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On Generalised Chen and k-minimal Immersions

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0. Introduction

B. Rouxel [5] and S. J. Li and C. S. Houh [3] have given extensions of the notion of an $\mathcal{A}$-submanifold (Chen submanifold) to an $\mathcal{A}_k$-submanifold. Here we investigate the relation between their definitions.

We introduce the idea of a k-minimal submanifold which satisfies a stronger condition than that of an $\mathcal{A}_k$-submanifold. By generalising a construction from [1] we give a method for constructing k-minimal submanifolds.

1. $\mathcal{A}_k$-immersions

Let $f : M^m \to \mathbb{R}^{m+d}$ be a smooth immersion of a connected, m-dimensional manifold without boundary into Euclidean $m+d$-space. For $x \in M$, the normal plane to f at x is defined by $N_x(f) = \{n \in \mathbb{R}^{m+d} : <n, df_x X> = 0, \ \forall \ X \in T_xM\}$, where $df_x : T_xM \to \{f(x)\} \times \mathbb{R}^{m+d} \equiv \mathbb{R}^{m+d}$ is the tangent mapping induced from f. For $n \in N_x(f)$, the shape operator $A_n : T_xM \to T_xM$ is defined to be the linear mapping such that for any tangent vector field $X : M \to TM$, $df_x(A_n X)$ is the tangent component of the directional derivative of n in direction X. The mean curvature vector is defined, at $x \in M$, by $H = \frac{1}{m}\sum_{i=0}^{d-1}(trace A_{e_i})e_i$ where $\{e_0, \ldots, e_{d-1}\}$ is an orthonormal basis for $N_x(f)$.

In [3], S. J. Li and C. S. Houh define normal fields $\mathcal{A}_1(H), \mathcal{A}_2(H), \ldots, \mathcal{A}_{d-1}(H)$ on M. At $x \in M$, if $H \neq 0$, $\mathcal{A}_1(H) = \sum_{i=1}^{d-1}(trace A_H A_{e_i})e_i$ where $\{e_0, \ldots, e_{d-1}\}$ is an orthonormal basis for $N_x(f)$ with $e_0 = H/\|H\|$. If $H = 0$, then $\mathcal{A}_1(H)$ is defined to be 0. So $\mathcal{A}_1(H)$ is orthogonal to H.

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Now assume that $\mathcal{A}_1(H), \dots, \mathcal{A}_{d-1}(H)$ have been defined on M and are pairwise orthogonal and such that at each $x \in M$, $\mathcal{A}_{k-1}(H) \neq 0 \Rightarrow \mathcal{A}_j(H) \neq 0$ for $j = 1, \dots, k-2$. Then, at $x \in M$, if $\mathcal{A}_{k-1}(H) \neq 0$ define $\mathcal{A}_k(H) = \sum_{i=k}^{d-1}(trace A_{\mathcal{A}_{k-1}(H)} A_{e_i})e_i$ where $\{e_0, \dots, e_{d-1}\}$ is an orthonormal basis for $N_x(f)$ with $e_0 = H/\|H\|$ and $e_i = \mathcal{A}_i(H)/\|\mathcal{A}_i(H)\|$, $i = 1, \dots, k-1$. If $\mathcal{A}_{k-1}(H) = 0$, define $\mathcal{A}_k(H) = 0$. Then $\mathcal{A}_k(H)$ is orthogonal to each of $H, \mathcal{A}_1(H), \dots, \mathcal{A}_{k-1}(H)$. This definition is independent of the choice of basis for $N_x(f)$.

In this way the normal fields $\mathcal{A}_1(H), \mathcal{A}_2(H), \dots, \mathcal{A}_{d-1}(H)$ are defined. For $k \geq d$, we make the convention that $\mathcal{A}_k(H) \equiv 0$. Also, we write $\mathcal{A}_0(H) = H$.

Following [3], f is said to be an $\mathcal{A}_k$-*immersion* if $\mathcal{A}_k(H) \equiv 0$ and is said to be a k-*Chen immersion* if $\mathcal{A}_k(H) \equiv 0$ and $\mathcal{A}_{k-1}(H) \not\equiv 0$. So $\mathcal{A}_0$-immersion is minimal. $\mathcal{A}_1$-immersions are also known as Chen immersions. Note that $f : M^m \to \mathbb{R}^{m+d}$ is a k-Chen for some $k \leq d$. Further, if f has flat normal bundle and $m < d$ then it follows from the next theorem that f is k-Chen for some $k \leq m$.

Theorem 1.1 *Let $f : M^m \to \mathbb{R}^{m+d}$ be an immersion with flat normal bundle. Then $\mathcal{A}_k(H) \equiv 0$ for some $k \leq min\{d, m\}$.*

We first prove a lemma :

Lemma 1.2 *Let $N_x(f) = \Lambda \oplus \Lambda^\perp$ where $\Lambda^\perp$ is the orthogonal complement of Λ in $N_x(f)$, and suppose that $\forall\ u \in \Lambda$, $\forall\ n \in \Lambda^\perp$, $trace A_n = 0$ and $trace A_u A_n = 0$ then, at x, $\mathcal{A}_k(H) = 0$ for some $k \leq dim\Lambda$.*

Proof. Let $dim\Lambda = \ell$. Take an orthonormal basis $\{e_0, \dots, e_{d-1}\}$ for $N_x(f)$ with $e_0, \dots, e_{\ell-1} \in \Lambda$ and $e_\ell, \dots, e_{d-1} \in \Lambda^\perp$. So $trace A_{e_j} = 0$ for $j = \ell, \dots, d-1$ and $trace A_{e_i} A_{e_j} = 0$ for $i = 0, \dots, \ell-1, j = \ell, \dots, d-1$.

Now $\mathcal{A}_0(H) = H = \frac{1}{m}\sum_{i=0}^{d-1}(trace A_{e_i})e_i = \frac{1}{m}\sum_{i=0}^{\ell-1}(trace A_{e_i})e_i \in \Lambda$. Suppose that $\mathcal{A}_0(H), \dots, \mathcal{A}_{r-1}(H)$ are nonzero and belong to Λ. We choose the basis for $N_x(f)$ so that $e_i = \mathcal{A}_i(H)/\|\mathcal{A}_i(H)\|$, $i = 0, \dots r-1$. Then

$$\mathcal{A}_r(H) = \sum_{i=r}^{d-1}(trace A_{\mathcal{A}_{r-1}(H)} A_{e_i})e_i = \sum_{i=r}^{\ell-1}(trace A_{\mathcal{A}_{r-1}(H)} A_{e_i})e_i \in \Lambda.$$

So, $\forall\ r$, $\mathcal{A}_r(H) \neq 0 \Rightarrow \mathcal{A}_r(H) \in \Lambda$.

Hence, at x, $\mathcal{A}_k(H) = 0$ for some $k \leq dim\Lambda$. □

Proof of Theorem 1.1. As remarked above, $\mathcal{A}_k(H) \equiv 0$ for some $k \leq d$. Let $x \in M$, then since f has flat normal bundle $\{A_n : n \in N_x(f)\}$ is a family of self-adjoint commuting operators on $N_x(f)$ and for some $t \in \mathbb{Z}$, $1 \leq t \leq m$, there exist $v_1, \dots, v_t \in N_x(f)$ and $\alpha_1, \dots, \alpha_t \in \mathbb{Z}$ with $\sum_{i=1}^t \alpha_i \leq m$ such that $\forall\ n \in N_x(f)$,

$$det(1 - A_n) = \prod_{i=1}^{t}(1 - \langle v_i, n\rangle)^{\alpha_i}, \quad [4].$$

Put $\Lambda = span\{v_1, \dots, v_t\}$. Now if $n \in N_x(f)$ with $\|n\| = 1$, the principal curvatures in direction n are $\langle v_1, n\rangle, \dots, \langle v_t, n\rangle$ where $\langle v_i, n\rangle$ has multiplicity α_i, $i = 1, \dots, t$, together with a zero principal curvature of multiplicity $m - \sum_{i=1}^t \alpha_i$. So, for any $n \in \Lambda^\perp$ the principal curvatures in direction $n/\|n\|$ are all zero and therefore $A_{n/\|n\|}$, and hence A_n, is the zero operator.

It follows from Lemma 1.2 that $\mathcal{A}_k(H) = 0$ for some $k \le dim\Lambda \le t \le m$. $\square$

B. Rouxel used a different approach in [5]. He considered the quadratic form on $N_x(f)$ defined by $Q(u,v) = (traceA_u)(traceA_v) - trace(A_uA_v)$. With respect to the inner product on $N_x(f)$ induced from $\mathbb{R}^{m+d}$, $Q(u,v) = <u, Qv>$ where $Q : N_x(f) \to N_x(f)$ is a linear map. Let $\lambda_1, \dots, \lambda_s$ be the distinct eigenvalues of this linear map with corresponding eigenspaces $E_1, \dots, E_s$, so that $N_x(f) = E_1 \oplus \cdots \oplus E_s$. Let $q(H)$ denote the number of nonzero projections of H on the eigenspaces $E_1, \dots, E_s$. Clearly $q(H) = 0$ if and only if $H = 0$. It is shown in [5] that $q(H) = 1$ if and only if $H \ne 0$ and $\mathcal{A}_1(H) = 0$. Here we will show that f is an $\mathcal{A}_k$-immersion if and only if $q(H) \le k$ at all points of M, and f is k-Chen if and only if f is $\mathcal{A}_k$ and $q(H) = k$ at some $x \in M$. This follows from the next theorem.

Theorem 1.3 *Let $f : M^m \to \mathbb{R}^{m+d}$ be an immersion and let $k \ge 1$. Then, at $x \in M$, $q(H) = k$ if and only if $\mathcal{A}_{k-1}(H) \ne 0$ and $\mathcal{A}_k(H) = 0$.*

The proof of this theorem depends on the following lemmas:

Lemma 1.4 *At $x \in M$, let $\mathcal{A}_{k-1}(H) \ne 0$ and $\mathcal{A}_k(H) = 0$, $k \ge 1$. Then $1 \le q(H) \le k$.*

Proof. Let $\{e_0, \dots, e_{d-1}\}$ be an orthonormal basis for $N_x(f)$ with $e_i = \mathcal{A}_i(H)/\|\mathcal{A}_i(H)\|$, $i = 0, \dots, k-1$. Then, writing $u = \sum_{i=0}^{d-1} u_ie_i$, $v = \sum_{i=0}^{d-1} v_ie_i$,

$$Q(u,v) = \sum_{i,j=0}^{d-1} u_iv_j(traceA_{e_i}traceA_{e_j} - traceA_{e_i}A_{e_j}).$$

Now let Q denote the $(d \times d)$-matrix with ij-th term

$$[Q]_{ij} = traceA_{e_i}traceA_{e_j} - traceA_{e_i}A_{e_j}.$$

From the definitions of $\mathcal{A}_0(H), \dots, \mathcal{A}_{k-1}(H)$,

$$traceA_{e_j} = 0 \text{ for } j = 1, \dots, d-1 \text{ and } traceA_{e_i}A_{e_j} = 0 \text{ for } i = 0, \dots, k-2,\ j \ge k.$$

Further $\mathcal{A}_k(H) = 0 \Rightarrow traceA_{e_{k-1}}A_{e_j} = 0$ for $j \ge k$.

Thus $[Q]_{ij} = 0$ for $i = 0, \dots, k-1$, $j \ge k$. Therefore the matrix Q has the form $\left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right)$ where B is a $(k \times k)$-matrix.

Now if λ is an eigenvalue of Q with corresponding eigenspace E and λ is not an eigenvalue of B then $E \subset span\{e_k, \dots, e_{d-1}\}$ and, since $H = \|H\|e_0$, H has zero projection on E. Hence $q(H)$ is at most the number of distinct eigenvalues of B. So $1 \le q(H) \le k$. $\square$

Lemma 1.5 *At $x \in M$, let $H, \mathcal{A}_1(H), \dots, \mathcal{A}_{k-1}(H)$ be nonzero, let $v \in N_x(f)$, $v \ne 0$, with v orthogonal to $span\{H, \mathcal{A}_1(H), \dots, \mathcal{A}_{k-1}(H)\}$ and $traceA_{\mathcal{A}_{k-1}(H)}A_v = 0$. Then v is orthogonal to $\mathcal{A}_k(H)$.*

Proof. Let $\{e_0, \ldots, e_{d-1}\}$ be an orthonormal basis for $N_x(f)$ with $e_i = \mathcal{A}_i(H)/\|\mathcal{A}_i(H)\|$, $i = 0, \ldots, k-1$ and $e_k = v/\|v\|$. Then $\mathcal{A}_k(H) = \sum_{i=k}^{d-1}(trace A_{\mathcal{A}_{k-1}(H)}A_{e_i})e_i$. So the projection of $\mathcal{A}_k(H)$ in direction e_k is $trace A_{\mathcal{A}_{k-1}(H)}A_{e_k} = \frac{1}{\|v\|}trace A_{\mathcal{A}_{k-1}(H)}A_v = 0$. Therefore $\mathcal{A}_k(H)$ is orthogonal to v. □

We now take a different point of view. Suppose Q has distinct eigenvalues $\lambda_1, \ldots, \lambda_s$, $1 \leq s \leq d$, with corresponding eigenspaces $E_1, \ldots, E_s \subset N_x(f)$. For $u \in N_x(f)$, write $u = u_1 + \cdots + u_s$ where u_j is the projection of u on E_j, $j = 1, \ldots, s$.

By a change of basis for $N_x(f)$, if necessary, we can assume that $Q(u, v) = \sum_{j=1}^{s} \lambda_j <u_j, v_j>$.

Lemma 1.6 *At $x \in M$, let $q(H) = k$, then $\mathcal{A}_k(H) = 0$.*

Proof. By reordering the eigenspaces if necessary we can suppose that $H = H_1 + \cdots + H_k \in E_1 \oplus \cdots \oplus E_k$ with $H_1, \ldots, H_k$ nonzero. Put $H^* = span\{H_1, \ldots, H_k\}$. We show that $\forall\, r = 0, 1, \ldots,\ \mathcal{A}_r(H) \in H^*$.

Now $\mathcal{A}_0(H) = H \in H^*$. Suppose that $\mathcal{A}_{r-1}(H) \in H^*$. If $\mathcal{A}_{r-1}(H) = 0$, then $\mathcal{A}_r(H) = 0 \in H^*$. If $\mathcal{A}_{r-1}(H) \neq 0$ then $\mathcal{A}_{r-1}(H) = \alpha_1 H_1 + \cdots + \alpha_k H_k$ for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Let $v \in N_x(f)$, $v \neq 0$ with v orthogonal to H^*, say $v = v_1 + \cdots + v_s \in E_1 \oplus \cdots \oplus E_s$. Then

$$\begin{aligned} Q(\mathcal{A}_{r-1}(H), v) &= \sum_{j=1}^{s} \lambda_j <(\mathcal{A}_{r-1}(H))_j, v_j> \\ &= \sum_{j=1}^{k} \lambda_j \alpha_j <H_j, v_j> = 0 \end{aligned}$$

since for $j = 1, \ldots, k$, $<H_j, v_j> = <H_j, v> = 0$. Also

$$\begin{aligned} Q(\mathcal{A}_{r-1}(H), v) &= trace A_{\mathcal{A}_{r-1}(H)} trace A_v - trace A_{\mathcal{A}_{r-1}(H)}A_v \\ &= -trace A_{\mathcal{A}_{r-1}(H)}A_v, \quad \text{since } trace A_v = 0 \text{ as } <H, v> = 0. \end{aligned}$$

Therefore $trace A_{\mathcal{A}_{r-1}(H)}A_v = 0$. So, by Lemma 1.5, v is orthogonal to $\mathcal{A}_r(H)$. This holds for all $v \in N_x(f)$ with v orthogonal to H^*. Hence $\mathcal{A}_r(H) \in H^*$.

Since $dim H^* = k$ it follows that $\mathcal{A}_k(H) = 0$. □

Proof of Theorem 1.3. The theorem is true for $k = 1$, since by Lemma 1.4, $\mathcal{A}_0(H) \neq 0$ and $\mathcal{A}_1(H) = 0 \Rightarrow q(H) = 1$. Also, $q(H) = 1 \Rightarrow H \neq 0$ and, by Lemma 1.6, $q(H) = 1 \Rightarrow \mathcal{A}_1(H) = 0$.

Now assume that the statement of the theorem holds for $k \leq r-1$. Then by Lemma 1.4, $\mathcal{A}_{r-1}(H) \neq 0$ and $\mathcal{A}_r(H) = 0 \Rightarrow q(H) \leq r$. But, by the induction hypothesis, $q(H) < r \Rightarrow \mathcal{A}_{r-1}(H) = 0$. Hence $q(H) = r$.

Conversely, if $q(H) = r$ then, by Lemma 1.6, $\mathcal{A}_r(H) = 0$. Again, by the induction hypothesis, if $\mathcal{A}_{r-1}(H) = 0$ then $q(H) < r$ and hence $\mathcal{A}_{r-1}(H) \neq 0$. Hence the statement of the theorem holds for all $k \geq 1$. □

2. k-minimal immersions

In this section we give methods for constructing $\mathcal{A}_k$-immersions by generalising a construction from [1]. Let $f : M^m \to \mathbb{R}^p$ be an immersion such that for some s, $2 \leq s \leq p-m$, f has an orthonormal set of parallel normal fields $\{u_1, \dots, u_s\}$ and let $g : W^w \to \mathbb{R}^s$ be an immersion. We consider $h : M \times W \to \mathbb{R}^p$ defined by $h(x,y) = f(x) + \sum_{i=1}^{s} g_i(y)u_i(x)$, where $g(y) = (g_1(y), \dots, g_s(y))$. It was proved in Theorem 1.1 of [1] that

(1) h is an immersion if and only if, $\forall\, x \in M$, $\forall\, y \in W$, the linear map $(1 - \sum_{i=1}^{s} g_i(y)A_{u_i}) : T_xM \to T_xM$ is nonsingular,

(2) $\xi \in N_{(x,y)}(h)$ if and only if $\xi = n + \sum_{i=1}^{s} \eta_i u_i(x)$ where $n \in N_x(f)$, $<n, u_i(x)> = 0$ for $i = 1, \dots, s$ and $\eta = (\eta_1, \dots, \eta_s) \in N_y(g)$,

(3) the shape operator of h, $C_\xi : T_{(x,y)}(M \times W) \to T_{(x,y)}(M \times W)$, is given in terms of the shape operator of f, $A_n : T_xM \to T_xM$, and the shape operator of g, $B_\eta : T_yW \to T_yW$ by the formula
$$C_\xi(X,Y) = ((1 - \textstyle\sum_{i=1}^{s} g_i(y)A_{u_i})^{-1}A_\xi X, B_\eta Y)$$
where $X : M \to TM$, $Y : W \to TW$ are tangent fields.

We will use this construction to give examples of $\mathcal{A}_k$-immersions for different values of k.

In [1] we introduced the idea of a 1-minimal immersion. We generalise this to a k-minimal immersion. We consider a parallel, flat k-plane subbundle $\Lambda(f)$ of the normal bundle $N(f)$ of an immersion f, that is, locally there is a set of k parallel normal fields which at each point x determines an orthonormal basis for the fibre $\Lambda_x(f) \subset N_x(f)$.

Definition 2.1 *Let $f : M^m \to \mathbb{R}^p$ be an immersion with a parallel, flat k-plane subbundle, $\Lambda(f)$, of $N(f)$. We say that f is k-minimal with respect to $\Lambda(f)$ if $\forall x \in M$, $\forall n \in N_x(f)$ with n orthogonal to $\Lambda_x(f)$, $\forall u \in \Lambda_x(f)$ with $1 - A_u$ nonsingular, $trace(1 - A_u)^{-1}A_n = 0$.*

From Proposition 2.2 of [1], f is k-minimal with respect to $\Lambda(f)$ if and only if $\forall x \in M$, $\forall n \in N_x(f)$ with n orthogonal to $\Lambda_x(f)$, $\forall u \in \Lambda_x(f)$, $\forall \ell = 0, 1, \dots$ $trace(A_u)^\ell A_n = 0$.

For most applications we will need $\Lambda(f)$ to satisfy the global condition that there exist a set of parallel normal fields $u_i : M^m \to \mathbb{R}^p$, $i = 1, \dots, k$ such that $\forall x \in M$, $\{u_1(x), \dots, u_k(x)\}$ is an orthonormal basis for $\Lambda_x(f)$. In this case we will say $\Lambda(f)$ is global parallel flat. In the definition of 1-minimal in [1] we only considered this stronger condition.

Proposition 2.2 *Let $f : M^m \to \mathbb{R}^p$ be k-minimal with respect to $\Lambda(f)$, then f is an $\mathcal{A}_k$-immersion.*

Proof. Since, $\forall x \in M$, $\forall n \in N_x(f)$ with n orthogonal to $\Lambda_x(f)$, $\forall u \in \Lambda_x(f)$ $traceA_n = 0$ and $traceA_uA_n = 0$, this result follows immediately from Lemma 1.2. □

A relation between k-minimal and minimal immersions is given in the next theorem.

Theorem 2.3 *Let f be an immersion and let $\Lambda(f)$ be a parallel, flat k-plane subbundle of $N(f)$. Let $\Omega = \{(x,u) \in \Lambda(f) : 1 - A_u$ is nonsingular at $x\}$ and let $F : \Omega \to \mathbb{R}^p$ be defined by $F(x,u) = f(x) + u(x)$.*
Then f is k-minimal with respect to $\Lambda(f)$ if and only if F is minimal.

Proof. In a neighbourhood V of $x_0 \in M$ we can take a set of parallel normal fields $\{u_1, \dots, u_k\}$ which at each point of V determines an orthonormal basis for the fibre of $\Lambda(f)$. Then, locally, Ω can be identified with an open subset of $V \times \mathbb{R}^k$, by $(x, \sum_{i=1}^k t_i u_i(x)) \equiv (x, t)$, where $t = (t_1, \dots, t_k)$ and $F(x,t) = f(x) + \sum_{i=1}^k t_i u_i(x)$. Now use the construction described at the start of this section with M replaced by V, $s = k$, $g : \mathbb{R}^k \to \mathbb{R}^k$ the identity function and $h = F$. So F is an immersion and $n \in N_{(x,t)}(F)$ if and only if n is orthogonal to $\Lambda_x(f)$ in $N_x(f)$. Further if C_n denotes the shape operator of F then, at (x,t), $traceC_n = trace(1 - \sum_{i=1}^k t_i A_{u_i})^{-1} A_n$ at x. Writing $u = \sum_{i=1}^k t_i u_i(x)$, $traceC_n = trace(1 - A_u)^{-1} A_n$.

Now F is minimal if and only if $\forall x \in M, \forall n$ orthogonal to $\Lambda_x(f)$, $traceC_n = 0$. This holds if and only if f is k-minimal with respect to $\Lambda(f)$. □

In [1] we constructed many examples of 1-minimal immersions. The next theorem gives a way of using these examples to build up examples of k-minimal (and hence $\mathcal{A}_k$-) immersions for different values of $k > 1$.

Theorem 2.4 *Let $f : M^m \to \mathbb{R}^p$ be an immersion with an orthonormal set of parallel normal fields $\{u_1, \dots, u_k\}$ for some $1 \le k \le p - m$, and let $g : W^w \to \mathbb{R}^q$, $q - w \ge k$ be an immersion with an orthonormal set of parallel normal fields $\{\psi_1, \dots, \psi_{q-w}\}$ and such that $\forall x \in M$, $1 - \sum_{i=1}^k g_i(y) A_{u_i}$ is nonsingular at x, where $g(y) = (g_1(y), \dots, g_q(y))$. Let $\{e_1, \dots, e_{q-k}\}$ be the standard basis for $\mathbb{R}^{q-k}$ and let $h : M \times W \to \mathbb{R}^{p+q-k} \equiv \mathbb{R}^p \times \mathbb{R}^{q-k}$ be defined by*

$$h(x,y) = (f(x) + \textstyle\sum_{i=1}^k g_i(y) u_i(x), \sum_{i=1}^{q-k} g_{k+i}(y) e_i).$$

Then

(1) h is an immersion.

(2) $\xi \in N_{(x,y)}(h)$ if and only if $\xi = (n + \sum_{i=1}^k \eta_i u_i(x), \sum_{i=1}^{q-k} \eta_{k+i} e_i) \in N_x(f) \times \mathbb{R}^{q-k}$, for some $n \in N_x(f)$ with n orthogonal to $span\{u_1(x), \dots, u_k(x)\}$, $\eta = (\eta_1, \dots, \eta_q) \in N_y(g)$.

(3) If $\mu_j : M \times W \to \mathbb{R}^p \times \mathbb{R}^{q-k}$ is defined by $\mu_j(x,y) = (\sum_{i=1}^k \psi_{j,i}(y) u_i(x), \sum_{i=1}^{q-k} \psi_{j,k+i}(y) e_i)$, where $\psi_j(y) = (\psi_{j,1}(y), \dots, \psi_{j,q}(y))$, $j = 1, \dots, q - w$, then $(\mu_1, \dots, \mu_{q-w})$ is an orthonormal set of parallel normal fields for h.

(4) $C_\xi(X,Y) = ((1 - \sum_{i=1}^k g_i(y) A_{u_i})^{-1} A_v X, B_\eta Y)$ and $traceC_\xi = trace((1 - \sum_{i=1}^k g_i(y) A_{u_i})^{-1} A_v) + traceB_\eta$, where $\xi = (n + \sum_{i=1}^k \eta_i u_i(x),\ \sum_{i=1}^{q-k} \eta_{k+i} e_i)$, $v = n + \sum_{i=1}^k \eta_i u_i(x)$, $C_\xi : T_{(x,y)}(M \times W) \to T_{(x,y)}(M \times W)$ is the shape operator of h and $B_\eta : T_y W \to T_y W$ is the shape operator of g.

(5) Let $\Lambda(f)$ be the global parallel flat subbundle of $N(f)$ with fibre $\Lambda_x(f) = span\{u_1(x), \dots, u_k(x)\}$ and let $\widetilde{\Lambda}(h)$ be the global parallel flat subbundle of $N(h)$ with fibre $\widetilde{\Lambda}_{(x,y)}(h) = span\{\mu_1(x,y), \dots, \mu_{q-w}(x,y)\}$. Then if f is k-minimal with respect to $\Lambda(f)$, h is k-minimal with respect to $\widetilde{\Lambda}(h)$.

Proof. Define $\hat{f} : M^m \to \mathbb{R}^{p+q-k} \equiv \mathbb{R}^p \times \mathbb{R}^{q-k}$ and $\hat{u}_i : M^m \to \mathbb{R}^{p+q-k} \equiv \mathbb{R}^p \times \mathbb{R}^{q-k}$, $i = 1, \dots, k$ by $\hat{f}(x) = (f(x), 0)$ and $\hat{u}_i(x) = (u_i(x), 0)$. For $i = 1, \dots, q - k$ put $\hat{e}_i = (0, e_i) \in$

$\mathbb{R}^p \times \mathbb{R}^{q-k}$, where $\{e_1, \dots, e_{q-k}\}$ is the standard basis for $\mathbb{R}^{q-k}$. Then

$$h(x,y) = \widehat{f}(x) + \sum_{i=1}^{k} g_i(y)\hat{u}_i(x) + \sum_{i=1}^{q-k} g_{k+i}(y)\hat{e}_i.$$

and

$$\mu_j(x,y) = \sum_{i=1}^{k} \psi_{j,i}(y)\hat{u}_i(x) + \sum_{i=1}^{q-k} \psi_{j,k+i}(y)\hat{e}_i,\ j = 1, \dots, q-w.$$

Now h is defined from $\widehat{f}$ and g as in the construction described at the start of this section, since $\{\hat{u}_1, \dots, \hat{u}_k, \hat{e}_1, \dots, \hat{e}_{q-k}\}$ is an orthonormal set of parallel normal fields for $\widehat{f}$. For $(n,z) \in N_x(\widehat{f}) \equiv N_x(f) \times \mathbb{R}^{q-k}$, the shape operator of $\widehat{f}$ in the direction (n,z), $\widehat{A}_{(n,z)} : T_xM \to T_xM$ is given by $\widehat{A}_{(n,z)} = A_n$, the shape operator of f in the direction n. So $\forall i = 1, \dots, k$, $\widehat{A}_{\hat{u}_i} = A_{u_i}$, and $\forall i = 1, \dots, q-k$, $\widehat{A}_{\hat{e}_i} = 0$ is the zero map.

(1): By Theorem 1.1 of [1], h is an immersion, since $1 - \sum_{i=1}^{k} g_i(y)\widehat{A}_{\hat{u}_i} - \sum_{i=1}^{q-k} g_{k+i}(y)\widehat{A}_{\hat{e}_i} = 1 - \sum_{i=1}^{k} g_i(y)A_{u_i}$, and this map is nonsingular, by hypothesis.

(2): Again using Theorem 1.1 of [1], $\xi \in N_{(x,y)}(h)$ if and only if $\xi = \hat{n} + \sum_{i=1}^{k} \eta_i\hat{u}_i(x) + \sum_{i=1}^{q-k} \eta_{k+i}\hat{e}_i$, where $\hat{n} \in N_x(\widehat{f})$, $<\hat{n}, \hat{u}_i(x)> = 0\ \forall i = 1, \dots, k$ and $<\hat{n}, \hat{e}_i> = 0\ \forall i = 1, \dots, q-k$, and $\eta = (\eta_1, \dots, \eta_q) \in N_y(g)$.
Now $\hat{n} = (n, z)$ for some $n \in N_x(f)$, $z \in \mathbb{R}^{q-k}$. So the given condition on $\hat{n}$ is satisfied if and only if $\hat{n} = (n, 0)$ and $<n, u_i(x)> = 0$ for $i = 1, \dots, k$. Hence
$\xi = (n + \sum_{i=1}^{k} \eta_i u_i(x), \sum_{i=1}^{q-k} \eta_{k+i}e_i) \in N_x(f) \times \mathbb{R}^{q-k}$.

(3): From (2), for $j = 1, \dots, q-w$, $\mu_j(x,y) \in N_{(x,y)}(h)$. Also μ_j is parallel in the normal bundle of h since $u_1, \dots, u_k$ are parallel in the normal bundle of f and $\psi_1, \dots, \psi_{q-w}$ are parallel in the normal bundle of g.
Further $<\mu_j(x,y), \mu_i(x,y)> = \sum_{\ell=1}^{q}(\psi_{j,\ell}(y), \psi_{i,\ell}(y)) = <\psi_j(y), \psi_i(y)> = \delta_{ji}$.
Therefore $\{\mu_1, \dots, \mu_{q-w}\}$ is an orthonormal set of parallel normal fields for h.

(4): Now ξ is as in (2), and Theorem 1.1 of [1] gives

$$\begin{aligned} C_\xi(X,Y) &= ((1 - \textstyle\sum_{i=1}^{k} g_i(y)\widehat{A}_{\hat{u}_i} - \sum_{i=1}^{q-k} g_{k+i}(y)\widehat{A}_{\hat{e}_i})^{-1}\widehat{A}_v X, B_\eta Y) \\ &= ((1 - \textstyle\sum_{i=1}^{k} g_i(y)A_{u_i})^{-1}A_v X, B_\eta Y) \end{aligned}$$

and as in [1], $traceC_\xi = trace((1 - \sum_{i=1}^{k} g_i(y)A_{u_i})^{-1}A_v) + traceB_\eta$.

(5): Note that (3) shows that $\widetilde{\Lambda}(h)$ is a global parallel flat $(q-w)$-plane subbundle of the normal bundle of h. Let $\mu \in \widetilde{\Lambda}_{(x,y)}(h) = span\{\mu_1(x,y), \dots, \mu_{q-w}(x,y)\}$.
So $\mu = (\sum_{i=1}^{k} \eta_i u_i(x), \sum_{i=1}^{q-k} \eta_{k+i}e_i)$ for some $\eta = (\eta_1, \dots, \eta_q) \in N_y(g)$. Therefore from (4), $C_\mu(X,Y) = ((1 - \sum_{i=1}^{k} g_i(y)A_{u_i})^{-1}A_v X, B_\eta Y)$, where $v = \sum_{i=1}^{k} \eta_i u_i$.
Now ξ is orthogonal to $\widetilde{\Lambda}_{(x,y)}(h)$ if and only if $\xi = \hat{n} = (n, 0) \in N_x(f) \times \mathbb{R}^{q-k}$ and from (4),

$$C_{\hat{n}}(X,Y) = ((1 - \sum_{i=1}^{k} g_i(y)A_{u_i})^{-1}A_n X, 0).$$

Hence, for $\ell = 1, 2, \dots$,

$$(C_\mu)^\ell C_{\hat{n}} = ((1 - \sum_{i=1}^{k} g_i(y)A_{u_i})^{-1}A_v)^\ell(1 - \sum_{i=1}^{k} g_i(y)A_{u_i})^{-1}A_n), 0).$$

So, as in [1], $trace(C_\mu)^\ell C_{\hat{n}} = trace((1-\sum_{i=1}^k g_i(y)A_{u_i})^{-1}A_v)^\ell(1-\sum_{i=1}^k g_i(y)A_{u_i})^{-1}A_n$. Now $A_v = \sum_{i=1}^k \eta_i A_{u_i}$, therefore, as trace is linear, $trace(C_\mu)^\ell C_{\hat{n}}$ is a sum of terms of the form $\alpha(i_1,\dots,i_k)\sum traceDA_{u_{j_1}}DA_{u_{j_2}}\cdots DA_{u_{j_k}}DA_n$, where $D=(1-\sum_{i=1}^k g_i(y)A_{u_i})^{-1}$, $i_1,\dots,i_k\in\{1,\dots,k\}$, $\alpha(i_1,\dots,i_k)\in\mathbb{R}$ and the sum is taken over all permutations $j_1,\dots,j_k$ of $i_1,\dots,i_k$.
But, up to a constant multiple, this term is $\frac{\partial^\ell}{\partial t_{i_1}\cdots\partial t_{i_k}}trace(1-\sum_{i=1}^k t_iA_{u_i})^{-1}A_n$ at $(t_1,\dots,t_k)=(g_1(y),\dots,g_k(y))$, and, since f is k-minimal with respect to $\Lambda(f)$, $trace(1-\sum_{i=1}^k t_iA_{u_i})^{-1}A_n=0$, for all $(t_1,\dots,t_k)$ for which $(1-\sum_{i=1}^k t_iA_{u_i})$ is nonsingular. Hence $trace(C_\mu)^\ell C_{\hat{n}}=0$ and therefore h is $(q-k)$-minimal with respect to $\widetilde{\Lambda}(h)$. □

If M and W are compact, $f: M^m\to\mathbb{R}^p$ is a 1-minimal immersion with respect to a parallel normal field and $g: W^w\to\mathbb{R}^{w+k}$ has codimension k and flat normal bundle then, replacing g by ϵg for some sufficiently small $\epsilon>0$ if necessary, as in [1], to ensure that the map $h: M\times W\to\mathbb{R}^{p+w}$ is an immersion it follows from Theorem 2.4 that h is k-minimal with respect to a global parallel flat normal subbundle. This immersion can then be used as the base immersion f for further constructions. For example, if f is k-minimal and g has codimension k then the constructed h will be k-minimal.

3. Umbilical normal directions

Immersions with an umbilical normal direction can be used to construct $\mathcal{A}_k$-immersions and k-minimal immersions for different values of k.

Let $f: M^m\to\mathbb{R}^p$ be an immersion and let $v: M^m\to\mathbb{R}^p$ be a parallel normal field for f such that $A_v=\lambda 1$, $\lambda: M\to\mathbb{R}$. Let $g: W^w\to\mathbb{R}^q$ be an immersion such that, $\forall\, x\in M$, $1-g_1(y)A_v$ is nonsingular at x, and let $h: M\times W\to\mathbb{R}^{p+q-1}$ be defined by

$$h(x,y)=(f(x)+g_1(y)v(x),\sum_{i=1}^{q-k}g_{1+i}(y)e_i)$$

where $\{e_1,\dots,e_{q-1}\}$ is the standard basis for $\mathbb{R}^{q-1}$. By Theorem 2.4, h is an immersion.

Let Π be a linear subspace of $N_x(f)=\Pi\oplus\Pi^\perp$ and, for $(x,y)\in M\times W$, let $\widetilde{\Pi}=\{(u+\eta_1 v(x),\sum_{i=1}^{q-k}\eta_{1+i}e_i) : u\in\Pi,\ <u,v(x)>=0,\ \eta\in N_y(g)\}$. Then, by Theorem 2.4, $\widetilde{\Pi}\subset N_{(x,y)}(h)=\widetilde{\Pi}\oplus\widetilde{\Pi}^\perp$.

Theorem 3.1 *With $f,v,g,h,\Pi,\widetilde{\Pi}$ as above, for $x\in M$ let the shape operators of f satisfy the condition :*

$\forall u\in\Pi,\ \forall n\in\Pi^\perp,\ traceA_u^rA_n=0$ *for* $r=0,1,\dots,\ell$.

Then, for $(x,y)\in M\times W$ the shape operators of h satisfy the condition :

$\forall\tilde{u}\in\widetilde{\Pi},\ \forall\tilde{n}\in\widetilde{\Pi}^\perp,\ traceC_{\tilde{u}}^rC_{\tilde{n}}=0$ *for* $r=0,1,\dots,\ell$.

Proof. Let $\tilde{u}=(u+\eta_1v(x),\sum_{i=1}^{q-1}\eta_{1+i}e_i)\in\widetilde{\Pi}$, where $u\in\Pi,\ <u,v(x)>=0,\ \eta\in N_y(g)\}$. As in Theorem 2.4,

$$C_{\tilde{u}}(X,Y)=((1-g_1(y)A_v)^{-1}A_{u+\eta_1v}X,B_\eta Y)=(\frac{1}{1-\lambda g_1(y)}A_{u+\eta_1v}X,B_\eta Y).$$

Let $\tilde{n} \in \widetilde{\Pi}^{\perp}$, then $\tilde{n} = (n, 0)$ for some $n \in \Pi^{\perp}$. So

$$C_{\tilde{n}}(X, Y) = ((1 - g_1(y)A_v)^{-1}A_nX, 0) = (\frac{1}{1 - \lambda g_1(y)}A_nX, 0).$$

Hence

$$\begin{aligned} traceC_{\tilde{u}}^{\ell}C_{\tilde{n}} =& \frac{1}{(1 - \lambda g_1(y))^{\ell+1}} traceA_{u+\eta_1 v}^{\ell}A_n \\ =& \frac{1}{(1 - \lambda g_1(y))^{\ell+1}} \sum_{r=0}^{\ell} \binom{\ell}{r} (\lambda\eta_1)^{\ell-r} traceA_u^r A_n = 0 \text{ since } u \in \Pi,\ n \in \Pi^{\perp}. \quad \square \end{aligned}$$

Corollary 3.2 *With f, v, g, h as at the start of this section, let f be an $\mathcal{A}_k$-immersion. Then h is an $\mathcal{A}_{k+q-w}$-immersion. Further, if $\forall x \in M$, $v(x) \in span\{\mathcal{A}_0(H), \ldots, \mathcal{A}_{k-1}(H)\}$, where H is the mean curvature vector of f, then h is an $\mathcal{A}_{k+q-w-1}$-immersion.*

Proof. At $x \in M$, put $\Pi = span\{\mathcal{A}_0(H), \ldots, \mathcal{A}_{k-1}(H), v(x)\}$. Then for $u \in \Pi$, $u = \zeta + \alpha v(x)$, where $\zeta \in span\{\mathcal{A}_0(H), \ldots, \mathcal{A}_{k-1}(H)\}$, $\alpha \in \mathbb{R}$. For $n \in \Pi^{\perp}$, $traceA_n = 0$, since $<n, H> = 0$ and $traceA_uA_n = traceA_{\zeta}A_n + \lambda\alpha\, traceA_n = 0$, since $\zeta \in span\{\mathcal{A}_0(H), \ldots, \mathcal{A}_{k-1}(H)\}$ and $\mathcal{A}_k(H) = 0$. Therefore by Theorem 3.1 with $\ell = 1$, $\forall \tilde{u} \in \widetilde{\Pi}$, $\tilde{n} \in \widetilde{\Pi}^{\perp}$, $traceC_{\tilde{n}} = 0$ and $traceC_{\tilde{u}}C_{\tilde{n}} = 0$.

Hence by Lemma 1.2 h is an $\mathcal{A}_{k+q-w}$-immersion, since, $\forall x \in M$, $dim\widetilde{\Pi} \leq k + q - w$. If, $\forall x \in M$, $v(x) \in span\{\mathcal{A}_0(H), \ldots, \mathcal{A}_{k-1}(H)\}$ then, $\forall x \in M$, $dim\widetilde{\Pi} \leq k + q - w - 1$ and therefore h is an $\mathcal{A}_{k+q-w-1}$-immersion. $\square$

In the particular case where g is a codimension one immersion and, $\forall x \in M$, $v(x) \in span\{\mathcal{A}_0(H), \ldots, \mathcal{A}_{k-1}(H)\}$ then h is an $\mathcal{A}_k$-immersion. If f is $\mathcal{A}_1$, g has codimension one and $\forall x \in M$, $v(x) \notin \{\alpha H : \alpha \in \mathbb{R}\}$ then h is an $\mathcal{A}_2$-immersion. For examples of $\mathcal{A}_k$-immersions with umbilical normal directions, see [2]. A product of circles $\mathbb{S}^1(a) \times \mathbb{S}^1(b) \times \mathbb{S}^1(c)$ with radii a, b, c is 1-Chen if $a = b = c$, 2-Chen if two of a, b, c are equal and 3-Chen if no two of a, b, c are equal. This manifold lies on a 5-sphere in $\mathbb{R}^6$ and the radial normal direction is umbilical. Taking the inclusion of this embedding into $\mathbb{R}^{\ell}$, $\ell > 6$, gives more umbilical parallel normal directions. These examples can also be used in the next Corollary.

Corollary 3.3 *With f, v, g, h as at the start of this section, let f be k-minimal with respect to a parallel flat subbundle $\Lambda(f)$ of $N(f)$ and let g have flat normal bundle. If, $\forall x \in M$ $v(x) \notin \Lambda_x(f)$ then h is $(k+q-w)$-minimal. If, $\forall x \in M$ $v(x) \in \Lambda_x(f)$ then h is $(k+q-w-1)$-minimal.*

Proof. For $x \in M$, put $\Pi_x = \{\zeta + \alpha v(x) : \zeta \in \Lambda_x(f),\ \alpha \in \mathbb{R}\}$, then $\widetilde{\Pi}_{(x,y)} = \{(u + \eta_1 v(x), \sum_{i=1}^{q-1} \eta_{1+i}e_i) : u \in \Pi_x(f),\ \eta \in N_y(y)\}$. For $u \in \Pi_x$, $u = \zeta + \alpha v(x)$, and $n \in \Pi_x^{\perp}$, $traceA_u^{\ell}A_n = \sum_{r=0}^{\ell} \binom{\ell}{r} (\lambda\alpha)^{\ell-r} traceA_{\zeta}^r A_n = 0$ for $\ell = 0, 1, \ldots$, since $\zeta \in \Lambda_x(f)$ and $n \in \Lambda_x^{\perp}(f)$ and f is k-minimal with respect to $\Lambda(f)$.

Therefore, by Theorem 3.1, $\forall \tilde{u} \in \widetilde{\Pi}_{(x,y)},\ \tilde{n} \in \widetilde{\Pi}^{\perp}_{(x,y)}$, $traceC^{\ell}_{\tilde{u}}C_{\tilde{n}} = 0$ for $\ell = 0, 1, \ldots$. Let $\widetilde{\Pi}(h)$ denote the subbundle of h with fibre $\widetilde{\Pi}_{(x,y)}$ at $(x, y) \in M \times W$. Note that $dim\widetilde{\Pi}_{(x,y)} = k+q-w$ if $v(x) \notin \Lambda_x(f)$ and $dim\widetilde{\Pi}_{(x,y)} = k+q-w-1$ if $v(x) \in \Lambda_x(f)$. Also $\widetilde{\Pi}(h)$ is a parallel flat subbundle of $N_x(f)$. Hence h is $(k + q - w)$-minimal (resp. $(k + q - w - 1)$-minimal) with respect to $\widetilde{\Pi}(h)$ if, $\forall x \in M$, $v(x) \notin \Lambda_x(f)$ (resp. $v(x) \in \Lambda_x(f)$). $\square$

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