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On the Generalized Theta Divisor

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Abstract. We denote by $SU_X(n)$ the moduli space of semistable rank n bundles with trivial determinant on a complex curve X. Let L be the ample generator of the Picard group of $SU_X(n)$. In this paper we show the base point freeness of the linear system $|L^{\otimes k}|$, for any integer $k > (n-1)^2(g-1)$.

1. Introduction

Let X be a smooth complete complex curve of genus g. We consider the moduli space $SU_X(n)$ of semistable rank n vector bundles with determinant $\mathcal{O}_X$, which is a projective scheme. This space was constructed in [9] by C. S. Seshadri. If we fix an arbitrary line bundle $L \in \mathrm{Pic}_X^{g-1}$ and define $\Theta_L = \{E \in SU_X(n) \mid h^0(L \otimes E) > 0\}$, then we have $\mathrm{Pic}(SU_X(n)) = \mathbb{Z}\mathcal{O}(\Theta_L)$ (cf. [3]). The linear system $|k\Theta_L|$ is base point free, for $k \gg 0$. But can we give concrete bounds? For the case $n = 2$ it is well known, that $|\Theta_L|$ is base point free (see [8]). We give a new proof of this fact in §3. For $n > 2$, $|\Theta_L|$ may have base points. But up to now there are no examples of base points for $k = 2$ (see [1]). The first affirmative answer to this problem gave J. Le Potier in [6]. He showed that for $k > \frac{n^3}{4}(g-1)$ the linear system $|k\Theta_L|$ is base point free. We will show in §2, that this bound can be replaced by $k > (n-1)^2(g-1)$.

We use the following idea of J. Le Potier [6] to construct sections of $\mathcal{O}(\Theta_L)^{\otimes k}$: From a vector bundle F of rank k that satisfies $H^*(E \otimes F) = 0$, for one $E \in SU_X(n)$, we obtain a canonical section (called Theta-function) in $\mathcal{O}(\Theta_L)^{\otimes k}$. Its divisor of zeros Θ_F is supported on the set $\{E \in SU_X(n) \mid h^0(E \otimes F) > 0\}$. Consequently a vector bundle E is not a base

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point of $|k\Theta_L|$, if there exists a vector bundle F of rank k such that $H^*(E \otimes F) = 0$. This implies, moreover, that F must be semistable of degree $k(g-1)$.

G. Faltings gives in §1 of [4] a construction of the moduli space $SU_X(n)$ which is based on Theta divisors. In particular this construction does not need the Geometric Invariant Theory.

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2. A bound for base point freeness

In this section we need the following lemma due to H. Lange (see [5]). We need it in the following form (see [2]).

Lemma 2.1. *If E is a general point in the moduli space $U_X(n,d)$, then for any $F \subset E$ holds:*

$$\mu(F) \leq \mu(E) - \frac{\mathrm{rk}(E) - \mathrm{rk}(F)}{\mathrm{rk}(E)}(g-1) \qquad .$$

We need the following technical fact:

Lemma 2.2. *Let S be a reduced scheme, s_0 a geometric point of S and $\tilde{s} \cong spec(k[\varepsilon]/\varepsilon^2)$ an infinitesimal neighbourhood of s_0 in S. Let, moreover, $\mathcal{E}$ be a vector bundle over $X \times S$ with $h^0(\mathcal{E}_s) = m$, for all $s \in S$. Then $h^0(\mathcal{E}_{\tilde{s}}) = 2m$.*

Proof. We denote the projection of $X \times S$ to its second factor by q. By [7] Cor. II.5.2 it follows that $q_*\mathcal{E}$ is a vector bundle on S. Now the assertion follows from the base change theorem. □

Theorem 2.3. *Let $E \in U_X(n,0)$ be a stable bundle, and $k > (n-1)^2(g-1)$ an integer. Then there exists a stable $F \in U_X(k, k(g-1))$, such that $h^0(E \otimes F) = 0$.*

Proof. We use the isomorphism $H^0(E \otimes F) = \mathrm{Hom}(E^\vee, F)$.
Step 1: Here we show that for a general $F \in U(k, k(g-1))$ (in the sense of 2.1) any nontrivial morphism $\varphi : E^\vee \to F$ is an embedding. So let F be a general bundle and suppose $\varphi : E^\vee \to F$ were not an embedding. We set $M = \mathrm{im}\varphi$. Since $E^\vee$ is stable, we obtain $\deg M \geq 1$. From $\mathrm{rk}M \leq n-1$ and 2.1 we conclude:

$$\begin{array}{ccccc} \frac{1}{\mathrm{rk}M} & \leq & \mu(M) & \leq & g-1-\frac{k-\mathrm{rk}M}{k}(g-1) \\ \text{and so} & & k & \leq & (n-1)^2(g-1) \quad . \end{array}$$

But this contradicts our assumption on k.

Step 2: Let now $U \subset U(k, k(g-1))$ be the open set of general stable bundles. This set is stratified into locally closed subsets U_m, defined by $U_m = \{F \in U \mid h^0(F \otimes E) = m\}$. We show, that the codimension of U_m in U is at least m.

We consider a bundle $F \in U_m$ and $\eta \in \mathrm{Ext}^1(F,F)$ corresponding to an extension $0 \to F \to \tilde{F} \to F \to 0$. If η is in the tangent space $T_{U_m,F}$, then by lemma 2.2 we have $h^0(E \otimes \tilde{F}) = 2m$.

In particular the boundary map $\partial : \mathrm{Hom}(E^\vee, F) \to \mathrm{Ext}^1(E^\vee, F)$ is the zero homomorphism. Thus for any nontrivial $\varphi \in \mathrm{Hom}(E^\vee, F)$ the element η is in the kernel of $\pi : \mathrm{Ext}^1(F, F) \to Ext^1(E^\vee, F)$. By step 1, any nontrivial φ is an embedding an d hence π is surjective. This shows that $T_{U_m,F} \subset \ker(\pi)$ and $\mathrm{codim}(T_{U_m,F}, T_{U,F}) \geq \dim(\mathrm{Ext}^1(E^\vee, F)) = m$. □

Corollary 2.4. *Let Θ_L be a Theta divisor on $SU_X(n)$. Then the linear system $|k\Theta_L|$ is base point free, for $k > (n-1)^2(g-1)$.*

Proof. Given a semistable bundle E. If E is not stable, then we have a Jordan-Hölder filtration $0 \subset E_1 \subset \ldots \subset E_k = E$, such that the quotients E_l/E_{l-1} are stable bundles. By theorem 2.3 there exist semistable vector bundles F_l on X with $H^*(F_l \otimes (E_l/E_{l-1})) = 0$, $\mathrm{rk}(F_l) = k$ and $\deg(F_l) = k(g-1)$, for $l = 1, \ldots, k$. We denote by $U_X(k, k(g-1))$ the moduli space of rank-k vector bundles on X of degree $k(g-1)$. The subsets $\Theta_{E_l/E_{l-1}} = \{F \in U_X(k, k(g-1)) \mid H^*((E_l/E_{l-1}) \otimes F) \neq 0\}$ are properly closed subsets. Hence we can find a vector bundle F which is not in their union. For this bundle F holds $H^*(E \otimes F) = 0$. □

By the same method we prove:

Theorem 2.5. *Let E be a semistable bundle of rank n and degree d. If (k, m) is a pair of integers with $k > \frac{n(n-1)^2(g-1)}{(n,d)}$ and $kd + nm = kn(g-1)$, then there exists a stable bundle F of degree m and rank k such that $h^0(F \otimes E) = 0$.*

Corollary 2.6. *Let $U_X(n, L)$ be the moduli space of semistable rank n bundles on X with determinant L. We denote by h the ample generator of its Picard group. Then the linear system $|h^{\otimes k}|$ is base point free for $k > (n-1)^2(g-1)$.*

Corollary 2.7. *Let E be a vector bundle of rank n and degree d. Then the following conditions are equivalent.*

1. *E is semistable.*
2. *For any $r = \frac{n}{(n,d)}k$ with $k > (n-1)^2(g-1)$, there exists a vector bundle F of rank r, such that $H^*(E \otimes F) = 0$.*
3. *There exists a vector bundle F, such that $H^*(E \otimes F) = 0$.*

3. The case $SU_X(2)$

Theorem 3.1. *Let F be a stable bundle of rank 2 and $\deg F = 0$. Then there exists a line bundle $L \in \mathrm{Pic}_X^{g-1}$, such that $h^0(F \otimes L) = 0$.*

Proof. We define $U_F^k = \{L \in \mathrm{Pic}_X^{g-1+k} \mid h^1(L \otimes F) = 0\}$, for any $k \geq 0$. These are open subsets of the Jacobians. For $k \geq g-1$ they are nonempty. We set $m = min\{k \mid U_F^k \neq \emptyset\}$.

If $m = 0$, then we are done, so we suppose $m > 0$. We take a $L \in U_F^m$. We have $H^0(F \otimes L) \cong \mathrm{Hom}(F^\vee, L)$. We consider two linearly independent elements φ_1 and φ_2 of $\mathrm{Hom}(F^\vee, L)$. If $\ker\varphi_1 \neq \ker\varphi_2$, then we choose a point $P \in X$, such that: $\ker\varphi_1(P) \neq \ker\varphi_2(P)$ and $P \notin \mathrm{supp}(L/\mathrm{im}\varphi_i)$.

Then the short exact sequence $0 \to L(-P) \to L \to k(p) \to 0$ yields a surjection $\mathrm{Hom}(F^\vee, L) \to \mathrm{Hom}(F^\vee, k(p))$. Hence $L(-P) \in U_F^{m-1}$. So if $U_F^{m-1} = \emptyset$, this would imply, that there exists a $L_0 \subset F$ with $\ker \varphi = L_0$, for all nonzero $\varphi \in \mathrm{Hom}(F^\vee, L)$. So we obtain an exact sequence $0 \to L_0 \to F^\vee \to L_1 \to 0$ and an isomorphism $\mathrm{Hom}(L_1, L) \cong \mathrm{Hom}(F^\vee, L)$.

We set $k = \deg L_1$ and write $L = L_1(D)$, where D is an effective divisor of degree $g - 1 + m - k$. From the above isomorphism we obtain $h^0(\mathcal{O}_X(D)) = 2m$. But this must hold for almost all effective divisors of degree $g - 1 + m - k$. This implies $m + k = 0$ which contradicts the semistability of F. □

Corollary 3.2. *The Theta divisor on $SU_X(2)$ is base point free.*

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