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An Isoperimetric Partition Problem

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Abstract. In the present paper pairs of plane figures are considered. Each of them is inscribed into a unit circular sector with the same central angle. These indomains are linked together by the fixed total perimeter length. Pairs of figures with minimal total area and the corresponding partition of the total perimeter are determined. In general, the solution is not symmetrical. The basis for this problem forms the generalized Favard problem [6] and the analytical properties of its optimal value function.

1. Introduction

In the first half of this century one tried to find optimal parameters of convex figures while fixing other parameters. For a large number of problems this was done successfully using problem-specific methods [2].
Investigating such geometrical extremal problems we have met the isoperimetric partition problem presented in this paper. The generalized Favard problem published recently in ref. [6] forms the basis for its solution. It deals with the search for a plane figure of fixed perimeter length, inscribed into a circular sector and having minimal area.
We denote by S_ω the circular sector with radius 1 and angle $\omega \in [0, 2\pi]$. Let Δ_ω be the corresponding isosceles triangle.

Definition 1. *Any closed plain set $\mathcal{B}$ with $S_\omega \cap \Delta_\omega \subseteq \mathcal{B} \subseteq S_\omega$ and $\Delta_\omega \cup \mathcal{B}$ convex is called ω-sector indomain of S_ω.*

In this paper we consider a pair of plain domains, each inscribed into a unit circular sector S_ω. We pose the following question:

> *What is the pair of ω-sector indomains with fixed total perimeter length, having minimal total area?*

For a pair of unit circle indomains this problem was formulated and investigated by J. Focke and R. Klötzler [5] from 1987 to 1989. Generalizations for sector indomains were studied by A. Brach [1].

This extremal problem shows a hierarchic structure. It can be divided into two generalized Favard problems if we parametrically fix the perimeter length of each sector indomain. Through the given total perimeter length these two problems are linked together. We know the solutions of the partial problems. The corresponding optimal value function is the so-called *fonction penetrante*. Using these informations we determine minimizers over disjunct parts of the feasible domain. A certain convexity defect of the *fonction penetrante* involves a structural branching of the global minimal solution in dependence on the total perimeter length.

For some parameter values we demonstrate the solution and its branching by numerical calculations.

2. General Basic Problem

In ref. [6] we have shown that the solution of the generalized Favard problem is polyhedral. The boundary of a polyhedral sector indomain is composed by the legs of Δ_ω and a polygon p_ω.

Definition 2. *A polyhedral ω-sector indomain $\mathcal{P}_\omega$ is called ω-inpolyeder of S_ω if all vertices of $\mathcal{P}_\omega$ except the vertex of S_ω belong to the sector arc. The corresponding polygon p_ω is called ω-inpolygon of S_ω.*

Because we seek the solution of the generalized Favard problem among ω-inpolyeders this problem was reduced to the following basic problem:

$$(\mathrm{P}^{\omega,\mathrm{L}}): \qquad F_\omega(\varphi, n) := \frac{1}{2}\sum_{i=1}^{n} \sin\varphi_i \to \text{Min!}$$

$$L_\omega(\varphi, n) := 2\sum_{i=1}^{n} \sin\frac{\varphi_i}{2} = L$$

$$\sum_{i=1}^{n} \varphi_i = \omega$$

$$0 < \varphi_i \leq \pi \quad \text{for} \quad i = 1, ..., n$$

In this, the central angles φ_i describe the inpolygon p_ω. For the inpolyeder $\mathcal{P}_\omega$ we use the representation

$$\mathcal{P}_\omega = [\varphi_1, ..., \varphi_n].$$

The corresponding inpolygon p_ω represents the variable part of its perimeter. We call it p-perimeter and denote its length by $\mathcal{L}(\mathcal{P}_\omega)$ and the area of $\mathcal{P}_\omega$ by $\mathcal{F}(\mathcal{P}_\omega)$, where $\mathcal{L}(\mathcal{P}_\omega) = L_\omega(\varphi, n)$ and $\mathcal{F}(\mathcal{P}_\omega) = F_\omega(\varphi, n)$.
It is useful to regard this problem for all $\omega \geq 0$. This means considering multiple embraced inpolygons. For this, we have added the upper bound π for φ_i. In ref. [6] it was shown

that optimal solutions of the generalized Favard problem ($\omega \in (0, 2\pi]$) fulfil this condition automatically.
The minimal objective function value depends on the parameters L and ω. The lower bound of L is fixed by

$$\underline{L}_\omega := 2\left(k + \sin\frac{\hat{\omega}}{2}\right) \tag{2.1}$$

where k is a non-negative integer and $\hat{\omega} \in [0, \pi)$ the angle according $\omega = k\pi + \hat{\omega}$. To describe optimal solutions we give the following definition.

Definition 3. *An inpolygon p_ω of S_ω is called almost regular if all but one side of p_ω are equal, the remaining one is not greater than the others. The ω-inpolygon is called regular if all sides are equal. The corresponding inpolyeder $\mathcal{P}_\omega$ is called almost regular and regular, resp.*

A regular n-sided ($n \geq 2$) inpolygon p_n^ω and the corresponding regular inpolyeder $\mathcal{P}_n^\omega$ are described by the n equal central angles

$$\alpha_n^\omega := \begin{cases} \mathrm{Min}\,\{\omega, \pi\} & \text{for } n = 1 \\ \frac{\omega}{n} & \text{for } n \geq 2. \end{cases} \tag{2.2}$$

The length of its p-perimeter is given by

$$l_n^\omega := \begin{cases} \underline{L}_\omega & \text{for } n = 1 \\ 2n \sin\frac{\omega}{2n} & \text{for } n \geq 2. \end{cases} \tag{2.3}$$

The case $n = 1$ corresponds to an ω-inpolygon of smallest length. An almost regular n-sided ω-inpolyeder is described by $n - 1$ equal central angles α and one *defect angle* $\delta = \omega - (n-1)\alpha$ with $0 < \delta \leq \alpha$.
Almost regular and regular inpolyeders of S_ω are optimal figures of the generalized Favard problem for $L \in [\underline{L}_\omega, \omega)$.

Theorem 1. *Problem* $(\mathrm{P}^{\omega,\mathrm{L}})$ *is uniquely solvable for each $\omega \geq 0$ and $L \in [\underline{L}_\omega, \omega]$ where*
a) the number n of central angles is determined by the relation

$$L \in \left(l_{n-1}^\omega, l_n^\omega\right],$$

b) the optimal central angles are determined by a unique angle $\alpha_\omega(L) \in [\alpha_n^\omega, \alpha_{n-1}^\omega)$ according to

$$\begin{aligned} &0 < \varphi_1 = \dots = \varphi_n = \alpha_\omega(L) < \pi, \\ &0 < \varphi_n = \delta_\omega(L) = \omega - (n-1)\alpha_\omega(L) \leq \alpha_\omega(L). \end{aligned}$$

Sketch of the proof. The existence of the solution results from that of the geometrical extremal problem.

For $\omega \in (0, 2\pi]$ and restrictions $\varphi_i \leq \omega$ instead of $\varphi_i \leq \pi$ we sketch the proof of ref. [6] in step 1 and step 2.

Step 1: We fix the optimal yet unknown number n and evaluate necessary and sufficient optimality conditions for the optimization problem $(\mathrm{P}^{\omega,\mathrm{L}})$. We get

$$\alpha := \varphi_1 = ... = \varphi_{n-1} \geq \varphi_n = \omega - (n-1)\alpha =: \delta > 0 \tag{2.4}$$

as structural statement about the optimal solution.

Step 2: We verify the isoperimetric condition for $L > \underline{L}_\omega$. For this purpose we consider the piecewise composed function

$$l_\omega(\alpha) = \left\{ {}^j l_\omega(\alpha) \quad \text{for } \alpha \in \left[\alpha_j^\omega, \alpha_{j-1}^\omega\right], \quad j = 2, 3, ... \right. \tag{2.5}$$

with α_j^ω from (2.2) and

$${}^j l_\omega(\alpha) = 2(j-1)\sin\frac{\alpha}{2} + 2\sin\frac{\delta}{2}. \tag{2.6}$$

Then, we have to solve the nonlinear equation

$$L = l_\omega(\alpha) \tag{2.7}$$

for

$$\alpha \in \cup_{j=2}^{\infty} \left[\alpha_j^\omega, \alpha_{j-1}^\omega\right) = (0, \mathrm{Min}\,\{\omega, \pi\}). \tag{2.8}$$

Because the function l_ω is a strictly monotonously decreasing function of α, equation (2.7) has a unique solution $\alpha = \alpha_\omega(L) \in (0, \mathrm{Min}\{\omega, \pi\})$. In dependence on L, α_ω is a strictly monotonously decreasing function, too. From (2.8) we get a unique number n with $\alpha \in [\alpha_n^\omega, \alpha_{n-1}^\omega)$. This relation corresponds via (2.7) and $l_n^\omega = l_\omega(\alpha_n^\omega)$ to

$$L \in \left(l_{n-1}^\omega, l_n^\omega\right]. \tag{2.9}$$

Step 3: For $\omega > 2\pi$ and $L > \underline{L}_\omega$ there are at least two central angles of the optimal ω-inpolyeder $\mathcal{P}_\omega$ smaller than π, say $\varphi_1, \varphi_2 < \pi$. We assume that in the optimal solution of $(\mathrm{P}^{\omega,\mathrm{L}})$ occurs one central angle π, say $\varphi_3 = \pi$. Then, we consider the partial sector $S_{\tilde{\omega}}$ with $\tilde{\omega} = \varphi_1 + \varphi_2 + \varphi_3$. The partial inpolyeder $\mathcal{P}_{\tilde{\omega}}$ with $\mathcal{L}(\mathcal{P}_{\tilde{\omega}}) = \tilde{L} = 2\sum_{i=1}^{3} \sin\frac{\varphi_i}{2}$ has to be area-minimal. Let

$$\phi(\varphi, \lambda) = \lambda_0 \cdot \frac{1}{2}\sum_{i=1}^{3} \sin\varphi_i + \lambda_1 \left(2\sum_{i=1}^{3}\sin\frac{\varphi_1}{2} - \tilde{L}\right)$$
$$+ \lambda_2 \left(\sum_{i=1}^{3} \varphi_i - \tilde{\omega}\right) + \lambda_3(\varphi_3 - \pi)$$

be the Lagrangian function for $(\mathrm{P}^{\tilde{\omega},\tilde{\mathrm{L}}})$. The gradients $\left(\cos\frac{\varphi_1}{2},\cos\frac{\varphi_2}{2},0\right)^T$ and $(1,1,1)^T$ of the constrained functions at $\varphi=(\varphi_1,\varphi_2,\pi)$ are linearly independent. Therefore, there exist Lagrangian multipliers λ_i with $\lambda_3\geq 0$, $\lambda_0=1$ and $\phi_\varphi(\varphi,\lambda)=0$ (see [4]). This implies

$$\lambda_1=-\left(\cos\frac{\varphi_1}{2}+\cos\frac{\varphi_2}{2}\right)$$

$$\lambda_2=\frac{1}{2}+\cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2}$$

and from $\varphi_3=\pi$

$$\lambda_3=\frac{1}{2}-\lambda_2=-\cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2}<0$$

in contradiction to $\lambda_3\geq 0$. This means that all angles φ_i are less than π. Now, we can proceed as in step 1 and step 2. □

We return to $\omega\in[0,2\pi]$. In the following we use the notations $I_n:=[l^\omega_{n-1},l^\omega_n]$ and $I^0_n:=(l^\omega_{n-1},l^\omega_n)$.

Corollary 1. *For any given parameter $L\in\left(l^\omega_{n-1},l^\omega_n\right]$ there exists one and only one almost regular inpolyeder of S_ω with p-perimeter length L. This one has n central angles and solves the general basic problem.*

Proof. The assertion follows from Theorem 1. The isoperimetric condition in the shape of formula (2.7) is written for almost regular and regular inpolyeders, resp. It is uniquely solvable for an $\alpha\in[\alpha^\omega_n,\alpha^\omega_{n-1})$ representing the solution of the general basic problem. □

The area of an almost regular inpolyeder depends on L. This functional connection can be described implicitely by means of

$$f_\omega(L):=\frac{n-1}{2}\sin\alpha_\omega(L)+\frac{1}{2}\sin\delta_\omega(L) \tag{2.10}$$

with $\delta_\omega(L)=\omega-(n-1)\alpha_\omega(L)$ and $\alpha_\omega(L)$ from

$$L=2(n-1)\sin\frac{\alpha}{2}+2\sin\frac{1}{2}(\omega-(n-1)\alpha) \tag{2.11}$$

for $L\in\left(l^\omega_{n-1},l^\omega_n\right]$.The function f_ω is the optimal value function of problem $(\mathrm{P}^{\omega,\mathrm{L}})$ and called *fonction penetrante*. We summarize some properties of f_ω [7].

Lemma 1. *The function $f_\omega:[\underline{L}_\omega,\omega]\to R$ is*
a) continuous and strictly monotonously increasing,
b) strictly convex on $[l^\omega_1,l^\omega_2]$, and for $n\geq 3$ there exists one turning point $\lambda^\omega_n\in(l^\omega_{n-1},l^\omega_n)$ such that f_ω is strictly convex on $[l^\omega_{n-1},\lambda^\omega_n]$ and strictly concave on $[\lambda^\omega_n,l^\omega_n]$.

The proof of this lemma follows from sign properties of the first three derivatives of f_ω calculated by implicit differentiation [1] and summarized in the following lemma.

Lemma 2. *The function f_ω is continuously differentiable on each interval I_n^0 where*
a) $f'_\omega > 0$ *and strictly concave on* I_n^0,
f''_ω *has a unique zero* λ_n^ω *on* I_n^0,
$f'''_\omega < 0$ *and* $f_\omega^{(4)} < 0$ *on* I_n^0 *for all* $n \geq 3$,
b) $f'_\omega > 0$ *and linear on* I_2^0, $f'' > 0$ *on* I_2^0.

Further we use that any almost regular inpolyeder is decomposable in a regular and an almost regular inpolyeder. This is carried over to the corresponding p-perimeters.

Lemma 3. *The p-perimeter of an almost regular inpolyeder satisfies the decomposition rule*

$$l_{k\cdot\alpha+\omega}(\alpha) = l_{k\cdot\alpha}(\alpha) + l_\omega(\alpha).$$

Proof. Let $\omega' = k \cdot \alpha + \omega$. $\mathcal{P}_{\omega'}$ is the almost regular inpolyeder of $S_{\omega'}$ with p-perimeter $\mathcal{L}(\mathcal{P}_{\omega'}) = L' = l_{\omega'}(\alpha)$. The parameter L' corresponds uniquely to the angle α. This implies $\omega' = k' \cdot \alpha + \delta$ with $k' > k$ and $\omega = (k' - k) \cdot \alpha + \delta$ for any $0 < \delta \leq \alpha$. From (2.5) and (2.6) follows the decomposition rule. □

Furthermore, there are two possibilities for composing a regular $(n-1)$-sided and an almost regular n-sided inpolyeder.

Lemma 4. *Let $\mathcal{P}_{2\omega}$ be composed of a regular $(n-1)$-sided ω-inpolyeder $\mathcal{P}_{\omega,1}$ and an almost regular n-sided ω-inpolyeder $\mathcal{P}_{\omega,2}$. Then, the p-perimeter of $\mathcal{P}_{2\omega}$ satisfies the composition rule*

$$\mathcal{L}(\mathcal{P}_{2\omega}) = l_\omega(\alpha) + l_\omega(\alpha_{n-1}^\omega) = l_{(n-1)\alpha}(\alpha) + l_{\omega+\delta}(\alpha_{n-1}^\omega)$$

for $\alpha \in [\alpha_n^\omega, \alpha_{n-1}^\omega)$, $\delta = \omega - (n-1)\alpha$.

Proof. Because $\mathcal{P}_{\omega,1}$ is a regular $(n-1)$-sided ω-inpolyeder its p-perimeter is given by $\mathcal{L}(\mathcal{P}_{\omega,1}) = l_\omega(\alpha_{n-1}^\omega)$. The p-perimeter of the n-sided almost regular inpolyeder $\mathcal{P}_{\omega,2}$ is given by $\mathcal{L}(\mathcal{P}_{\omega,2}) = l_\omega(\alpha)$ with a unique $\alpha \in [\alpha_n^\omega, \alpha_{n-1}^\omega)$. This yields

$$\mathcal{L}(\mathcal{P}_{2\omega}) = l_\omega(\alpha) + l_\omega(\alpha_{n-1}^\omega)$$

where $\omega = (n-1)\alpha + \delta = (n-1)\cdot\alpha_{n-1}^\omega$. The composed inpolyeder $\mathcal{P}_{2\omega}$ has the representation

$$\mathcal{P}_{2\omega} = \Big[\, \underbrace{\alpha, ..., \alpha}_{n-1},\ \delta,\ \underbrace{\alpha_{n-1}^\omega, ..., \alpha_{n-1}^\omega}_{n-1} \,\Big].$$

On the other hand, for $\omega' = (n-1)\alpha$ we understand $\mathcal{P}_{2\omega}$ as the composition of a regular $(n-1)$-sided ω'-inpolyeder $\mathcal{P}_{(n-1)\cdot\alpha}$ and an almost regular n-sided $(\omega+\delta)$-inpolyeder $\mathcal{P}_{\omega+\delta}$. The p-perimeter of $\mathcal{P}_{2\omega}$ remains unchanged and can be represented via (2.5) and (2.6) by

$$\mathcal{L}(\mathcal{P}_{2\omega}) = l_{(n-1)\alpha}(\alpha) + l_{\omega+\delta}(\alpha_{n-1}^\omega).$$ □

3. Partition Problem

Now, we ask for an area-minimal pair of ω-sector indomains where the total perimeter length is fixed. For each feasible decomposition of the total perimeter length, both ω-sector indomains have to be optimal solutions of a generalized Favard problem. This means that only pairs $(\mathcal{P}_{\omega,1}, \mathcal{P}_{\omega,2})$ of almost regular ω-inpolyeders are under consideration as optimal solutions. These inpolyeders are linked through the total perimeter. Using the *fonction penetrante* to describe the area of $\mathcal{P}_{\omega,i}$ $(i = 1, 2)$ in dependence upon the p-perimeters $L_i = \mathcal{L}(\mathcal{P}_{\omega,i})$ we get the following equivalent formulation of the partition problem:

$$({}^2\mathrm{P}^{\omega,\Lambda}) \qquad {}^2f_\omega(L_1, L_2) := f_\omega(L_1) + f_\omega(L_2) \to \mathrm{Min!}_{L_1,L_2}$$

$$L_1 + L_2 = 2\Lambda$$

$$\underline{L}_\omega \le L_1 \le L_2 \le \omega$$

where the parameter $\Lambda \in [\underline{L}_\omega, \omega]$ is chosen in advance. The symmetry of ${}^2f_\omega$ with respect to $l_1 = l_2$ justifies the condition $L_1 \le L_2$. A feasible solution is given by the symmetric pair (Λ, Λ). Optimal solutions of this nonlinear parametric problem exist because the objective function ${}^2f_\omega$ is continuous and the feasible domain is nonempty and compact.

To solve this problem we distinguish two cases with respect to the partition of the total length 2Λ onto L_1 and L_2. For this, we fix $n \ge 2$ according $\Lambda \in (l^\omega_{n-1}, l^\omega_n]$ with l^ω_n from (2.3) and investigate

Case 1: $L_1, L_2 \in I^0_n$,

Case 2: $L_1 \le l^\omega_{n-1}$ or $L_2 \ge l^\omega_n$.

The cases $L_1 > l^\omega_n$ and $L_2 \le l^\omega_{n-1}$ cannot occur because we have $L_1 \le L_2$ and $\frac{1}{2}(L_1 + L_2) = \Lambda \in (l^\omega_{n-1}, l^\omega_n]$.

The second case can be formulated somewhat sharper by using the implications

$$\Lambda \in \left(l^\omega_{n-1}, \frac{1}{2}(l^\omega_{n-1} + l^\omega_n)\right], \; L_2 \ge l^\omega_n \;\; \Rightarrow \;\; L_1 \le l^\omega_{n-1} \tag{3.1}$$

and

$$\Lambda \in \left(\frac{1}{2}(l^\omega_{n-1} + l^\omega_n), l^\omega_n\right], \; L_1 \le l^\omega_{n-1} \;\; \Rightarrow \;\; L_2 \ge l^\omega_n. \tag{3.2}$$

We get the subcases

Case 2.1: $\Lambda \in \left(\frac{1}{2}(l^\omega_{n-1} + l^\omega_n), l^\omega_n\right] =: I^+_n, \;\; L_2 \ge l^\omega_n$

Case 2.2: $\Lambda \in \left(l^\omega_{n-1}, \frac{1}{2}(l^\omega_{n-1} + l^\omega_n)\right] =: I^-_n, \;\; L_1 \le l^\omega_{n-1}$.

3.1. Almost Regular Twin Solutions

We consider
Case 1: $L_1, L_2 \in I_n^0$,
and study necessary optimality conditions for problem $({}^2\mathrm{P}^{\omega,\Lambda})$. Replacing L_1 by L this problem is equivalent to the problem

$$(\mathrm{Q}_0^{\omega,\Lambda}): \qquad {}^2f_\omega(L, 2\Lambda - L) = f_\omega(L) + f_\omega(2\Lambda - L) \to \text{Min!}$$
$$L \in I_n^0 \cap (2\Lambda - l_n^\omega, \Lambda]$$

Solutions of $(\mathrm{Q}_0^{\omega,\Lambda})$ correspond to local minimizers of $({}^2\mathrm{P}^{\omega,\Lambda})$ on $I_n^0 \times I_n^0$.

Theorem 2. *If $l_{n-1}^\omega < L < 2\Lambda - L < l_n^\omega$ then L is not a local minimal solution of problem $(\mathrm{Q}_0^{\omega,\Lambda})$.*

Proof. At first, we consider the general case $n \geq 3$. We use the Lagrangian method and prove that there is no feasible $L \in I_n^0 \cap (2\Lambda - l_n^\omega, \Lambda)$ satisfying the necessary optimality conditions

$$f'_\omega(L) - f'_\omega(2\Lambda - L) = 0, \tag{3.3}$$

$$f''_\omega(L) + f''_\omega(2\Lambda - L) \geq 0. \tag{3.4}$$

Because f'_ω is strictly concave on I_n^0 with the unique maximal point $\lambda_n^\omega \in I_n^0$ (see Lemma 1) we have $L < \lambda_n^\omega < 2\Lambda - L$ from (3.3). For such L we show the implication

$$f'_\omega(L) = f'_\omega(2\Lambda - L) \Rightarrow f''_\omega(L) + f''_\omega(2\Lambda - L) < 0 \tag{3.5}$$

which is equivalent to each of the implications

$$f'_\omega(L) = f'_\omega(2\Lambda - L) \Rightarrow 0 < f''_\omega(L) < |f''_\omega(2\Lambda - L)|, \tag{3.6}$$

$$f'_\omega(L) - f'_\omega(\lambda_n^\omega) = f'_\omega(2\Lambda - L) - f'_\omega(\lambda_n^\omega) \Rightarrow 0 < {f''_\omega}^2(L) < {f''_\omega}^2(2\Lambda - L). \tag{3.7}$$

We have used that $L < \lambda_n^\omega < 2\Lambda - L$ and $f''_\omega(L) > 0$, $f''_\omega(2\Lambda - L) < 0$ because of Lemma 2. If the inequality

$$\mu\,(f'_\omega(L) - f'_\omega(\lambda_n^\omega)) + {f''_\omega}^2(L) < \mu\,(f'_\omega(2\Lambda - L) - f'_\omega(\lambda_n^\omega)) + {f''_\omega}^2(2\Lambda - L) \tag{3.8}$$

is satisfied for a suitable Lagrangian multiplier μ then we get (3.7). For this, we prove the sharpened inequality

$$\mu\,(f'_\omega(L) - f'_\omega(\lambda_n^\omega)) + {f''_\omega}^2(L) < 0 < \mu\,(f'_\omega(2\Lambda - L) - f'_\omega(\lambda_n^\omega)) + {f''_\omega}^2(2\Lambda - L) \tag{3.9}$$

studying the function

$$Q(L) := \mu\,(f'_\omega(L) - f'_\omega(\lambda_n^\omega)) + {f''_\omega}^2(L)$$

for $L \in I_n^0$ and the special multiplier $\mu := -2f_\omega'''(\lambda_n^\omega)$. Obviously, we have $Q(\lambda_n^\omega) = 0$ and $Q'(L) = (\mu + 2f_\omega'''(L))f_\omega''(L)$. Applying the mean-value theorem on f_ω'' and f_ω''' with respect to λ_n^ω and using Lemma 2 we get for the chosen parameter μ

$$Q'(L) = 2(L - \lambda_n^\omega)^2 f_\omega^{(4)}(\tilde{L}) f_\omega''(\hat{L}) < 0$$

where $\tilde{L}, \hat{L} \in (L, \lambda_n^\omega)$ are intermediate values. Hence, the inequality (3.9) follows from $Q(L) < Q(\lambda_n^\omega) = 0 < Q(2\Lambda - L)$.
Now, let be $n = 2$ and $l_1^\omega < L < 2\Lambda - L < l_2^\omega$. Because f_ω' is linear on (l_1^ω, l_2^ω) the first order necessary optimality condition (3.3) is satisfied for $L = \Lambda$ only. □

We denote the symmetric partition (Λ, Λ) as *twin solution* of problem $(^2\mathrm{P}^{\omega,\Lambda})$.

Corollary 2. *For $L_1, L_2 \in (l_{n-1}^\omega, l_n^\omega)$ the twin solution is the only local minimal solution of Problem $(^2\mathrm{P}^{\omega,\Lambda})$ on $I_n^0 \times I_n^0$ if*
a) $n = 2$ or
b) $n \geq 3$ and $\Lambda \in (l_{n-1}^\omega, \lambda_n^\omega)$.

Proof. Because of Theorem 2, only the symmetric pair (Λ, Λ) may be a local minimal solution of problem $(^2\mathrm{P}^{\omega,\Lambda})$ on $I_n^0 \times I_n^0$. Condition (3.3) holds for $L_1 = L_2 = \Lambda$. The sufficient optimality condition $2f_\omega''(\Lambda) > 0$ is satisfied for $n = 2$ and for $n \geq 3$ if $\Lambda < \lambda_n^\omega$ because of Lemma 2. □

The twin solution corresponds to a pair of almost regular ω-inpolyeders. In case $n \geq 3$, it is locally minimal if the defect angle is sufficiently small in the sense of $(n-1)\sin\frac{\delta(\Lambda)}{2} < \sin\frac{\alpha(\Lambda)}{2}$. This corresponds to $f_\omega''(\Lambda) > 0$ for $\Lambda < \lambda_n^\omega$.

Corollary 3. *Let $n \geq 3$ and $\lambda_n^\omega < \Lambda < l_n^\omega$. Then the twin solution is a local maximum of $(^2\mathrm{P}^{\omega,\Lambda})$ with*

$$2f_\omega(\Lambda) > \begin{cases} f_\omega(l_{n-1}^\omega) + f_\omega(2\Lambda - l_{n-1}^\omega) & \text{if } \Lambda \in I_n^- \\ f_\omega(l_n^\omega) + f_\omega(2\Lambda - l_n^\omega) & \text{if } \Lambda \in I_n^+. \end{cases}$$

Proof. Let $L_1 = L$. The objective function ${}^2f_\omega(L, 2\Lambda - L)$ of problem $(Q_0^{\omega,\Lambda})$ is continuous on the interval

$$I_n^0 \cap (2\Lambda - l_n^\omega, \Lambda] = \begin{cases} (l_{n-1}^\omega, \Lambda] & \text{for } \Lambda \in I_n^- \\ (2\Lambda - l_n^\omega, \Lambda] & \text{for } \Lambda \in I_n^+. \end{cases}$$

For $\Lambda \in (\lambda_n^\omega, l_n^\omega)$ we have $f_\omega''(\Lambda) < 0$ from Lemma 2. This means that $L = \Lambda$ is a local maximum of ${}^2f_\omega(L, 2\Lambda - L)$. Theorem 2 says that there are no local minimal points on $I_n^0 \cap (2\Lambda - l_n^\omega, \Lambda)$. Therefore, the function ${}^2f_\omega(L, 2\Lambda - L)$ is strictly monotonously increasing on $I_n^0 \cap (2\Lambda - l_n^\omega, \Lambda]$ which completes the proof. □

In the case $\Lambda \in (\lambda_n^\omega, l_n^\omega)$ we have found the better but nonsymmetric partitions $(l_{n-1}^\omega,\ 2\Lambda - l_{n-1}^\omega)$ and $(2\Lambda - l_n^\omega, l_n^\omega)$, resp.

3.2. Nonsymmetric and Regular Symmetric Solutions

We consider case 2 and investigate the nonsymmetric partitions of Corollary 3 with respect to global minimality of ${}^2f_\omega$ on $[\underline{L}_\omega, \Lambda] \times [\Lambda, \omega] \backslash (I_n^0 \times I_n^0)$.
For this, we interprete a pair of ω-inpolyeders as one 2ω-inpolyeder and embed the partition problem $({}^2\mathrm{P}^{\omega.\Lambda})$ in the basic problem $(\mathrm{P}^{2\omega,2\Lambda})$.

Case 2.1: $\Lambda \in I_n^+,\ L_2 \geq l_n^\omega,\ n \geq 2.$

We modify $(\mathrm{P}^{2\omega,2\Lambda})$ by adding condition (+) and get the problem

$$(\mathrm{P}_+^{2\omega,2\Lambda}) \qquad F_{2\omega}(\varphi, m) = \frac{1}{2}\sum_{i=n}^{m} \sin\varphi_i \to \text{Min!}$$

$$L_{2\omega}(\varphi, m) = 2\sum_{i=1}^{m} \sin\frac{\varphi_i}{2} = 2\Lambda$$

$$\sum_{i=1}^{m} \varphi_i = 2\omega$$

$$0 < \varphi_i < \pi \qquad \text{for } i = 1, ..., m$$

$$(+) \qquad \varphi_i \leq \alpha_n^\omega \qquad \text{for } i = 1, ..., n+1.$$

This problem results from problem $(\mathrm{P}^{2\omega,2\Lambda})$ by adding the condition (+) which ensures that an almost regular twin solution does not correspond to a feasible solution of $(\mathrm{P}_+^{2\omega,2\Lambda})$. Moreover, feasible solutions of $({}^2\mathrm{P}^{\omega.\Lambda})$ with $L_2 < l_n^\omega$ are excluded.

Lemma 5. *A feasible solution of* $({}^2\mathrm{P}^{\omega.\Lambda})$ *corresponds to a feasible solution of* $(\mathrm{P}_+^{2\omega,2\Lambda})$ *iff* $L_2 \geq l_n^\omega$.

Proof. a) Let $L_2 < l_n^\omega$. Then $L_1 > l_{n-1}^\omega$ follows from (3.2). Each pair $(\mathcal{P}_{\omega,1}, \mathcal{P}_{\omega,2})$ with $\mathcal{L}(\mathcal{P}_{\omega,i}) = L_i$ $(i = 1, 2)$ is associated with $2(n-1)$ central angles $\alpha_\omega(L_i) \in (\alpha_n^\omega, \alpha_{n-1}^\omega)$ and two defect angles $0 < \delta_\omega(L_i) < \alpha_\omega(L_i)$ (see Theorem 1). Therefore, condition (+) cannot be fulfilled.

b) Let $L_2 = l_n^\omega$. From $\Lambda \in I_n^+$ and $L_1 = 2\Lambda - L_2 \leq L_2$ follows $L_1 > l_{n-1}^\omega$. We get the representations

$$\mathcal{P}_{\omega,1} = \Big[\underbrace{\alpha, ..., \alpha}_{n-1}, \delta\Big],\ \mathcal{P}_{\omega,2} = \Big[\underbrace{\alpha_n^\omega, ..., \alpha_n^\omega}_{n}\Big]$$

with $\alpha_n^\omega \leq \alpha < \alpha_{n-1}^\omega$ and $\delta = \omega - (n-1)\alpha \leq \alpha_n^\omega$. Condition (+) is fulfilled.

c) Let $L_2 > l_n^\omega$. Then, condition (+) follows from

$$\mathcal{P}_{\omega,2} = \Big[\underbrace{\alpha, ..., \alpha}_{n}, \delta\Big]$$

with $\alpha < \alpha_n^\omega$ and $\delta \leq \alpha$. $\square$

Now, we discuss necessary conditions for an optimal solution (φ, m) of problem $(\mathrm{P}_+^{2\omega,2\Lambda})$. Each such (φ, m) represents a 2ω-inpolyeder $\mathcal{P}_{2\omega}$ with q central angles $\varphi_1 = ... = \varphi_q = \alpha_n^\omega$ for some $q \geq 0,\ n+1-q$ central angles $\varphi_i < \alpha_n^\omega$ for $i = q+1, ..., n+1$ and remaining $\varphi_i \in (0,\pi)$. We decompose $\mathcal{P}_{2\omega}$ into the inpolyeders $\mathcal{P}_{q\cdot\alpha_n^\omega}$ and $\mathcal{P}_\xi$ with $\xi := 2\omega - q\cdot\alpha_n^\omega$. The $q\cdot\alpha_n^\omega$-inpolyeder is a regular one with $\mathcal{L}(\mathcal{P}_{q\cdot\alpha_n^\omega}) = l_q^{q\cdot\alpha_n^\omega} = 2q\sin\frac{\omega}{2n}$ (if $q \geq 1$). The central angles of the ξ-inpolyeder $\mathcal{P}_\xi$ are free variables with respect to the inequality restrictions (+) of $(\mathrm{P}_+^{2\omega,2\Lambda})$. Hence, $\mathcal{P}_\xi$ corresponds to an optimal solution of problem (P^{ξ,L_ξ}) with $L_\xi := 2\Lambda - l_q^{q\cdot\alpha_n^\omega}$. This means that $\mathcal{P}_\xi$ is a well-determined almost regular $(p+1)$-sided ξ-inpolyeder with p from $L_\xi \in (l_p^\xi, l_{p+1}^\xi]$. For $\mathcal{P}_{2\omega}$ we get the representation

$$\mathcal{P}_{2\omega} = \Big[\underbrace{\alpha_n^\omega, ..., \alpha_n^\omega}_{q},\ \delta,\ \underbrace{\alpha, ..., \alpha}_{p}\Big] \tag{3.10}$$

$$2\omega = q\cdot\alpha_n^\omega + \delta + p\cdot\alpha, \tag{3.11}$$

$$\mathcal{L}(\mathcal{P}_{2\omega}) = l_q^{q\cdot\alpha_n^\omega} + l_\xi(\alpha) \tag{3.12}$$

where α, δ are uniquely determined by $L_\xi = l_\xi(\alpha)$ and $\delta = \xi - p\cdot\alpha,\ 0 < \delta \leq \alpha$ (see (2.7) and Theorem 1). Therefore, the structure of an optimal solution (φ, m) of $(\mathrm{P}_+^{2\omega,2\Lambda})$ is given by

$$\varphi = \Big(\underbrace{\alpha_n^\omega, ..., \alpha_n^\omega}_{q},\ \delta,\ \underbrace{\alpha, ..., \alpha}_{p}\Big), \quad m = q+1+p. \tag{3.13}$$

Condition (+) leads to the alternative

$$\begin{aligned} \text{either}\quad & \alpha \leq \alpha_n^\omega \\ \text{or}\quad & \alpha > \alpha_n^\omega \quad \text{and} \quad q \geq n. \end{aligned} \tag{3.14}$$

For such central angles we discuss the isoperimetric condition.

Lemma 6. *Let* (φ, m) *satisfy* (3.13). *Then the following implication holds:*

$$\alpha \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \alpha_n^\omega \Longrightarrow L_{2\omega}(\varphi, m) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 2l_n^\omega.$$

Proof. For $\alpha = \alpha_n^\omega = \frac{\omega}{n}$ we get $\delta = \alpha_n^\omega$ from (3.11) and $0 < \delta \leq \alpha$. This implies $m = 2n,\ 2\omega = 2n\cdot\alpha_n^\omega$ and $L_{2\omega}(\varphi, m) = 4n\sin\frac{\omega}{2n} = 2l_n^\omega = l_{2\omega}(\alpha_n^\omega)$. For each $q \geq 0$ we consider the decomposition $l_{2\omega}(\alpha_n^\omega) = l_{q\cdot\alpha_n^\omega}(\alpha_n^\omega) + l_\xi(\alpha_n^\omega)$ with $\xi = 2\omega - q\cdot\alpha_n^\omega$. Because the function l_ξ is strictly monotonously decreasing for increasing α the required implication is proved. □

Lemma 5 and Lemma 6 allow to derive the minimal partition (L_1, L_2) in case 2.1.

Theorem 3. *Let $\Lambda \in I_n^+$ and $L_2 \geq l_n^\omega$. Then the optimal solution of problem $(P_+^{2\omega,2\Lambda})$ is uniquely determined by $m = 2n$ and φ according* (3.13) *with $q = n$, $p = n-1$, $\alpha = \alpha_\omega(2\Lambda - l_n^\omega) \in [\alpha_n^\omega, \alpha_{n-1}^\omega)$. This solution corresponds to the minimal solution (L_1, L_2) of ${}^2f_\omega$ on $[\underline{L}_\omega, 2\Lambda - l_n^\omega) \times [l_n^\omega, \omega]$ with $L_2 = l_n^\omega$ and $L_1 \in (l_{n-1}^\omega, l_n^\omega]$.*

Proof. From $L_{2\omega}(\varphi, m) = 2\Lambda \leq 2l_n^\omega$ we can exclude $\alpha < \alpha_n^\omega$ because of Lemma 6. For $L_{2\omega}(\varphi, m) = 2l_n^\omega$ follows $\alpha = \alpha_n^\omega$, $\delta = \alpha_n^\omega$ and $m = 2n$ from Lemma 6. These angles correspond to the feasible pair $(L_1, L_2) = (l_n^\omega, l_n^\omega)$ of problem $({}^2\mathrm{P}^{\omega,\Lambda})$. For $L_{2\omega}(\varphi, m) = 2\Lambda < 2l_n^\omega$ we get $\alpha > \alpha_n^\omega$ from Lemma 6 and $q \geq n$ from (3.14). Assuming $q > n$. Then, the corresponding $(\tilde{\varphi}, m)$ from (3.13) cannot be optimal. To show this we consider the decomposition $\tilde{\varphi} = (\Psi^1, \tilde{\Psi}^2)$ with

$$\Psi^1 = \Big(\underbrace{\alpha_n^\omega, ..., \alpha_n^\omega}_{n}\Big), \quad \tilde{\Psi}_2 = \Big(\underbrace{\alpha_n^\omega, ..., \alpha_n^\omega}_{q-n},\ \delta,\ \underbrace{\alpha, ..., \alpha}_{p}\Big)$$

and $\omega = n \cdot \alpha_n^\omega = (q-n) \cdot \alpha_n^\omega + \delta + p \cdot \alpha$. A corresponding decomposition is possible for the objective function

$$F_{2\omega}(\tilde{\varphi}, m) = F_\omega(\Psi^1, n) + F_\omega(\tilde{\Psi}^2, m-n)$$

and the restriction function

$$L_{2\omega}(\tilde{\varphi}, m) = L_\omega(\Psi^1, n) + L_\omega(\tilde{\Psi}^2, m-n).$$

Obviously, $L_\omega(\Psi^1, n) = l_n^\omega$ and $F_\omega(\Psi^1, n) = f_\omega(l_n^\omega)$. We use the notation $\tilde{L} := L_\omega(\tilde{\Psi}^2, m-n)$. Taking into account $L_{2\omega}(\tilde{\varphi}, m) = 2\Lambda$ we get $\tilde{L} = 2\Lambda - l_n^\omega \in (l_{n-1}^\omega, l_n^\omega]$ because of $\Lambda \in I_n^+$. The pair $(\tilde{\Psi}^2, m-n)$ is a feasible solution of the basic problem $(\mathrm{P}^{\omega,\tilde{L}})$. It is not optimal because Theorem 1 yields the unique optimal solution

$$\Psi^2 = \Big(\underbrace{\alpha, ..., \alpha}_{n-1}, \delta\Big), \quad \alpha \in [\alpha_n^\omega, \alpha_{n-1}^\omega).$$

The corresponding composition $(\varphi, m) = ((\Psi^1, \Psi^2), 2n)$ is feasible in problem $(\mathrm{P}_+^{2\omega,2\Lambda})$ because of $\delta = \omega - (n-1)\alpha < \alpha_n^\omega$. This pair (φ, m) is one of structure (3.13) with $q = n$, $m = 2n$ and

$$F_{2\omega}(\tilde{\varphi}, m) > f_\omega(l_n^\omega) + f_\omega(\tilde{L}) = F_{2\omega}(\varphi, 2n).$$

This means that $(\varphi, 2n)$ represents the optimal solution of $(\mathrm{P}_+^{2\omega,2\Lambda})$. The corresponding 2ω-inpolyeder $\mathcal{P}_{2\omega}$ is composed by two almost regular ω-inpolyeders. Hence the pair $(\tilde{L}, l_n^\omega)$ is feasible and therefore optimal in $({}^2\mathrm{P}^{\omega,\Lambda})$ because of Lemma 5. □

Case 2.2: $\Lambda \in I_n^-$, $L_1 \leq l_{n-1}^\omega, n \geq 2$.

This case is treated in an analogous way as case 2.1.

Problem $(\mathrm{P}^{2\omega,2\Lambda})$ is modified by adding condition $(-)$ which excludes feasible solutions of $({}^2\mathrm{P}^{\omega,\Lambda})$ with $L_1 > l_{n-1}^\omega$.

We consider the problem

$$(\mathrm{P}_{-}^{2\omega,2\Lambda}) \qquad F_{2\omega}(\varphi, m) = \frac{1}{2}\sum_{i=n}^{m} \sin\varphi_i \to \text{Min!}$$

$$L_{2\omega}(\varphi, m) = 2\sum_{i=1}^{m} \sin\frac{\varphi_i}{2} = 2\Lambda$$

$$\sum_{i=1}^{m} \varphi_i = 2\omega$$

$$0 < \varphi_i < \pi \qquad \text{for } i = 1, ..., m$$

$$(-) \qquad \varphi_i \le \alpha_{0,n} \qquad \text{for } i = 1, ..., n.$$

In this, $\alpha_{0,n}$ is implicitely defined by

$$l_\omega(\alpha_{0,n}) = 2\Lambda - l_{n-1}^{\omega} =: l_{0,n}. \tag{3.15}$$

Condition $\Lambda \in I_n^-$ implies

$$l_{n-1}^{\omega} < \Lambda < l_{0,n} \le l_n^{\omega} \tag{3.16}$$

and therefore

$$\alpha_n^{\omega} \le \alpha_{0,n} < \alpha_\omega(\Lambda) < \alpha_{n-1}^{\omega} \tag{3.17}$$

and

$$0 < \delta_{0,n} := \omega - (n-1)\alpha_{0,n} \le \alpha_{0,n}. \tag{3.18}$$

The effect of the additional restriction $(-)$ describes the following lemma.

Lemma 7. *A feasible solution of* $({}^2\mathrm{P}^{\omega,\Lambda})$ *corresponds to a feasible solution of* $(\mathrm{P}_{-}^{2\omega,2\Lambda})$ *iff* $L_1 \le l_{n-1}^{\omega}$.

Proof. a) If $L_1 > l_{n-1}^{\omega}$ then $L_1 \le L_2 = 2\Lambda - L_1 < l_{0,n} \le l_n^{\omega}$. Each pair $(\mathcal{P}_{\omega,1}, \mathcal{P}_{\omega,2})$ with $\mathcal{L}(\mathcal{P}_{\omega,i}) = L_i$ corresponds to $2(n-1)$ central angles $\alpha_\omega(L_i) > \alpha_{0,n}$ and two defect angles $\delta_\omega(L_i) < \delta_{0,n}$. Restriction $(-)$ is violated.

b) If $L_1 \le l_{n-1}^{\omega}$ then $L_2 \ge l_{0,n} > l_{n-1}^{\omega}$. In this case, a feasible solution of $({}^2\mathrm{P}^{\omega,\Lambda})$ contains at least $n-1$ central angles α with $\alpha_n^{\omega} \le \alpha \le \alpha_{0,n}$ and one defect angle δ with $\delta \le \alpha_n^{\omega} \le \alpha_{0,n}$. Restriction $(-)$ is fulfilled. $\square$

The additional condition $(-)$ enables us to investigate case 2.2. by means of problem $(\mathrm{P}_{-}^{2\omega,2\Lambda})$. A feasible solution of this problem contains $q \ge 0$ angles $\varphi_i = \alpha_{0,n}$ and $m-q$ free angles with respect to restriction $(-)$. For an optimal solution the last ones describe an almost regular inpolyeder $\mathcal{P}_\xi$ with $L_\xi := 2\Lambda - l_q^{q\cdot\alpha_{0,n}}$. The corresponding 2ω-inpolyeder has the representation

$$\mathcal{P}_{2\omega} = \Big[\underbrace{\alpha_{0,n}, ..., \alpha_{0,n}}_{q},\ \delta,\ \underbrace{\alpha, ..., \alpha}_{p}\Big] \tag{3.19}$$

$$2\omega = q \cdot \alpha_{0,n} + \delta + p \cdot \alpha \tag{3.20}$$

$$\mathcal{L}(\mathcal{P}_{2\omega}) = l_q^{q\cdot\alpha_{0,n}} + l_\xi(\alpha) \tag{3.21}$$

where p, α, δ are uniquely determined by $L_\xi \in (l_p^\xi, l_{p+1}^\xi]$, $L_\xi = l_\xi(\alpha)$ and $\delta = \xi - p \cdot \alpha$, $0 < \delta \leq \alpha$. An optimal solution of $(\mathrm{P}_-^{2\omega,2\Lambda})$ is of the structure

$$\varphi = \Big(\underbrace{\alpha_{0,n}, ..., \alpha_{0,n}}_{q},\ \delta,\ \underbrace{\alpha, ..., \alpha}_{p}\Big), \quad m = q + 1 + p \tag{3.22}$$

with $L(\varphi, m)$ from (3.21). Restriction (–) implies the alternative

$$\begin{aligned} &\text{either } \ \alpha \leq \ \alpha_{0,n} \\ &\qquad \text{or } \ \alpha > \ \alpha_{0,n} \text{ and } \quad q \geq n - 1. \end{aligned} \tag{3.23}$$

Lemma 8. *For* (φ, m) *from* (3.22) *the following implication holds:*

$$\alpha \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \alpha_{0,n} \Longrightarrow L_{2\omega}(\varphi, m) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} l_{2\omega}(\alpha_{0,n})$$

Proof. For $\alpha = \alpha_{0,n}$ we get $L_{2\omega}(\varphi, m) = l_{2\omega}(\alpha_{0,n})$. Lemma 3 yields $l_{2\omega}(\alpha_{0,n}) = l_{q\alpha_{0,n}}(\alpha_{0,n}) + l_\xi(\alpha_{0,n})$ with $\xi = \delta + p \cdot \alpha_{0,n}$. Taking into account that l_ξ is strictly monotonously decreasing in α the proof is complete. □

From Lemma 7 and Lemma 8 we get an analogous result to Theorem 3.

Theorem 4. *Let* $\Lambda \in I_n^-$ *and* $L_1 \leq l_{n-1}^\omega$.
Then the optimal solution of problem $(\mathrm{P}_-^{2\omega,2\Lambda})$ *is uniquely determined by* $m = 2n$ *and* φ *according* (3.22) *with* $q = n - 1$, $p = n - 1$, $\alpha = \alpha_{0,n} \in [\alpha_n^\omega, \alpha_{n-1}^\omega)$. *This solution corresponds to the minimal solution* (L_1, L_2) *of* ${}^2f_\omega$ *on* $[\underline{L}_\omega, l_{n-1}^\omega] \times [2\Lambda - l_{n-1}^\omega, \omega]$ *with* $L_1 = l_{n-1}^\omega$ *and* $L_2 \in (l_{n-1}^\omega, l_n^\omega]$.

Proof. From $\omega = (n-1)\alpha_{n-1}^\omega = (n-1)\alpha_{0,n} + \delta_{0,n}$ (see (3.18)) follows

$$2\omega = (n-1)\alpha_{n-1}^\omega + \delta_{0,n} + (n-1)\alpha_{0,n}. \tag{3.24}$$

Lemma 4, (3.15), (3.17) and the monotonity of $l_{\omega+\delta_{0,n}}(\alpha)$ imply the estimate

$$\begin{aligned} 2\Lambda &= l_\omega(\alpha_{0,n}) + l_\omega(\alpha_{n-1}^\omega) \\ &= l_{(n-1)\alpha_{0,n}}(\alpha_{0,n}) + l_{\omega+\delta_{0,n}}(\alpha_{n-1}^\omega) \\ &< l_{(n-1)\alpha_{0,n}}(\alpha_{0,n}) + l_{\omega+\delta_{0,n}}(\alpha_{0,n}) = l_{2\omega}(\alpha_{0,n}). \end{aligned} \tag{3.25}$$

By means of Lemma 8 we can exclude the case $\alpha \leq \alpha_{0,n}$.

For $\alpha > \alpha_{0,n}$ we have $q \geq n-1$ from (3.23). The corresponding $\tilde{\varphi}$ is decomposed according $\tilde{\varphi} = (\Psi^1, \tilde{\Psi}^2)$ with

$$\Psi^1 = (\underbrace{\alpha_{0,n}, ..., \alpha_{0,n}}_{n-1}), \ \tilde{\Psi}^2 = (\underbrace{\alpha_{0,n}, ..., \alpha_{0,n}}_{q-n+1}, \ \delta, \ \underbrace{\alpha, ..., \alpha}_{p}).$$

This leads to the decompositions

$$F_{2\omega}(\tilde{\varphi}, m) = F_{(n-1)\alpha_{0,n}}(\Psi^1, n-1) + F_\rho(\tilde{\Psi}^2, m-n+1)$$

and

$$L_{2\omega}(\tilde{\varphi}, m) = L_{(n-1)\alpha_{0,n}}(\Psi^1, n-1) + L_\rho(\tilde{\Psi}_2, m-n+1)$$

with $\rho = 2\omega - (n-1)\alpha_{0,n} = \omega + \delta_{0,n}$ (see (3.24)). From $L_{(n-1)\alpha_{0,n}}(\Psi^1, n-1) = l_{(n-1)\alpha_{0,n}}(\alpha_{0,n}) = l_{n-1}^{(n-1)\alpha_{0,n}}$ we get $F_{(n-1)\alpha_{0,n}}(\Psi^1, n-1) = f_{(n-1)\alpha_{0,n}}(l_{n-1}^{(n-1)\alpha_{0,n}})$. For $\tilde{L} := 2\Lambda - l_{(n-1)\alpha_{0,n}}(\alpha_{0,n}) = l_\rho(\alpha_{n-1}^\omega)$ we consider the problem $(\mathrm{P}^{\rho,\tilde{L}})$ with the feasible solution $(\tilde{\Psi}_2, m-n+1)$. It follows

$$\alpha_n^\rho = \frac{\rho}{n} = \frac{2\omega - (n-1)\alpha_{0,n}}{n} = \frac{(n-1)\alpha_{n-1}^\omega + \delta_{0,n}}{n} < \alpha_{n-1}^\omega$$

and

$$\alpha_{n-1}^\rho = \frac{\rho}{n-1} = \alpha_{n-1}^\omega + \frac{\delta_{0,n}}{n-1} > \alpha_{n-1}^\omega$$

from (3.17), (3.18) and (3.24). This implies $\tilde{L} \in (l_{n-1}^\rho, l_n^\rho)$. Hence, from Theorem 1 it follows that the pair (Ψ^2, n) with $\Psi^2 = (\underbrace{\alpha_{n-1}^\omega, ..., \alpha_{n-1}^\omega}_{n-1}, \ \delta_{0,n})$ is optimal in $(\mathrm{P}^{\rho,\tilde{L}})$. The composed pair $(\varphi, 2n-1)$ with $\varphi = (\Psi^1, \Psi^2)$ satisfies

$$F_{2\omega}(\tilde{\varphi}, m) \geq f_{(n-1)\alpha_{0,n}}(2\Lambda - \tilde{L}) + f_\rho(\tilde{L}) = F_{2\omega}(\varphi, 2n-1)$$

for $\tilde{\varphi}$ from (3.22). This means that $(\varphi, 2n-1)$ of structure (3.22) with $\alpha = \alpha_{n-1}^\omega$ and $q = n-1$ represents the optimal solution of $(\mathrm{P}_-^{2\omega,2\Lambda})$. It corresponds to a pair of almost regular ω-inpolyeders with p-perimeters l_{n-1}^ω, $2\Lambda - l_{n-1}^\omega$. This pair yields a feasible and therefore optimal solution of $({}^2\mathrm{P}^{\omega,\Lambda})$ on $[\underline{L}_\omega, l_{n-1}^\omega] \times [2\Lambda - l_{n-1}^\omega, \omega]$ because of Lemma 7. □

In case 2, nonsymmetric partitions turned out to be minimal. Moreover, a regular $(n-1)$- or n-sided ω-inpolyeder belongs to this partition.

4. Solution Branching and Global Solution

In the previous section we have shown that the symmetric as well as a nonsymmetric partition come into consideration as solution of our isoperimetric partition problem. Let $\Lambda \in (l_{n-1}^\omega, l_n^\omega]$, then the following partitions may be optimal:

(TS) a pair $(\mathcal{P}_\omega, \mathcal{P}_\omega)$ of two equal n-sided almost regular ω-inpolyeders,
(NS$_+$) a pair $(\mathcal{P}_\omega, \mathcal{P}_n^\omega)$ of a regular n-sided ω-inpolyeder $\mathcal{P}_n^\omega$ and an almost regular n-sided ω-inpolyeder $\mathcal{P}_\omega$ for $\Lambda \in I_n^+$,
(NS$_-$) a pair $(\mathcal{P}_{n-1}^\omega, \mathcal{P}_\omega)$ of a regular $(n-1)$-sided ω-inpolyeder $\mathcal{P}_{n-1}^\omega$ and an almost regular n-sided ω-inpolyeder $\mathcal{P}_\omega$ for $\Lambda \in I_n^-$.

In these partitions the number of chords between $\mathcal{P}_{\omega,1}$ and $\mathcal{P}_{\omega,2}$ differs at most by one. Always a regular ω-inpolyeder belongs to the nonsymmetric partition.
We have found these solutions as minimal points of the objective function ${}^2f_\omega$ over disjunct parts of the feasible domain, namely

Case 1: $L_1 = L_2 = \Lambda$
with ${}^2f_\omega(L_1, L_2) = 2f_\omega(\Lambda)$ for $n = 2$ or $\Lambda < \lambda_n^\omega$,
Case 2.1: $L_2 = l_n^\omega,\ L_1 = 2\Lambda - l_n^\omega \in (l_{n-1}^\omega, l_n^\omega]$
with ${}^2f_\omega(L_1, L_2) = f_\omega(2\Lambda - l_n^\omega) + f_\omega(l_n^\omega)$,
Case 2.2: $L_1 = l_{n-1}^\omega,\ L_2 = 2\Lambda - l_{n-1}^\omega \in (l_{n-1}^\omega, l_n^\omega]$
with ${}^2f_\omega(L_1, L_2) = f_\omega(l_{n-1}^\omega) + f_\omega(2\Lambda - l_{n-1}^\omega)$.

For taking a decision about the global minimal solution of problem $({}^2P^{\omega,\Lambda})$ we consider the following *defect function*:

$$D_\omega^n(\Lambda) := \begin{cases} f_\omega(l_{n-1}^\omega) + f_\omega(2\Lambda - l_{n-1}^\omega) - 2f_\omega(\Lambda) & \text{for } \Lambda \in I_n^- \\ f_\omega(2\Lambda - l_n^\omega) + f_\omega(l_n^\omega) - 2f_\omega(\Lambda) & \text{for } \Lambda \in I_n^+ \,. \end{cases} \tag{4.1}$$

The sign of this continuous function determines the solution of our problem.

4.1. Convexity defect

The function $\frac{1}{2}D_\omega^n$ describes a convexity defect of the *fonction penetrante* f_ω over I_n in the following sense.

Definition 4. *Let $f : [0,1] \to R$ be a continuous function. Then the functions*

$$\varphi_{f,0}(x) := \frac{1}{2}(f(0) + f(2x)) - f(x) \qquad \text{for } x \in \left[0, \frac{1}{2}\right]$$

$$\varphi_{f,1}(x) := \frac{1}{2}(f(1) + f(2x-1)) - f(x) \qquad \text{for } x \in \left[\frac{1}{2}, 1\right]$$

are called the left-sided and the right-sided convexity defect of f, respectively. The function

$$\varphi_f(x) := \begin{cases} \varphi_{f,0}(x) & \textit{for } x \in [0, \frac{1}{2}] \\ \varphi_{f,1}(x) & \textit{for } x \in [\frac{1}{2}, 1] \end{cases}$$

is called convexity defect of f.

We study zeros of φ_f under well-defined conditions on f.

Lemma 9. *Let $f : [0,1] \to R$ be a twice differentiable function on (0,1) and $\xi \in (0,1)$ a unique turning point of f with $f''(x) > 0$ for $x < \xi$ and else $f''(x) < 0$. Then, the convexity defect φ_f of f has a unique zero $x_0 \in (0,1)$ with*

a) $x_0 \in \left(0, \frac{1}{2}\right]$ for $\varphi_f(\frac{1}{2}) \le 0$,

b) $x_0 \in \left[\frac{1}{2}, 1\right]$ for $\varphi_f(\frac{1}{2}) \ge 0$.

Proof. a) Let $x \in (0, \frac{1}{2}]$. From $f''(x) > 0$ on $(0, \xi)$ it follows $f'(2x) > f'(x)$ on $(0, \frac{1}{2}\xi)$ and therefore $\varphi'_f(x) > 0$ on $(0, \frac{1}{2}\xi)$.
Together with $\varphi(0) = 0$ we get $\varphi_f(x) > 0$ on $(0, \frac{1}{2}\xi)$. The function φ_f is strictly concave on $(\frac{1}{2}\xi, \xi) \cap (\frac{1}{2}\xi, \frac{1}{2})$ because $\varphi''_f(x) < 0$ for the corresponding x. If $\xi < \frac{1}{2}$ then we use $\varphi'_f(x) < 0$ on $(\xi, \frac{1}{2})$. Thus, there exists a unique zero $x_0 \in (\frac{1}{2}\xi, \frac{1}{2}]$ of φ_f in case $\varphi_f(\frac{1}{2}) \le 0$.
b) Let $x \in [\frac{1}{2}, 1]$. We consider the reflected function $g(y) = -f(1-y)$ for $y \in [0,1]$ and use the connection $\varphi(y) = -\varphi_f(x)$ for $y \in [0, \frac{1}{2}], x = 1-y$. The existence of a unique zero $y_0 \in (0, \frac{1}{2}]$ in case $\varphi_g(\frac{1}{2}) \le 0$ follows from part a). This proves part b). □

4.2. Global Minimum

The analytical properties of f_ω described in Lemma 1 and Lemma 2 are transfered to the function $f(x) = f_\omega(\Lambda(x))$ with $\Lambda(x) = l^\omega_{n-1} + x(l^\omega_n - l^\omega_{n-1})$, $n \ge 3$ and $x \in [0,1]$. The convexity defect is given by $\varphi_f(x) = \frac{1}{2} D^n_\omega(\Lambda(x))$. Lemma 9 ensures the existence of a unique zero ${}^2\lambda^\omega_n \in I^0_n$ of $D^n_\omega(\Lambda)$ on I^0_n where

$$D^n_\omega(\Lambda) \begin{cases} > 0 & \text{for } \Lambda \in (l^\omega_{n-1}, {}^2\lambda^\omega_n) \\ < 0 & \text{for } \Lambda \in ({}^2\lambda^\omega_n, l^\omega_n) \end{cases} \tag{4.2}$$

and

$$\begin{aligned} &\text{either} \quad {}^2\lambda^\omega_n \in I^-_n \quad \text{for } D^n_\omega\Big(\frac{1}{2}(l^\omega_{n-1} + l^\omega_n)\Big) \le 0 \\ &\text{or} \quad {}^2\lambda^\omega_n \in I^+_n \quad \text{for } D^n_\omega\Big(\frac{1}{2}(l^\omega_{n-1} + l^\omega_n)\Big) > 0. \end{aligned} \tag{4.3}$$

The defect function D^n_ω from (4.1) compares parametrically in Λ objective values of ${}^2f_\omega$ coming into consideration as global minimal solutions of $({}^2\mathrm{P}^{\omega,\Lambda})$. Relations (4.2) and (4.3) show that ${}^2\lambda^\omega_n$ describes a *branch point* for the solution structure if $n \ge 3$. It is related to the turning point λ^ω_n as follows.

Lemma 10. *The branch point ${}^2\lambda^\omega_n$ $(n \ge 3)$ satisfies*

$$\frac{1}{2}(l^\omega_{n-1} + \lambda^\omega_n) \le {}^2\lambda^\omega_n \le \lambda^\omega_n.$$

Proof. Corollary 3 implies $D^n_\omega(\Lambda) < 0$ for every $\Lambda \in (\lambda^\omega_n, l^\omega_n]$ and therefore ${}^2\lambda^\omega_n \le \lambda^\omega_n$ for the zero ${}^2\lambda^\omega_n$ of D^n_ω.

To verify the lower bound we distinguish two cases.

a) $D^n_\omega(\frac{1}{2}(l^\omega_{n-1}+l^\omega_n))\le 0$:

We consider $f(x)=f_\omega(\Lambda(x))$ with $\Lambda(x)=l^\omega_{n-1}+x(l^\omega_n-l^\omega_{n-1})$, $x\in[0,1]$ and $\lambda^\omega_n=\Lambda(\xi)$, ${}^2\lambda^\omega_n=\Lambda(x_0)$. Using $x_0\ge\frac{1}{2}\xi$ from the proof of Lemma 9 we get ${}^2\lambda^\omega_n\ge\frac{1}{2}(l^\omega_{n-1}+\lambda^\omega_n)$.

b) $D^n_\omega(\frac{1}{2}(l^\omega_{n-1}+l^\omega_n))\ge 0$:

The lower bound results from $x_0\ge\frac{1}{2}$ and $l^\omega_n\ge\lambda^\omega_n$. □

For $n=2$, the strict convexity of f_ω on I_2 implies

$$D^2_\omega(\Lambda)>0\quad\text{for }\Lambda\in I^0_2. \tag{4.4}$$

This means that there is no branch point for $\Lambda\in I_2$.

These results allow us to identify the global minimal solution of the partition problem $({}^2\mathrm{P}^{\omega,\Lambda})$.

Theorem 5. *Let n be determined according $\Lambda\in(l^\omega_{n-1},l^\omega_n]$. If $n=2$ then the optimal solution of the partition problem is given by the twin solution (TS).*
If $n\ge 3$ then there exists a unique zero ${}^2\lambda^\omega_n$ of the defect function D^ω_n on I^0_n describing a branching of the solution of the partition problem in the following manner:

- *(i) for $\Lambda\in(l^\omega_{n-1},{}^2\lambda^\omega_n)$ the twin solution (TS) is optimal,*
- *(ii) for $\Lambda={}^2\lambda^\omega_n$ the double solution (TS) and either (NS$_+$) for ${}^2\lambda^\omega_n\in I^+_n$ or (NS$_-$) for ${}^2\lambda^\omega_n\in I^-_n$ is optimal,*
- *(iii) for $\Lambda\in({}^2\lambda^\omega_n,l^\omega_n)$ the solution (NS$_+$) is optimal for $\Lambda\in I^+_n$ and else (NS$_-$),*
- *(iv) for $\Lambda=l^\omega_n$ the coinciding solutions (TS) and (NS$_+$) are optimal.*

Proof. From the Theorems 2, 3 and 4 it follows that only solutions (TS), (NS$_+$), (NS$_-$) may be optimal. These solutions are compared with each other via the defect function D^ω_n. If $n=2$ then Corollary 2 and (4.4) yield the twin solution (TS) as optimal solution. For $n\ge 3$ the existence of the unique zero ${}^2\lambda^\omega_n$ of D^ω_n on I^0_n $(n\ge 3)$ results from Lemma 9. This parameter value describes the change of the symmetric solution (TS) to a nonsymmetric solution (NS$_+$) or (NS$_-$), resp. Because of Corollary 2 and Lemma 10 solution (TS) is optimal for $\Lambda\in(l^\omega_{n-1},{}^2\lambda^\omega_n)$. The other cases follow immediately. □

We complete this section with a study about the asymptotic behaviour of the branch points ${}^2\lambda^\omega_n$. Lemma 10 shows that the values ${}^2\lambda^\omega_n$ are associated with the turning points λ^ω_n. We denote $\tau^\omega_n=\alpha_\omega(\lambda^\omega_n)$ and $\sigma^\omega_n=\delta_\omega(\lambda^\omega_n)$ for $L=\lambda^\omega_n$ (see (2.11)) where $(n-1)\tau^\omega_n+\sigma^\omega_n=\omega$, $\frac{\omega}{n}\le\tau^\omega_n\le\frac{\omega}{n-1}$, $0\le\sigma^\omega_n\le\tau^\omega_n$. From $f''(\lambda^\omega_n)=0$ it follows

$$(n-1)\sin\frac{\sigma^\omega_n}{2}=\sin\frac{\tau^\omega_n}{2}. \tag{4.5}$$

We use $\sin x=x+O(x^3)$ for sufficiently small x where $\frac{O(h)}{h}$ is bounded for $h\to 0$. Because of $\tau^\omega_n=O(n^{-1})$ and $\sigma^\omega_n\le\tau^\omega_n$ relation (4.5) implies

$$\sigma^\omega_n=O(n^{-2}) \tag{4.6}$$

for sufficiently large n.

Lemma 11. λ_n^ω, ${}^2\lambda_n^\omega \in I_n^-$ *for sufficiently large* n.

Proof. We compare $\lambda_n^\omega - l_{n-1}^\omega$ with $l_n^\omega - l_{n-1}^\omega$ under consideration of (2.3) and $\lambda_n^\omega = 2(n-1)\sin\frac{\tau_n^\omega}{2} + 2\sin\frac{\sigma_n^\omega}{2}$. Using (4.5) and $\sin x = x - \frac{x^3}{3} + O(x^5)$ we get

$$\begin{aligned}\frac{\lambda_n^\omega - l_{n-1}^\omega}{l_n^\omega - l_{n-1}^\omega} &= \frac{-(n-1)(\tau_n^\omega)^3 - (\sigma_n^\omega)^3 + \frac{\omega^3}{(n-1)^2} + O(n^{-4})}{\omega^3(-\frac{1}{n^2} + \frac{1}{(n-1)^2}) + O(n^{-4})} \\ &= \frac{\omega^3 - (n-1)^3(\tau_n^\omega)^3 + O(n^{-2})}{\omega^3(\frac{2}{n} - \frac{1}{n^2}) + O(n^{-2})} \\ &= O(n^{-1})\end{aligned} \tag{4.7}$$

and therefore ${}^2\lambda_n^\omega \in I_n^-$ for sufficiently large n. Lemma 10 implies the same asymptotic behaviour for ${}^2\lambda_n^\omega$. □

The optimal solution structure depending on the parameter Λ shows a jump at $\Lambda = {}^2\lambda_n^\omega$. For sufficiently large n the nonsymmetric solutions (NS_-) and (NS_+) both dominate. The numerical calculations show that ${}^2\lambda_n^\omega \in I_n^+$ occurs for small n. In these cases the twin solution (TS) dominates.

5. Numerical Treatment

The branch point ${}^2\lambda_n^\omega$ enables us to draw a decision about the solution pair of the partition problem. Because ${}^2\lambda_n^\omega$ is the unique zero of the defect function we have to solve the nonlinear equation $D_n^\omega(\Lambda) = 0$ on I_n^0.

However D_ω^n depends only implicitely on Λ. Therefore, we substitute this one equation by the following system of two nonlinear equations for corresponding central angles $\alpha_\Lambda,\ \alpha_L \in [\alpha_n^\omega, \alpha_{n-1}^\omega]$. The angle α_Λ corresponds to $\Lambda, \alpha_{n-1}^\omega$ to l_{n-1}^ω and α_L to $L = 2\Lambda - l_{n-1}^\omega$. We have to distinguish two cases according to (4.2), (4.3). For $D_n^\omega(\frac{1}{2}(l_{n-1}^\omega + l_n^\omega) \le 0$ we get

$$f^\omega(\alpha_{n-1}^\omega) + f^\omega(\alpha_L) - 2f\omega(\alpha_\Lambda) = 0 \tag{5.1}$$

$$l_\omega(\alpha_{n-1}^\omega) + l_\omega(\alpha_L) - 2l\omega(\alpha_\Lambda) = 0 \tag{5.2}$$

with $f^\omega(\alpha) = \frac{n-1}{2}\sin\alpha + \frac{1}{2}\sin(\omega - (n-1)\alpha)$ and l_ω from (2.2). In order to solve system (5.1), (5.2) the Newton method is used.

	n	$D_\omega^n(\frac{1}{2}(l_{n-1}^\omega + l_n^\omega))$	${}^2\lambda_n^\omega$
$\omega = 2\pi$	3	2.825E-02	4.781149802531
	4	-1.422E-03	5.421987271198
	5	-1.968E-03	5.744385342918
	6	-1.137E-03	5.917855737554
	7	-6.256E-04	6.020537706570
	8	-3.550E-04	6.085898726379
	9	-2.104E-04	6.129868389893
	10	-1.300E-04	6.160769675756
	20	-4.720E-06	6.254828582382
$\omega = \frac{\pi}{3}$	3	1.121 E-05	1.040450413358
	4	1.766 E-07	1.043114785701
	5	-2.245 E-07	1.044661940327
	6	-1.443 E-07	1.045485013600
	7	-8.096 E-08	1.045969679515
	8	-4.614 E-08	1.046277086067
	9	-2.735 E-08	1.046483299368
	10	-1.689 E-08	1.046627899579
	20	-6.090 E-10	1.047066101157

table 1

To specify a suitable initial point we follow the discussion in [1] where the algorithm is implemented in a Pascal program. For a suitable initial value Λ^0 we determine α_Λ^0 from $l_\omega(\alpha_\Lambda^0) = \Lambda^0$ and α_L^0 from $l_\omega(\alpha_L^0) = 2\Lambda^0 - l_{n-1}^\omega$ by means of the Newton method. Consequently, the second equation (5.2) is fulfilled approximately for the initial point $(\alpha_\Lambda^0, \alpha_L^0)$. Some results are given in table 1 for the central angles 2π and $\frac{\pi}{3}$, resp.

Concluding we illustrate the solution branching for $\omega = 2\pi$ and $n = 4$. The numerical computation yields $D_{2\pi}^4\left(\frac{1}{2}(l_3^{2\pi} + l_4^{2\pi})\right) < 0$. This implies ${}^2\lambda_4^{2\pi} \in (l_3^{2\pi}, \frac{1}{2}(l_3^{2\pi} + l_4^{2\pi}))$. If the parameter Λ is choosen according to the three cases

1. $\Lambda \in (l_3^{2\pi}, {}^2\lambda_4^{2\pi})$

2.1. $\Lambda \in ({}^2\lambda_4^{2\pi}, \frac{1}{2}(l_3^{2\pi} + l_4^{2\pi}))$

2.2. $\Lambda \in (\frac{1}{2}(l_3^{2\pi} + l_4^{2\pi}), l_4^{2\pi})$

we get three different solution structures (TS), (NS–), (NS+) of the partition problem as shown in figure 1.

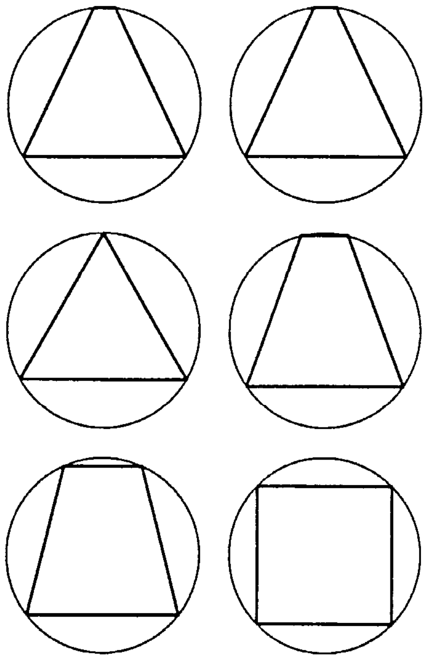

figure 1

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