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Harmonic Maps of Lorentz Surfaces, Quadratic Differentials and Paraholomorphicity

Sadettin Erdem

Matematik Bölümü, Orta Doğu Teknik Üniversitesi
06531 Ankara, TURKEY
e-mail: serdem@rorqual.cc.metu.edu.tr

Abstract. Some relations are established between harmonic maps f of Lorentz surfaces M into indefinite Riemannian manifolds and paracomplex analysis including a theory of quadratic and higher order differentials associated to f as in the Riemannian case.

Introduction

There has been some research in classifying and characterizing harmonic maps in terms of holomorphic data. Among many other works [12], [15], [16] and [17] may be cited. In [16], [17] harmonicity of maps of Riemann surfaces M is linked up with holomorphicity of certain quadratic differentials and sections of certain holomorphic bundles over M by J. C. Wood. In [13], Tilla Klotz Weinstein characterized harmonic maps of Lorentz surfaces in terms of holomorphic differentials. In this paper we established similar relations between harmonic maps of Lorentz surfaces and "paraholomorphic" differentials. In doing that we use the ring of paracomplex numbers. These numbers play a role similiar to that played by ordinary complex numbers in the work of Wood and Weinstein.

Paracomplex analysis and paracomplex geometry resting on these paracomplex numbers are at an early stage of development and this field of research is quite promising [1]. Physicists as well as mathematicians have been using paracomplex analysis and geometry in their field of research, see for example [5], [9], [10], [11].

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1. Paracomplex analysis [15], [1]

In $\mathbb{R}^2$, define a multiplication

$$(x, y)(u, v) = (xu + yv, xv + yu).$$

With this multiplication and the usual addition, $\mathbb{R}^2$ becomes a commutative, associative and unitary algebra over $\mathbb{R}$ of rank 2 with zero divisors (x, x) and $(x, -x)$. Identifying (x, y) with $x + \epsilon y$, where ϵ is an indeterminate with $\epsilon^2 = 1$ and following the notations in [1], we set

$$\mathbb{A} = \{x + \epsilon y : x, y \epsilon \mathbb{R}\}.$$

$\mathbb{A}$ is called the *algebra of paracomplex numbers.* (They are also called *double numbers* or *hyperbolic complex numbers.*) Note that with the above identification, $\epsilon = (0, 1)$. For a real vector space V of dimension n with a basis $\{v_1, \cdots, v_n\}$, set

$$V^A = V \otimes_{\mathbb{R}} \mathbb{A} = span_{\mathbb{R}}\{v_k \otimes 1, v_k \otimes \epsilon : k = 1, \cdots, n\}$$

V^A is the *paracomplexification* of V and may be viewed as a free module over $\mathbb{A}$ generated by $\{v_1, \cdots, v_n\}$, [9]. $\mathbb{A}$ may be canonically endowed with an indefinite Riemannian metric by

$$< z, \omega >= Re(z\overline{\omega}) = xu - yv$$

where $z = x + \epsilon y, \ \ \omega = u + \epsilon v \in \mathbb{A}$ with $\overline{\omega} = u - \epsilon v$.

An endomorphism $J : V \to V$ is called *a paracomplex structure* on V if $J^2 = I$, the identity map and the eigenspaces corresponding to 1 and -1 are of the same dimension. Let g be a nondegenerate symmetric bilinear form on V. Then J is said to preserve (or to be compatible with) the metric g if

$$g(x, y) = -g(Jx, Jy).$$

In this case g is called a *parahermitian metric.* A parahermitian metric is necessarily indefinite.

In $\mathbb{A} \cong \mathbb{R}^2$, the multiplication by ϵ induces a paracomplex structure compatible with the canonical indefinite metric on $\mathbb{A}$. The usual metric (which is indefinite of index (m, m)) and the paracomplex structure on $\mathbb{A}^m$ are obtained in the usual way from that on $\mathbb{A}$. Note that $\mathbb{A}^m$ is a free module over $\mathbb{A}$. In general every real vector space V of dimension $2m$ with an indefinite metric g of index (m, m) and a paracomplex structure J compatible with g can be turned into a free module with m generators over $\mathbb{A}$. Here the multiplication by a scalar $(x + \epsilon y)$ is given by $(x + \epsilon y)v = xv + yJ(v), \ \ v \in V$.

Definition 1.1 [15] *A differentiable map $f : (\mathbb{R}^{2m}, J) \to (\mathbb{R}^{2k}, J^{'})$ is said to be paraholomorphic (resp. anti-paraholomorphic) at $p \in \mathbb{R}^{2m}$ if*

$$df \circ J = J^{'} \circ df \qquad (resp.\ df \circ J = -J^{'} \circ df)$$

on a neighbourhood of p.

Notations: For $z = x + \epsilon y$ we set $\partial_z = \frac{1}{2}(\partial_x + \epsilon\partial_y)$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x - \epsilon\partial_y)$, $\partial_u = \frac{\partial}{\partial u}$, $f_u = \frac{\partial f}{\partial u}$.

For $z = (z^1, \cdots, z^m) \in \mathbb{A}^m$ with $z^r = x^r + \epsilon y^r$ and $f^r = u^r + \epsilon v^r$ we have

Lemma 1.2 *Let $f : \mathbb{A}^m \to \mathbb{A}^k$ be a real differentiable map with $f(z) = (f^1(z), \cdots, f^k(z))$. Then the following are equivalent on an open subset $D \subset \mathbb{A}^m$:*
i) *f is paraholomorphic;*
ii) *$u^r_{x^t} = v^r_{y^t}$ and $u^r_{y^t} = v^r_{x^t}$; $1 \le r \le k$, $1 \le t \le m$*
(These are called the para-Cauchy-Riemann equations);
iii) *$\partial_{\overline{z}^t} f = 0$, $1 \le t \le m$.*

Proof. As in the ordinary complex case. □

Definition 1.3 [1] *A map $f : \mathbb{A} \to \mathbb{A}$ is said to be paracomplex differentiable at a point $z \in \mathbb{A}$ if there exists a number $\alpha(z)$ such that*

$$f(z+h) - f(z) = \alpha(z)h + \sigma(z,h)||h||,$$

where $h = h_1 + \epsilon h_2$, $||h||^2 = h_1^2 + h_2^2$, and $\sigma(z,h) \to 0$ as $h \to 0$.

If the map α exists, it is unique and is called *paracomplex derivative of f* and we write $\alpha(z) = \frac{df}{dz}(z) = f'(z)$.

Lemma 1.4 [1] *A map $f : \mathbb{A} \to \mathbb{A}$ is paraholomorphic on an open subset D if and only if f is paracomplex differentiable on D.*

Lemma 1.5 *Let $f = u + \epsilon v : \mathbb{A} \to \mathbb{A}$ with u, v are of class C^2. Then f is paracomplex differentiable at z if and only if*

(I) $\quad lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = L(z)$ *exists and further $L(z) = \alpha(z) = f'(z)$.*

Here the limit is taken in the set of h with $h\overline{h} \neq 0$.

Proof. Let (I) be satisfied. For $h = h_1$ and $h = \epsilon h_2$ with h_1, h_2 real, (I) gives

$$\frac{\partial u}{\partial x} + \epsilon\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + \epsilon\frac{\partial u}{\partial y},$$

which gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

From Lemma (1.2) and (1.4) paracomplex differentiability of f follows. On the other hand (I) gives

$$f(z+h) - f(z) = L(z)h + \frac{\tilde{\sigma}(z,h)h}{||h||}||h||$$

with $\tilde{\sigma} \to 0, (h\overline{h} \neq 0)$. Since $h_k/||h||$ $(k = 1,2)$ is bounded, $\sigma(z,h) \equiv \tilde{\sigma}(z,h)h/||h|| \to 0$ as $h \to 0$. By the uniqueness of $L(z)$, $f'(z) = L(z)$.

Next let $f = u + \epsilon v$ be paracomplex differentiable. Then u, v satisfy the para-Cauchy-Riemann conditions; so setting $s = x + y$, $t = x - y$ we get

$$\frac{\partial^2 u}{\partial s \partial t} = 0 = \frac{\partial^2 v}{\partial s \partial t}$$

which have solutions $u(s,t) = F(s) + G(t)$ and $v(s,t) = F(s) - G(t)$ for some differentiable functions F and G. Thus one can write

$$\frac{f(z+h) - f(z)}{h} = \frac{\Delta f}{h} = \frac{(\Delta f)\overline{h}}{h\overline{h}} = \frac{(\Delta f)\overline{h}}{(\Delta s)(\Delta t)}$$

$$= (1+\epsilon)\frac{\Delta F}{\Delta s} + (1-\epsilon)\frac{\Delta G}{\Delta t}.$$

Since $\Delta s \to 0$, $\Delta t \to 0$ as $h \to 0$ and F and G are differentiable, (I) follows from above. That $f'(z) = L(z)$ is clear. □

Remark: The essence of the statement of the lemma and the proof of the fact that "(I) implies the para-Cauchy-Riemann equations" are provided in [15].

[1], [7] A function $f : \mathbb{A} \to \mathbb{A}$ is said to be *paracomplex analytic* if it has a power series expansion in $\mathbb{A}$. Here the multiplication is the one in $\mathbb{A}$. We remark in passing that every paracomplex analytic f is paraholomorphic. The converse is not true in general. However every paraholomorphic $f = u + \epsilon v$ is paracomplex analytic provided u and v are real analytic.

(Almost) paracomplex, (almost) parahermitian, (almost) parakaehler manifolds are defined in the usual way with a difference that the defining elements are replaced by the corresponding "para" ones.

The *torsion tensor* N of an almost paracomplex structure J of M is defined as (see for example [1], [15])

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y]$$

for smooth vector fields X, Y on M.

Lemma 1.6 [1] *An almost paracomplex structure J on an almost paracomplex manifold of dimension m is integrable (i.e. its torsion vanishes) if and only if J can be obtained from the paracomplex structure on $\mathbb{A}^m$ by means of the paraholomorphic charts.*

Definition 1.7 *For a metric g on a surface, a coordinate system (x,y) is said to be g-isothermal if $g(\partial_x, \partial_x) = kg(\partial_y, \partial_y)$ and $g(\partial_x, \partial_y) = 0$. Here $k = 1$ when g is positive definite, $k = -1$ when g is indefinite.*

Let (M, g) be a Lorentz surface, i.e an oriented, paracompact connected surface with a neutral metric (a metric of index (n, n), $n = 1, 2, \cdots$). The orientation and the metric determine a unique almost paracomplex structure J on M so that M becomes almost parahermitian manifold [1], (cf. [8]). Since every almost paracomplex structure on a manifold of dimension 2 is integrable, M is parahermitian (in fact it is parakaehler [15]). It can thus be proved that there exist g-isothermal coordinates (x, y) on M, [12].

The pseudosphere

$$S_1^2 = \{(x, y, z) \in \mathbb{R}^3 : -x^2 + y^2 + z^2 = 1\}$$

with the neutral metric h induced from that on Minkowski space $\mathbb{R}_1^3$ is orientable (since it is a level surface). So S_1^2 with the unique paracomplex structure J compatible with the metric h becomes a parahermitian surface and therefore possesses h-isothermal coordinates.

2. Harmonic maps and paraholomorphic differentials

Throughout, (M, h) and (N, g) will denote a Lorentz surface and n-manifold with an indefinite metric g of index $(r, n - r)$, respectively.

Let J be the canonical paracomplex structure on M so that M becomes a parahermitian manifold and let (x, y) be an h-isothermal coordinate system. The $\mathbb{A}$-linear extension J^A of J to the paracomplexified tangent bundle T^AM decomposes T^AM into $T'M$ and $T''M$ so that $T^AM = T'M \oplus T''M$, where

$$T'M = \{X + \epsilon JX : X \in TM\} \qquad \text{and} \qquad T''M = \overline{T'M} = \{X - \epsilon JX : X \in TM\}$$

are the eigenspaces of J^A corresponding to the eigenvalues ϵ and $-\epsilon$. Clearly $\partial_z = \frac{1}{2}(\partial_x + \epsilon\partial_y) \in T'M$ and $\partial_{\overline{z}} = \overline{\partial}_z \in T''M$.

Now let D be the metric connection on N. This connection D and the metric g on TN extend to an $\mathbb{A}$-linear connection and a paracomplex bilinear, symmetric, nondegenerate metric on T^AN, respectively, which we will denote by the same letters. It is easy to see that the bundle T^AN remains Riemannian connected, that is,

$$vg(z, w) = g(D_v z, w) + g(z, D_v w)$$

for $v \in T^AN$, and $z, w \in C(T^AN)$.

Let $f : (M, h) \to (N, g)$ be a smooth map. Then one may extend its differential df to $d^Af : T^AM \to T^AN$ by paracomplex linearity. For D_w let us write D' if $w = d^Af(\partial_z) = f_z$ and D'' if $w = d^Af(\partial_{\overline{z}})$. We also write $D'f = f_z$ etc. On an open subset U of M define $q : U \to \mathbb{A}$ by $q(p) = g((D'f)(p), (D'f)(p)) = g(f_z(p), f_z(p))$ for $p \in U$. Consider now the quadratic differential Q associated with f given by

$$Q(p) = q(p)dz^2.$$

Note that although q is only defined locally, Q is defined globally since it does not depend on the isothermal coordinates $z = x + \epsilon y$ choosen (cf. [16]).

Notation: For a C^1 map $f : M \to N$ we set $\tilde{M} = \{p \in M : df_p \neq 0\}$.

Definition 2.1 (cf. [4]) *For a C^1 map $f:(M,h)\to(N,g)$,*

1) *f is said to be weakly conformal if, for any $p\in\tilde{M}$ there is a real number $\lambda(p)\neq 0$ such that, for every $u,v\in T_pM$,*

$$g(df_p(u),df_p(v))=\lambda(p)h(u,v).$$

One may extend λ on $\tilde{M}$ to whole of M by setting $\lambda(p)=0$ for $p\in M-\tilde{M}$. Then the extended λ becomes a continuous function on M.

2) *f is said to be (image) nondegenerate if g is nondegenerate on $df(T_pM)$ for every $p\in\tilde{M}$.*

Unless otherwise stated, f will be smooth nondegenerate map throughout.

Proposition 2.2 *f is weakly conformal if and only if the associated quadratic differential $Q\equiv 0$.*

Proof. Observe that

$$q=g(D'f,D'f)=g(f_z,f_z)=\tfrac{1}{4}\Big[g(f_x,f_x)+g(f_y,f_y)+2\epsilon g(f_x,f_y)\Big]. \tag{1}$$

Let now f be weakly conformal. Then the "only if"-part of the result easily follows from the fact that (x,y) are isothermal coordinates. Conversely, by (1), $Q\equiv 0$ implies $g(f_x,f_x)=-g(f_y,f_y)$ and $g(f_x,f_y)=0$. This, together with the nondegeneracy of f, implies that $g_p(f_x,f_x)\neq 0$ for all $p\in\tilde{M}$. Note that then for $u=a\partial_x+b\partial_y$, $v=r\partial_x+k\partial_y$ and $s=ar-kb$ we have

$$g(f_u,f_v)=sg(f_x,f_x),\ \text{ and } sh(\partial_x,\partial_x)=h(u,v)$$

so

$g(f_u,f_u)=\dfrac{g(f_x,f_x)}{h(\partial_x,\partial_x)}\,sh\,(\partial_x,\partial_x)=\lambda h(u,v)$, with $\lambda=\frac{g(f_x,f_x)}{h(\partial_x,\partial_x)}\neq 0$ on $\tilde{M}$. This completes the proof. □

We now define *the second fundamental form* $\bigtriangledown df$ of $f:(M,h)\to(N,g)$ by, [2],

$$\bigtriangledown df(u,v)=D_{df(u)}(df(v))-df(D^M_u v).$$

Here D^M is the metric connection on M and $u,v\in C(TM)$. Note that $\bigtriangledown df$ is symmetric and extends to a symmetric, paracomplex bilinear form $(\bigtriangledown df)^A$.

Let (x^1,x^2) be any local coordinates on the Lorentz surface (M,h) and set $X_i=\partial_{x^i}\in TM$ and $Y_i=df(X_i)\in TN$.

Definition 2.3 *f is said to be harmonic if its tension field*

$$\tau_h(f)=\tau(f)=\sum_{i,j=1}^{2}h^{ij}\bigtriangledown df(X_i,X_j)$$

vanishes. Here h^{ij} is the (ij) th entry of $(h_{ij})^{-1}$.

Lemma 2.4 *Let (x^1, x^2) be h-isothermal coordinates on M. Then*

i) *the Christoffel symbols Γ_{ij}^k of M satisfies $\Gamma_{11}^1 - \Gamma_{22}^1 = \Gamma_{11}^2 - \Gamma_{22}^2 = 0$;*

ii) *for $\tau(f)$, the tension field of f, we have*

$$h_{11}\tau(f) = \sum_{i=1}^{2}(-1)^{i+1} \bigtriangledown df(X_i, X_i) = \sum_{i=1}^{2}(-1)^{i+1} D_{Y_i} df(X_i).$$

Proof. i) If we express Γ_{ij}^k in terms of the metric h, the result easily follows.

ii) (x^1, x^2) are h-isothermal, $h^{ij} = h^{11}\delta_{ij}(-1)^{i+1}$.

(Note here that without loss of generality we may assume that $h^{11} > 0$ on M.) So

$$\sum_{i=1}^{2}(-1)^{i+1} \bigtriangledown df(X_i, X_i) = h_{11}\tau(f).$$

Further note that

$$\sum_{i=1}^{2}(-1)^{i+1} D_{X_i}^M X_i = \sum_{i=1}^{2}(\Gamma_{11}^i - \Gamma_{22}^i)X_i = 0$$

by the first part (i). Thus we get

$$h_{11}\tau(f) = \sum_{i=1}^{2}(-1)^{i+1} D_{Y_i} df(X_i).$$

This completes the proof. □

Proposition 2.5 $\frac{1}{4h^{11}}\tau(f) = D''D'f = D'D''f$ *and therefore f is harmonic if and only if* $D''D'f = D'D''f = 0$.

Proof. $D'f = df(\partial_z) = \frac{1}{2}(df(X_1) + \epsilon df(X_2))$, where $X_i = \partial_{x^i}$. So

$$\begin{aligned} D''D'f &= \frac{1}{2}(D_{Y_1}D' - \epsilon D_{Y_2}D')f \\ &= \frac{1}{4}\Big\{\sum_{i=1}^{2}(-1)^{i+1} D_{Y_i} df(X_i) + \epsilon\Big[D_{Y_1} df(X_2) - D_{Y_2} df(X_1)\Big]\Big\}. \end{aligned}$$

But the term in the square bracket on the right hand side is equal to $\epsilon df([X_1, X_2])$, (cf. [2]), which vanishes since the Lie bracket is zero. So $D''D'f$ becomes equal to first part of the right hand side. Then by Lemma (2.4/ (ii)) the result follows. The treatment of $D'D''f$ is the same. □

For a smooth nondegenerate f, $\tau(f) \in TN$ decomposes into

$$\tau_h(f) = \mathcal{H}^h(f) + \nu^h(f)$$

on $\tilde{M}$, where $\nu^h(f) \in E = df(TM)$ and $\mathcal{H}^h(f) = E^\perp$ are the tangential and normal components of $\tau(f)$ respectively. If f is an immersion then we call $\mathcal{H}^h(f)$ the h-mean curvature vector field along f. (Note that $\mathcal{H}^I(f)$ is the ordinary mean curvature vector field. Here the metric on M is taken to be $I = g|_E$.)

We say that the quadratic differential $Q = qdz^2$ is paraholomorphic if $q = g(D'f, D'f)$ is paraholomorphic.

Theorem 2.6 *For a smooth nondegenerate map $f : (M,h) \to (N,g)$ we have*

i) *the associated quadratic differential $Q = qdz^2$ is paraholomorphic on M if and only if $\nu^h(f) = 0$ on $\tilde{M}$*

ii) *further if $\tilde{M}$ is dense in M, then f is harmonic if and only if Q is paraholomorphic and $\mathcal{H}^h(f) = 0$.*

Proof. Observe that

$$\begin{aligned} \frac{\partial q}{\partial \bar{z}} &= \frac{\partial}{\partial \bar{z}} g(D'f, D'f) = g(D''D'f, D'f) + g(D'f, D''D'f) = \frac{1}{2h^{11}} g(D'f, \tau(f)) \\ &= \frac{1}{4h^{11}} \Big[g(f_x, \tau(f)) + \epsilon g(f_y, \tau(f)) \Big] \end{aligned}$$

where $z = x + \epsilon y$. So

i) Q is paraholomorphic if and only if $g(f_x, \tau(f)) = 0$ and $g(f_y, \tau(f)) = 0$ if and only if $\tau(f) \in E^\perp$ if and only if $\nu^h(f) = 0$ since f is nondegenerate.

ii) If f is harmonic then $\tau(f) = 0$ so $\mathcal{H}^h(f) = 0$ and $\partial'' q = \partial_{\bar{z}} q = 0$ by the observation above, thus Q is paraholomorphic. Conversely if Q is paraholomorphic, then by part (i), $\tau(f) = \mathcal{H}^h(f)$ on $\tilde{M}$ which is zero by the hypothesis, so f is harmonic on $\tilde{M}$ and therefore, by continuity, harmonic on M. □

Corrollary 2.7 (cf. [4]) *For $f : (M,h) \to (N,g)$ a smooth non-degenerate map between Lorentz surfaces with $\tilde{M}$ dense in M, we have*

i) *f is harmonic if f is weakly conformal.*

ii) *Further let df_p be one-to-one for all $p \in \tilde{M}$. Then f is harmonic if and only if Q is paraholomorphic.*

Proof. i) By Proposition (2.2), if f is weakly conformal then $Q \equiv 0$ and therefore Q is trivially paraholomorphic. On the other hand, since f is weakly conformal and nondegenerate, df_p is $1-1$ for $p \in \tilde{M}$. Thus $E = TN$ and so $\mathcal{H}(f) = 0$. Hence f is harmonic by Theorem (2.6)/(ii).

ii) Observe that $\mathcal{H}^h(f) = 0$ since f is nondegenerate and df_p is $1-1$ for $p \in \tilde{M}$. By the theorem (2.6)/(ii) the result follows. □

We say that a map $f : (M,J) \to (N,J')$ between (almost) paracomplex manifolds is paraholomorphic (resp. antiparaholomorphic) if $df \circ J = J' \circ df$ (resp. *$df \circ J = -J' \circ df$*), [15].

Next we show that weak conformality implies paraholomorphicity under certain conditions.

Proposition 2.8 *Let $f : (M, h, J) \to (N, g, J')$ be a nondegenerate map of a Lorentz surface into a parahermitian manifold of dimension $2n$ satisfying the following two conditions:*

i) *$E = df(TM)$ is invariant under J' i.e. $J'(E) = E$.*

ii) *The zero set $K = \{p \in M : df_p = 0\}$ does not disconnect M i.e $M - K = \tilde{M}$ remains connected.*

Then f is weakly conformal if and only if f is paraholomorphic or anti-paraholomorphic.

Proof. Let f be weakly conformal. Then the associated quadratic differential Q of f is identically zero. Thus we have

(1) $\quad g(df(J\partial_x), df(J\partial_x)) = -g(df(\partial_x), df(\partial_x))$ and $g(df(\partial_x), df(J\partial_x)) = 0.$

Note that $df_p(\partial_x)$ and $df_p(\partial_y)$ are non-null vectors for $p \in M - K$ by the virtue of (1) and nondegeneracy of f, where $\partial_y = J\partial_x$. Observe also that $df(J\partial_x)$ is g-orthogonal to $df(\partial_x)$ by (1) and $J'df(\partial_x)$ is g-orthogonal to $df(\partial_x)$ since N is parahermitian. By the fact that $df(J\partial_x)$, $J'df(\partial_x)$, $df(\partial_x)$ are all in E, we deduce that $df(J\partial_x)$ and $J'df(\partial_x)$ are parallel. On the other hand $g(J'df(\partial_x), J'df(\partial_x)) = -g(df(\partial_x), df(\partial_x)) = g(df(J\partial_x), df(J\partial_x))$ by (1). So

(2) $\quad df(J\partial_x) = \pm J'df(\partial_x).$

Also note that

$$df(J\partial_y) = df(J^2\partial_x) = J'^2 df(\partial_x) = \pm J'df(J\partial_x) = \pm J'df(\partial_y) \text{ by (2).}$$

This, together with (2) gives $df \circ J = \pm J' o\, df$ at every $p \in M$. Now we need to show that

(3) $\quad df \circ J = J' \circ df$ on M

or

(4) $\quad df \circ J = -J' \circ df$ on M.

To do this let A and B be the sets of all points at which (3) is true and (4) is true respectively. Then A and B are closed by continuity of df and J. Next note that (3) and (4) give $M = A \cup B$ thus $M - (A \cap B) = (M - A) \cup (M - B)$ which is the union of two open disjoint sets. But it is easy to see that $A \cap B = K = \{p \in M : df_p = 0\}$. This leads to a contradiction with the assumption (ii). So we must have $M - A = \phi$ or $M - B = \phi$. That means that f is either paraholomorphic or antiparaholomorhic on M.

Conversely, let f be paraholomorphic then $J' \circ df = df \circ J$ so

$$g(df(J\partial_x), df(J\partial_x)) = g(J'df(\partial_x), J'df(\partial_x)) = -g(df(\partial_x), df(\partial_x))$$

and

$$g(df(J\partial_x), df(\partial_x)) = g(J'df(\partial_x), df(\partial_x)) = 0.$$

This amounts to saying that the associated quadratic differential $Q = g(f_z, f_z)dz^2$ is identically zero. If f is anti-paraholomorphic, in a similiar fashion, we get $Q \equiv 0$. This completes the proof. □

Corollary 2.9 *Let f be a local diffeomorphism between Lorentz surfaces. Then f is weakly conformal if and only if f is paraholomorphic or anti-paraholomorphic.*

Remark: Let (M, g) be an orientable Lorentz surface with g-isothermal coordinates (x, y). It is easy to see that the transition functions induced by the charts defining these g-isothermal coordinates are conformal and therefore they are paraholomorphic or anti-paraholomorphic. Replacing (x, y) by $(-x, -y)$ where necessary we obtain that the transition functions are all paraholomorphic. Thus (M, g) becomes a parahermitian manifold.

When N is a parakaehler space form (parakaehler manifold with constant paraholomorphic sectional curvature) we can define higher order paraholomorphic differentials associated with $f : M \to N$. For this, first we make the following observations: (cf. [15]). As for $T^A M$, J' splits $T^A N$ into $T'N$ and $T''N$ so that $T^A N = T'N \oplus T''N$. For the connection $\triangledown$ on $T'N$ induced by D via

$$\triangledown : T^A N \times T^A N \stackrel{D}{\to} T^A N \stackrel{\pi}{\to} T'N$$

where π is the projection along $T''N$, write $\triangledown' u = \pi \circ D'u$ and $\triangledown'' u = \pi \circ D''u$ for $u \in C(T^A N)$. Further put $\triangledown' f = \pi \circ D'f = \pi \circ d^A f(\partial_z)$ and $\triangledown'' f = \pi \circ D'' f$ so that $D'f = \triangledown' f + \overline{\triangledown'' f}$. Since N is parakaehler, D sends the sections of $T'N$ (resp. $T''N$) into sections of $T'N$ (resp. $T''N$). Further one can show that

1) $D'u = \triangledown' v + \overline{\triangledown'' w}$ for $u \in C(T^A N)$ with $u = v + \overline{w}$, where $v, w \in C(T'N)$. In particular $D'^k f = \triangledown'^k f + \overline{\triangledown''^k f}$ and $D'u = \triangledown' u$, $D''u = \triangledown'' u$ for $u \in C(T'N)$.

2) $g(u, v) = 0$ for all $u, v \in C(T'N)$ or $u, v \in C(T''N)$. So setting $G(u, v) = g(u, \overline{v})$, $u, v \in T'N$, gives a parahermitian inner product G on $T'N$. Thus the bundle $T'N$ over N together with G and the connection $\triangledown$ becomes parahermitian connected.

3) f is paraholomorphic (resp. antiparaholomorphic) if and only if $\triangledown'' f$ (resp. $\triangledown' f$) vanishes. Further f is harmonic if and only if $\triangledown' \triangledown'' f = \triangledown'' \triangledown' f = 0$. Thus if f is paraholomorphic or antiparaholomorphic then it is harmonic. In fact [15], every paraholomorphic or antiparaholomorphic map between parakaehler manifolds is harmonic. □

Set

$$Q_{s,t} = q_{s,t} dz^{s+t} \quad \text{and} \quad B_{s,t} = b_{s,t} dz^{s+t}$$

where

$$q_{s,t} = g(D'^s f, D'^t f) \quad \text{and} \quad b_{s,t} = G(\triangledown'^s f, \triangledown''^t f).$$

$Q_{s,t}$ and $B_{s,t}$ are differentials of order $s + t$.

Definition 2.10 (cf. [3]) *By the order of parareal (resp. paracomplex) isotropy of a map $f : M \to N$ with parakaehler N we mean the greatest value of $k \in \{1, 2, 3, \cdots, \infty\}$ such that, on each chart, $q_{s,t} \equiv 0$ (resp. $b_{s,t} \equiv 0$) for all integers $t, s \geq 1$ with $s + t \leq k$.*

For a map f let $\mu(f)$ (resp. $\tilde{\mu}(f)$) denote its order of parareal (resp. paracomplex) isotropy. Clearly $\mu(f) \geq \tilde{\mu}(f)$.

Lemma 2.11 *For a smooth map f and integers $s, t \geq 1$ let $\mu(f) = k$ and $\tilde{\mu}(f) = \ell$ and set $s + t = r + 1$. Then*

i) $q_{s,t} = (-1)^{t-1} q_{r,1}$, *when* $r \leq k$,

ii) $b_{s,t} = (-1)^{t-1} b_{r,1}$ *when* $r \leq \ell$,

iii) $k > \ell$ *when ℓ is even,*

iv) *if ℓ is odd then $q_{\ell,1} = \pm 2 b_{\ell,1}$ and therefore $k = \ell$.*

Proof. Note

$$q_{s,t} = \partial' g(D'^s f, D'^{t-1} f) - g(D'^{s+1} f, D'^{t-1} f) = -q_{s+1,t-1}$$

by the fact that $T^A N$ is a Riemannian connected bundle over N. So (i) follows from this easily. For (ii) the argument is similiar. For (iii) and (iv) first observe that

(I) $\quad q_{s,t} = b_{s,t} + b_{t,s} = (-1)^{t-1} b_{r,1} + (-1)^{s-1} b_{r,1}$ for all s,t with $1 \leq s+t = r+1 \leq \ell+1$.

Now if $s+t < \ell+1$ then (I) clearly gives $q_{s,t} = 0$ so $k \geq \ell$. Let ℓ be even, then $s+t = \ell+1$ is odd, thus one of s,t is odd and the other is even. So (I) gives $q_{s,t} = 0$. From this, (iii) follows.

For the case that ℓ is odd, $s+t = \ell+1$ is even and thus s,t are both even or both odd. In either case, (I) gives $q_{\ell,1} = \pm 2b_{\ell,1}$. This proves (iv). □

Notice that $q = q_{1,1} = 2b_{1,1}$ so that $Q = 2B_{1,1}$.

Proposition 2.12 *Let $f : M \to (N, J', g)$ be a harmonic map into a parakaehler space form N. Then*

i) *$Q_{2,1} = q_{2,1} dz^3$ is paraholomorphic, where $q_{2,1} = g(D'^2 f, D' f)$,*

ii) *if further f is weakly conformal with $\mu(f) = k$ (thus $k \geq 3$) and $g(J'(D'f), D'^{k-1} f) = 0$, then $Q_{k,1} = q_{k,1} dz^{k+1}$ is a globally defined paraholomorphic differential.*

Proof. For any smooth map f consider

$$\partial_{\overline{z}} q_{k,1} = \partial'' q_{k,1} = g(D'' D'^k f, D' f) + g(D'^k f, D'' D' f) = g(D'' D'^k f, D' f)$$

since $D'' D' f = 0$ by the harmonicity of f. Setting $X = df(\partial_x) = f_x$, $Y = f_y$, $P = D' f = X + \epsilon Y$ and $W = D'^{k-1} f$ for $k \geq 3$ we have

$$D'' D'^k f = D'' D' W - D' D'' W = 2\epsilon (D_X D_Y - D_Y D_X) W = 2\epsilon R(X,Y) W$$

where R is the Riemannian curvature tensor of N. Note here that we use the fact that

$$[X, Y] = D_X Y - D_Y X = df([\partial_x, \partial_y]) = 0.$$

Thus

(1) $\quad \partial'' q_{2,1} = 2\epsilon g(R(X,Y)P, P)$

(2) $\quad \partial'' q_{k,1} = 2\epsilon g(R(X,Y)W, P)$, for $k \geq 3$.

On the other hand (see [6])

$$(\star) \qquad R(X,Y)W = \frac{c}{4}\Big\{ g(X,W)Y - g(Y,W)X$$

$$+ g(X, J'W)J'Y - g(Y, J'W)J'X + 2g(X, J'Y)J'W \Big\}$$

where c is the paraholomorphic sectional curvature. So

(3) $$g(R(X,Y)W,P) = \frac{c}{4}\Big\{g(X,W)g(Y,P) - g(Y,W)g(X,P) + g(X,J'W)g(J'Y,P)$$
$$-g(Y,J'W)g(J'X,P) + 2g(X,J'Y)g(J'W,P)\Big\}.$$

Replacing W by P in (3), the equation (1) gives $\partial'' q_{2,1} = 0$. This proves (i).

For (ii), first observe that
$g(W,P) = g(W, X+\epsilon Y) = g(W,X) + \epsilon g(W,Y) = 0$
by the isotropy assumption. So we have

(4) $$g(W,X) = -\epsilon g(W,Y).$$

By (4), we get

(5) $$g(X,W)g(Y,P) - g(Y,W)g(X,P) = \epsilon g(X,W)g(P,P) = 0$$

since $g(P,P) = 0$ by the isotropy assumption. Next the hypothessis
$g(J'P,W) = 0$ gives

(6) $$\epsilon g(X,J'W) = -g(Y,J'W).$$

By (6) we get

(7) $$g(X,J'W)g(J'Y,P) - g(Y,J'W)g(J'X,P) = \epsilon g(X,J'W)g(J'P,P) = 0$$

since $g(J'P,P) = 0$. Combining (3), (5) and (7) we get $\partial'' q_{k,1} = 0$ so $Q_{k,1}$ is paraholomorphic. This completes the proof. □

Proposition 2.13 *Let $f : M \to (N, J', g)$ be a harmonic map into a parakaehler space form N with the paracomplex isotropy order $\tilde{\mu}(f) = k$. Then $B_{k,1} = b_{k,1} dz^{k+1}$ is a globally defined paraholomorphic differential of order $k+1$*

Proof. (cf. [17]) Note that

(1) $\partial'' b_{k,1} = \partial'' G(\nabla'^k f, \nabla'' f) = G(\nabla'' \nabla'^k f, \nabla'' f) + G(\nabla'^k f, \nabla' \nabla'' f) = G(\nabla'' \nabla'^k f, \nabla'' f)$. □

Claim: $\nabla'' \nabla'^\ell f \in S_{\ell-1} = span_{\mathbb{A}} \ \{\nabla' f, \cdots, \nabla'^{\ell-1} f\}$ for any $1 \le \ell \le k$.

This is true for $\ell = 1$ by the harmonic equation. Assume now that it is true for $\ell = r-1 < k$ and consider
$$\nabla'' \nabla'^r f = 2\epsilon R(X,Y) \nabla'^{r-1} f + \nabla'(\nabla'' \nabla'^{r-1} f).$$

For the second term on the right we have, by assumption,

$$\nabla'' \nabla'^{r-1} f \in S_{r-2} \quad \text{so that} \quad \nabla' (\nabla'' \nabla'^{r-1} f) \in S_{r-1}.$$

For the curvature term, first observe that for $X = f_x, \quad Y = f_y$,

$$\nabla'' f = \pi(df(\partial_{\overline{z}})) = \frac{1}{2}\pi(X - \epsilon Y) = \frac{1}{4}\Big[(X - J'Y) + \epsilon J'(X - J'Y)\Big] = \frac{1}{4}(V + \epsilon J'V)$$

where $V = X - J'Y$. Put $A = \bigtriangledown'^{r-1} f \in S_{r-1}$, then by $\tilde{\mu}(f) = k$, we have

$$G(A, \bigtriangledown'' f) = \frac{1}{4}\Big[g(A, V) - \epsilon g(A, J'V)\Big] = 0.$$

Then this, together with $g(A, \overline{\bigtriangledown'' f}) = 0$ gives

$$g(A, V) = g(A, X) - g(A, J'Y) = 0.$$

So

(2) $\quad g(A, X) = g(A, J'Y) = -\epsilon g(A, Y)$

since $J'A = \epsilon A$ for all $A \in T'N$.

Next, by $(\star)$ in the proof of Proposition (2.12), we have

$$R(X, Y)A = \frac{c}{4}\Big\{g(X, A)Y - g(Y, A)X + g(X, J'A)J'Y - g(Y, J'A)J'X + 2g(X, J'Y)J'A\Big\}$$

$$= \frac{c}{4}\Big\{g(X, A)Y + \epsilon g(X, A)X + \epsilon J'g(X, A)Y + J'g(X, A)X + 2\epsilon g(X, J'Y)A\Big\} \quad \text{(by (2))}$$

$$= \frac{c}{4}\Big\{g(X, A)[(X + J'Y) + \epsilon J'(X + J'Y)] + 2\epsilon g(X, J'Y)A\Big\}$$

where c is the constant paraholomorphic sectional curvature of N. That is, $R(X, Y)A$ is a linear combination of $\bigtriangledown' f = (X + J'Y) + \epsilon J'(X + J'Y)$ and A, thus it is in S_{r-1}. This completes the proof of the claim.

By the claim we have $\bigtriangledown'' \bigtriangledown'^k f \in S_{k-1}$. So by the assumption that $\tilde{\mu}(f) = k$, (1) gives $\partial'' b_{k,1} = G(\bigtriangledown'' \bigtriangledown'^k f, \bigtriangledown'' f) = 0$. That is, $B_{k,1}$ is paraholomorphic. □

Corollary 2.14 *For a weakly conformal harmonic map $f : M \to N$ to a parakaehler space form N,*

i) $Q_{2,1} = q_{2,1} dz^3 \equiv 0$

ii) $B_{2,1} = b_{2,1} dz^3$ *is paraholomorphic.*

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