(a), we have $Q(ab)1_J = ab\bar{a}\bar{b} = a\bar{a}b\bar{b} = Q(a)Q(b)1_J$ for $a, b \in K_0$. But K is in general not commutative and it is not clear whether Q is multiplicative on K. Also, I don't know whether Q is in general a unital quadratic form in the sense of [10], i.e., whether $Z.1_J$ is a free direct summand of the Z-module J.

5.4. Proposition. *Let (M,q) be a quadratic module of rank $2n$, $\mathfrak{D}(q)$ its discriminant algebra and $\psi_q\colon \mathfrak{D}(q) \to C_0$ as in 3.14.*
(a) *The map $a \mapsto (\psi_q(a), \psi_q(\bar a))$ is an algebra homomorphism $\Psi_q\colon \mathfrak{D}(q) \to K_0$.*
(b) *Let q be nonsingular. Then Ψ_q is an isomorphism, $K_1 = 0$, and $J_0 = K_0$. Furthermore, $\Delta\colon M \to J_1$ is an isomorphism and the quadratic form Q on J is isomorphic to $N \perp (-q)$ where N is the norm form of $\mathfrak{D}(q)$.*

Proof. (a) Since conjugation $a \mapsto \bar a$ is an automorphism of $\mathfrak{D}(q)$, 3.14(a) shows Ψ_q is an algebra homomorphism. By 3.10.1 it takes values in K_0.

(b) It is known [9, IV, §2, §4] that $Z = Z_0 = k.1$ and C^{C_0} is the center of C_0. By 3.14(c), $\psi_q\colon \mathfrak{D}(q) \to C^{C_0}$ is an isomorphism. Since $\psi_q = \mathrm{pr}_1 \circ \Psi_q$, we have $\mathrm{pr}_1 : K \to C^{C_0}$ surjective. By 5.2(a), pr_1 is injective so Ψ_q is an isomorphism. As the odd part of C^{C_0} is zero we obtain $K_1 = 0$.

Via χ, J_0/K_0 is isomorphic to a submodule of $\mathrm{Hom}(M, Z_1) = 0$ whence $J_0 = K_0$. Also, $K_1 = 0$ implies $\chi\colon J_1 \to \mathrm{Hom}(M, Z_0) = M^*$ injective. Since q is nonsingular, $b\colon M \to M^*$ is an isomorphism. Now 5.1.4 shows χ is surjective as well and Δ is an isomorphism. Finally, let $a = (u_0, v_0) + (u_1, v_1) \in J_0 \oplus J_1$. Then $a_0 = (u_0, v_0) \in K_0$ and $v_1 = -u_1 \in M$, and thus $Q(a) = (u_0 + u_1)(v_0 + v_1) = u_0v_0 - u_1^2 + u_1v_0 - u_0u_1 = N(a_0) - q(u_1)$ by 5.2(a). This completes the proof.

5.5. Theorem. *Let (M,q) be a quadratic module of rank $2n+1$, $\mathfrak{A}(q)$ its adjoint, and $\psi_q\colon E_{2n}(q) \to C_0$ as in 3.10(a).*
(a) *The k-linear map $a \mapsto (\psi_q(a), \psi_q(\bar a))$ is a morphism $\Psi_q\colon \mathfrak{A}(q) \to J_0$ of quadratic forms, i.e.,*

$$Q(\Psi_q(a)) = \psi_q(a)\psi_q(\bar a) = q^\sharp(a)1_C \quad (a \in \mathfrak{A}(q)), \tag{1}$$

and a homomorphism of Jordan algebras.
(b) *Let q be semiregular. Then $K_0 = k.1_J$, Ψ_q is an isomorphism, and $J \cong J_0 \otimes Z$ is extended from J_0. If M has rank at least 3 then $\mathrm{pr}_1 : J \to C$ and $\psi_q\colon \mathfrak{A}(q) \to C_0$ are injective.*

Proof. (a) By 3.9.2 and 3.10(c) we have $z_f(u)x - x\big(\tau_f(u)1 - z_f(u)\big) = xz_f(u) - \big(\tau_f(u)1 - z_f(u)\big)x = -z_q(x \wedge u) \in Z$ for all $x \in M$, $u \in W = \bigwedge^{2n} M$. Since $\mathfrak{A}(q) = k.1 \oplus s_f(W)$ and $\overline{s_f(u)} = \tau_f(u)1 - s_f(u)$ this implies that Ψ_q takes values in J_0 and also shows that the diagram

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] \arrow[d] & \mathfrak{A}(q) \arrow[r, "p"] \arrow[d, "\Psi_q"] & \bigwedge^{2n} M \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & K_0 \arrow[r] & J_0 \arrow[r, "\chi"'] & \mathrm{Hom}(M, Z_1) \arrow[r] & 0
\end{tikzcd} \tag{2}$$

commutes, where $\eta(u)(x) = -z_q(x \wedge u)$. Next, Ψ_q preserves quadratic forms since $T(\Psi_q(s_f(u))) = z_f(u) + \tau_f(u)1 - z_f(u) = T(s_f(u))1_C$ and $Q(\Psi_q(s_f(u))) = z_f(u)\big(\tau_f(u)1 - z_f(u)\big) = \gamma_f(u,u)1_C = q^\sharp(s_f(u))1_C$ for all $u \in W$, by 3.13.1, 4.1 and 4.2. From 4.2.3 and 5.2.1 it follows that Ψ_q is a Jordan homomorphism.

(b) By 3.14(c) the center of C is $Z = k.1 \oplus Z_1$ and $z_q: L = \bigwedge^{2n+1} M \to Z_1$ is an isomorphism. Since $W \cong \mathrm{Hom}(M, L)$ via $u \mapsto (x \mapsto x \wedge u)$, we see that η is an isomorphism. The top row of (2) is exact by 2.2.4. As both p and η are surjective so is χ, and thus the bottom row of (2) is exact. By 5.2(a) $\mathrm{pr}_1 : K \to C^{C_0}$ is injective. By [9, IV, (3.2.4)], $C^{C_0} = Z = k.1 \oplus Z_1$ whence $K_0 = k.1_J$. Since $k.1_J$ is a free direct summand of $C \times C$, the map $k \to K_0$ in (2) is an isomorphism, and by the Five Lemma so is Ψ_q. That J is extended from J_0 follows easily from $C_1 = C_0 Z_1 \cong C_0 \otimes Z_1$ and the fact that multiplication induces an isomorphism $Z_1 \otimes Z_1 \cong k$ [9, IV, (3.2.4)].

Finally, suppose M has rank ≥ 3. It suffices to show $\mathrm{pr}_1 : J_0 \to C_0$ injective. Since Ψ_q is an isomorphism and $\psi_q = \mathrm{pr}_1 \circ \Psi_q$ this is equivalent to ψ_q being injective. Thus assume $\psi_q(\lambda 1 + s_f(u)) = \lambda 1 + z_f(u) = 0$. Then by 3.9.2 and 3.10(c), $z_q(x \wedge u) = (2\lambda + \tau_f(u))x \in Z_1$ for all $x \in M$. This implies $\mu = 2\lambda + \tau_f(u) = 0$. Indeed, as M has rank ≥ 3, it contains a hyperbolic pair (e^+, e^-) after a faithfully flat base extension. Then $0 = e^+[e^-, \mu e^+] = \mu(e^+e^-e^+ - (e^+)^2 e^-) = \mu e^+$ shows $\mu = 0$. Hence $z_q(x \wedge u) = 0$ for all $x \in M$, i.e., $\eta(u) = 0$ and therefore $u = 0$ and $\lambda = 0$, as desired.

5.6. Corollary. *Let (M, q) be semiregular of odd rank $2n+1$ with adjoint form $q^\sharp$, and let $A = C_0(q)$. Then $C(q^\sharp) \cong \mathrm{Mat}_2(A)$ with the checkerboard grading. In particular, $C_0(q^\sharp) \cong A \times A$ and $\mathfrak{D}(q^\sharp) \cong k \times k$, i.e., $q^\sharp$ has trivial Arf invariant.*

Proof. Define a map $h: \mathfrak{A}(q) \to \mathrm{Mat}_2(A)$ by

$$h(a) = \begin{pmatrix} 0 & \psi_q(a) \\ \psi_q(\bar{a}) & 0 \end{pmatrix}.$$

Then 5.5.1 shows $h(a)^2 = q^\sharp(a)1$. Hence there exists a unique homomorphism $\tilde{h}: C(q^\sharp) \to \mathrm{Mat}_2(A)$ extending h, and $\tilde{h}$ is clearly compatible with the gradings. Since $q^\sharp$ is nonsingular by 4.4, $C(q^\sharp)$ is an Azumaya algebra of rank 2^{2n+2} and so is $\mathrm{Mat}_2(A)$, for A is Azumaya of rank 2^{2n}. Therefore, $\tilde{h}$ is an isomorphism. The rest is immediate from 3.14(c).

5.7. Example. Let us work out ψ_q for the standard form q as in 4.3. By 3.8, the center of $C(q)$ is $k.1 \oplus k.\varepsilon$ where $\varepsilon = e_0(1 - 2c)$. With the notations of 4.3, the map z_f: $\bigwedge^{2n} M \to C_0(q)$ is given by

$$z_f(v_0) = c, \tag{1}$$
$$z_f(v_{2i-1}) = e_{2i}\varepsilon, \tag{2}$$
$$z_f(v_{2i}) = -e_{2i-1}\varepsilon. \tag{3}$$

Indeed, (1) is clear from 3.8.1. For the second, let $u_i = \bigwedge_{j \neq i}(e_{2j-1} \wedge e_{2j})$. Then $v_{2i-1} = e_0 \wedge e_{2i} \wedge u_i$, and by 3.7,

$$\begin{aligned} z_f(v_{2i-1}) &= z_f(e_0) z_f(e_{2i} \wedge u_i) = e_0 z_f(e_{2i})\big(\tau_f(u_i).1 - 2 z_f(u_i)\big) \\ &= e_0 e_{2i}(1 - 2c_1 \bullet \ldots \bullet \hat{c}_i \bullet \ldots \bullet c_n). \end{aligned}$$

Now $(1 - 2c_i)^2 = 1$ and $(1 - 2c_i)(1 - 2c_j) = 1 - 2c_i \bullet c_j$, and furthermore $e_{2i}(1 - 2c_i) = e_{2i} - 2e_{2i}e_{2i-1}e_{2i} = e_{2i} - 2e_{2i} = -e_{2i}$. Hence $z_f(v_{2i-1}) = e_0 e_{2i}(1 - 2c_i)\prod_{j=1}^{n}(1 - 2c_j) = -e_0 e_{2i}(1 - 2c) = e_{2i}\varepsilon$. The proof of (3) is similar.

As an application, we obtain the following result, whose counterpart in the even rank case is well known [9, IV, (4.3.1)].

5.8. Corollary. *Let (M, q) be semiregular of rank $2n+1 \geq 3$, identify J_0 with $\mathrm{pr}_1(J_0) = \psi_q(\mathfrak{A}(q)) \subset A = C_0(q)$ as in 5.5(b) and let ρ be the standard involution of $C(q)$, characterized by $x^\rho = -x$ for $x \in M$.*
(a) *If n is even, $J_0 \subset S = \{a + a^\rho : a \in A\}$, the set of traces of ρ in A. In particular, the restriction of ρ to J_0 is the identity.*
(b) *If n is odd, the restriction of ρ to J_0 corresponds to the conjugation of $\mathfrak{A}(q)$ under the isomorphism $\psi_q\colon \mathfrak{A}(q) \to J_0$.*

Proof. By faithfully flat descent, we may assume that q is the standard form on k^{2n+1}. A direct computation with the formulas (1)–(3) of 4.3 shows that the conjugation in $\mathfrak{A}(q)$ is given by

$$\bar{1} = 1, \quad \overline{s_f(v_0)} = 1 - s_f(v_0), \quad \overline{s_f(v_i)} = -s_f(v_i) \quad \text{for } i > 0.$$

From $e_i^\rho = -e_i$ follows $c_j^\rho = (e_{2j-1}e_{2j})^\rho = e_{2j}e_{2j-1} = 1 - e_{2j-1}e_{2j} = 1 - c_j$, and hence $c_1 \bullet c_2 = c_1 + c_2 - 2c_1c_2 = c_1(1-c_2) + c_2(1-c_1) = c_1c_2^\rho + (c_1c_2^\rho)^\rho$ is a trace of ρ. Moreover, if d_1 and d_2 are commuting idempotents which are traces then d_1d_2 is fixed under ρ and therefore $d_1 \bullet d_2 = d_1 + d_2 - \big(d_1d_2 + (d_1d_2)^\rho\big)$ is a trace. By induction, $z_f(v_0) = c = c_1 \bullet \ldots \bullet c_n$ is a trace for n even. If n is odd, let $d = c_2 \bullet \ldots \bullet c_n$. Then $c = c_1 \bullet d$ and $c^\rho = c_1^\rho \bullet d^\rho = (1-c_1) \bullet d = 1 - c_1 + d - 2(1-c_1)d = 1 - c$. This implies

$$\varepsilon^\rho = (-e_0)(1 - 2c^\rho) = (-1)^{n+1}\varepsilon,$$

and by (2) and (3) of 5.7,

$$z_f(v_i)^\rho = (-1)^n z_f(v_i) \quad (i > 0).$$

Now (b) follows from these formulas, and it remains to show that the $e_l\varepsilon$ $(l = 1, \ldots, 2n)$ are traces when n is even. Suppose $l = 2i$ or $l = 2i-1$ and choose $j \in \{1, \ldots, n\}$ different from i, which is possible since $n \geq 2$. Then e_l commutes with c_j and c_j^ρ. Hence $e_l\varepsilon = 1.e_l\varepsilon = (c_j + c_j^\rho)e_l\varepsilon = c_je_l\varepsilon + (-\varepsilon)(-e_l)c_j^\rho = c_je_l\varepsilon + (c_je_l\varepsilon)^\rho$ is a trace. This completes the proof.

5.9. Forms of low rank. For M of rank 3, a similar argument as in the proof of 3.14(c) shows that $\psi_q\colon \mathfrak{A}(q) \to C_0$ is an isomorphism (of modules and of Jordan algebras), even when q is not semi-regular. If q is semiregular of rank 5, it is known that the standard involution on C_0 is of even symplectic type, and the space S of traces of ρ is of rank 6 [9, V, (5.4)]. This implies $\psi_q\colon \mathfrak{A}(q) \to S$ is an isomorphism. Indeed, since ψ_q is a homomorphism between finitely generated modules, it suffices that $\psi_q \otimes k/\mathfrak{m}$ be surjective for all maximal ideals $\mathfrak{m}$ of k. Since all constructions are compatible with change of base ring, this follows from injectivity of ψ_q and the equality of the ranks.

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