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# Regular Linear Spaces

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**Abstract.** We call a finite linear space regular, if all pencils of lines are similar. This means that the way how the lines through a point partition the complement of this point is equivalent for all points. We enumerate all finite regular linear spaces of order $\leq 14$ and, with some gaps, up to order 16. We comment on some of these spaces, point out interrelations between them and give presentations for several distinguished species.
MSC 1991: 51E26, 51E05, 05B05, 05B30

## 1. Introduction

An incidence geometry $(P, \mathcal{B})$ consists of a set $P$ of points, $|P| = v$, and a set $\mathcal{B}$ of subsets of $P$ which are called blocks (or lines), $|\mathcal{B}| = b \geq 1$. We assume at least the following:

$(*)$ each block has cardinality $\geq 2$ and each pair of points is on at most one block.

The geometry $(P, \mathcal{B})$ with $(*)$ is called a linear space if each pair of points is on at least (and hence on exactly) one block. The number of lines through a point $p$ is called the degree of $p$. The number of points on a block $B$ is called the length of $B$. An incidence geometry with $(*)$ is called a configuration if all points have the same degree, say $r$, and if all blocks have the same length, say $k$. There are linear spaces which are a configuration at the same time, for instance a projective plane or a Steiner triple system. But if the linear space has lines of different length, then it cannot be a configuration. We are interested in the case that for each existing line length $k$ the set of all blocks of length $k$ constitutes a configuration on $P$. So we have not one configuration but the incidence matrix is composed by several

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configurations, one for each $k$. We call a linear space of this type regular. Note that this condition is equivalent to the property stated in the abstract.

We will study regular linear spaces with small point numbers in the following. If we have specified all lines of length $\geq 3$ then all lines of length 2 will follow (as those pairs of points which are not yet joined). So we often drop the 2−lines. Also in the figures we do not draw the 2−lines.

## 2. Parameters

As usual let $v$ and $b$ be the number of points and of blocks (or lines) and denote by $b_2, b_3, b_4, ...$ the number of lines of length $2, 3, 4, ...$ . A $k$−line is a line of length $k$. To get the line length distribution we have to solve the equation

$$b_2 + 3b_3 + 6b_4 + 10b_5 + ... = \binom{v}{2}.$$

Regularity imposes the (necessary) condition

$$v|kb_k \quad \text{for all} \quad k = 2, 3, ....$$

We write the solutions in the form

$$(v|b_2, b_3, b_4, ...)$$

and get the parameter candidates for finite regular spaces. Note, however, that there may be parameters which have no realization. Take for example $(8|4, 0, 4)$; these are the parameters for a regular linear space on 8 points. But, obviously, one cannot place four 4−lines on 8 points. The number of blocks is $b = b_2 + b_3 + b_4 + ...$ and the constant $r$ (number of lines through a point) is $r = \frac{1}{v}(2b_2 + 3b_3 + 4b_4 + ...)$. Spaces with constant $r$ have been studied in [11] up to order 12. The condition there is slightly more general than regularity in our sense. In the following list we have already left out the parameters which are not realizable. Note that in some cases the 2−lines are omitted. For instance we describe the linear space $(10|15, 0, 5)$ by $\binom{5}{2}$ dually. This means: take the 10 2−lines on 5 points and dualize it. This gives 5 4−lines on 10 points. Here 15 pairs of points are not yet joined, for these we add 15 2−lines.

## 3. List of regular linear spaces

| parameter | $\sharp$ | comment |
|---|---|---|
| $(2\|1)$ | 1 | $\binom{2}{2}$ one line |
| $(3\|0,1)$ | 1 | one line |
| $(3\|3)$ | 1 | $\binom{3}{2}$ |
| $(4\|0,0,1)$ | 1 | one line |
| $(4\|6)$ | 1 | $\binom{4}{2}$ |
| $(5\|0,0,0,1)$ | 1 | one line |
| $(5\|10)$ | 1 | $\binom{5}{2}$ |
| $(6\|0,...,0,1)$ | 1 | one line |
| $(6\|3,4)$ | 1 | $\binom{4}{2}$ dually |
| $(6\|9,2)$ | 1 | 2 disj. 3-lines |
| $(6\|15)$ | 1 | $\binom{6}{2}$ |
| $(7\|0,...,0,1)$ | 1 | one line |
| $(7\|0,7)$ | 1 | 7 point plane, conf. $7_3$, see 4.4 |
| $(7\|21)$ | 1 | $\binom{7}{2}$ |
| $(8\|0,...,0,1)$ | 1 | one line |
| $(8\|4,8)$ | 1 | punctured affine plane ord. 9, conf. $8_3$, see 4.4 |
| $(8\|16,0,2)$ | 1 | 2 disj. 4-lines |
| $(8\|28)$ | 1 | $\binom{8}{2}$ |
| $(9\|0,...,0,1)$ | 1 | one line |
| $(9\|0,12)$ | 1 | aff. plane ord. 3 |
| $(9\|9,9)$ | 3 | conf. $9_3$, see 4.4 |
| $(9\|18,6)$ | 2 | cub. graph ord. 6 dually, derivation of lat. sq. 4 — see 4.1 |
| $(9\|27,3)$ | 1 | 3 disj. 3-lines |
| $(9\|36)$ | 1 | $\binom{9}{2}$ |
| $(10\|0,...,0,1)$ | 1 | one line |
| $(10\|15,0,5)$ | 1 | $\binom{5}{2}$ dually |
| $(10\|15,10)$ | 10 | conf. $10_3$, see 4.4 |
| $(10\|25,0,0,2)$ | 1 | two disj. 5-lines |
| $(10\|45)$ | 1 | $\binom{10}{2}$ |

| parameter | $\sharp$ | comment |
|---|---|---|
| $(11\|0,...,0,1)$ | 1 | one line |
| $(11\|22,11)$ | 31 | conf. $11_3$, see 4.4 |
| $(11\|55)$ | 1 | $\binom{11}{2}$ |
| $(12\|0,...,0,1)$ | 1 | one line |
| $(12\|0,4,9)$ | 1 | punctured proj. plane of order 3 |
| $(12\|0,16,3)$ | 2 | lat.sq. order 4, see 4.1 |
| $(12\|6,8,6)$ | 1 | see 4.6 |
| $(12\|6,20)$ | 5 | punctured STS(13), see 4.2 |
| $(12\|12,0,9)$ | 1 | aff. plane order 3, dually |
| $(12\|12,12,3)$ | 4 | conf. $12_3$ with parall. class<br>derivation of lat. sq. ord 5 see 4.1 and 4.4 |
| $(12\|18,4,6)$ | 1 | see 4.6 |
| $(12\|18,16)$ | 574 | conf. $12_4 16_3$ |
| $(12\|24,8,3)$ | 8 | cub.graph ord.8 with parall.class, dually |
| $(12\|30,0,6)$ | 1 | 4-graph on 6 points, dually, see also 4.6 |
| $(12\|30,12)$ | 229 | conf. $12_3$, see 4.4 |
| $(12\|36,0,0,0,2)$ | 1 | 2 disj. 6-lines |
| $(12\|36,4,3)$ | 1 | $3 \times 4$ grid |
| $(12\|42,8)$ | 6 | cubic graph ord. 8 , dually |
| $(12\|48,0,3)$ | 1 | 3 disj. 4-lines |
| $(12\|54,4)$ | 1 | 4 disj. 3-lines |
| $(12\|66)$ | 1 | $\binom{12}{2}$ |
| $(13\|0,...,0,1)$ | 1 | one line |
| $(13\|0,0,13)$ | 1 | proj. plane ord. 4 = conf. $13_4$, see 4.4 |
| $(13\|0,26)$ | 2 | STS(13), see 4.2 |
| $(13\|39,13)$ | 2036 | conf. $13_3$, see 4.4 |
| $(13\|78)$ | 1 | $\binom{13}{2}$ |
| $(14\|0,...,0,1)$ | 1 | one line |
| $(14\|7,0,14)$ | 1 | conf. $14_4$, see 4.4 and figure 8 in 4.6 |
| $(14\|7,28)$ | 787 | punctured STS(15), see 4.2 |
| $(14\|49,0,0,0,0,2)$ | 1 | 2 disj. 7-lines |
| $(14\|49,0,7)$ | 2 | see 4.6 and figure 9 |
| $(14\|49,14)$ | 21399 | conf. $14_3$, see 4.4 |
| $(14\|91)$ | 1 | $\binom{14}{2}$ |

| parameter | $\sharp$ | comment |
|---|---|---|
| $(15\|0,....,0,1)$ | 1 | one line |
| $(15\|0,5,15)$ | 1 | punctured aff. plane of ord. 4 |
| $(15\|0,15,0,6)$ | 1 | $\binom{15}{2}$+15 par.cl. 2+2+2(dually), see 4.3 |
| $(15\|0,25,0,3)$ | 2 | lat. sq.(5), see 4.1 |
| $(15\|0,35)$ | 80 | STS(15), see 4.2 |
| $(15\|15,0,15)$ | 4 | conf. $15_4$, see 4.4 |
| $(15\|15,10,0,6)$ | 1 | see 4.3 |
| $(15\|15,20,0,3)$ | 40 | conf. $15_3$ with a par. class<br>derivation of lat. sq. order 6 see 4.1 |
| $(15\|15,30)$ | | conf.$15_6 30_3$, see 4.5 |
| $(15\|30,5,0,6)$ | 1 | see 4.3 |
| $(15\|30,15,0,3)$ | 251 | |
| $(15\|30,25)$ | | conf.$15_5 25_3$ |
| $(15\|45,0,0,6)$ | 1 | $\binom{6}{2}$ dually |
| $(15\|45,10,0,3)$ | 23 | |
| $(15\|45,20)$ | | conf.$15_4 20_3$ |
| $(15\|60,5,0,3)$ | 1 | $3 \times 5$ grid |
| $(15\|60,15)$ | 245342 | conf. $15_3$, see 4.4 |
| $(15\|75,0,0,3)$ | 1 | 3 disj. 5-lines |
| $(15\|75,10)$ | 21 | cubic graph on 10 points, dually, see 4.6 |
| $(15\|90,5)$ | 1 | 5 disj. 3-lines |
| $(15\|105)$ | 1 | $\binom{15}{2}$ |
| $(16\|0,...,0,1)$ | 1 | one line |
| $(16\|0,0,20)$ | 1 | affine plane order 4 |
| $(16\|0,16,12)$ | 1 | see 4.6 and figure 10 |
| $(16\|0,32,4)$ | 23 | see 4.6 |
| $(16\|24,0,16)$ | 19 | conf. $16_4$, see 4.4 |
| $(16\|24,16,8)$ | 300880 | see 4.5 |
| $(16\|24,32)$ | | conf. $16_6 32_3$ |
| $(16\|48,0,12)$ | 574 | conf. $16_3 12_4$ |
| $(16\|48,16,4)$ | 88 | conf. $16_3$ with a parallel class |
| $(16\|64,0,...,0,2)$ | 1 | 2 disj. 8-lines |
| $(16\|72,0,8)$ | 6 | 4-graph on 8 points, dually, see 4.5 |
| $(16\|72,16)$ | | conf. $16_3$ |
| $(16\|96,0,4)$ | 1 | four disj. 4-lines |
| $(16\|120)$ | 1 | $\binom{16}{2}$ |

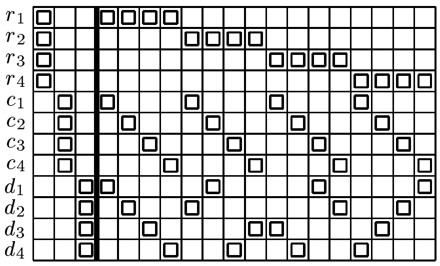
Note: Most of the geometries of this list are well known configurations, cp. [11]. But some of these geometries may be new, for instance the parameter cases $(15|60,15)$, $(16|24,0,16)$ or $(16|24,16,8)$. In the list the following five parameter cases are still missing:

$$(15|15,30),(15|30,25),(15|45,20),(16|24,32),(16|72,16).$$

## 4. Some Comments

### 4.1. Linear spaces related to latin squares

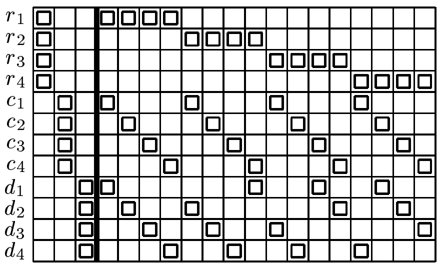
We may regard a latin square of order n as a regular linear space having parameters $(3n|0,n^2,0,...,0,3)$. For this we take the n rows of the latin square as an n-block, furthermore the n columns as another n-block and the n digits as a third one. We assume that these three blocks are disjoint thus defining $3n$ points. Now each of the $n^2$ elements of the latin square defines a triple of points, namely the number of its row, the number of its column and the digit of this element. Hence we get $n^2$ 3-blocks. It is easily seen that we have a regular linear space with parameters stated above. We take $n \geq 4$, otherwise we get a mixing of the three distinguished n-blocks with the other $n^2$ ones. We call two latin squares isomorphic if the corresponding linear spaces are isomorphic. This means that we do not only take permutations of the $n^2$ elements which bring rows to rows, columns to columns and digits to digits (inner automorphisms), but we also allow that rows, columns and digits are interchanged. For instance there are 22 "conjugation types" of latin squares of order 6, [5], but only 12 isomorphism types, [2].
Let us illustrate this procedure for the two latin squares of order 4; the parameters of the associated linear space are $(12|0,16,3)$, see figure 1.




$\cong$

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $r_1$ | 1 | 2 | 3 | 4 |
| $r_2$ | 2 | 1 | 4 | 3 |
| $r_3$ | 3 | 4 | 1 | 2 |
| $r_4$ | 4 | 3 | 2 | 1 |

, $|\mathrm{Aut}| = 576$




$\cong$

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $r_1$ | 1 | 2 | 3 | 4 |
| $r_2$ | 2 | 3 | 4 | 1 |
| $r_3$ | 3 | 4 | 1 | 2 |
| $r_4$ | 4 | 1 | 2 | 3 |

, $|\mathrm{Aut}| = 192$

Figure 1: the two LSQ(4) (right) as linear spaces $(12|0,16,3)$ (left)

In both cases the automorphism group acts transitively on the set of 3-blocks. Derivation with respect to one 3-block defines a regular space $(9|9,9)$ with a distinguished parallel class. There are three geometries $(9|9,9)$, the symmetric configurations $9_3$. One of them has no parallel class at all. The second one has exactly one parallel class and a group of order 12. Hence the original latin square has an automorphism group of order $12 \cdot 16 = 192$. The third $9_3$ is the Pappus configuration which has three parallel classes and a group of order 108. If one specifies one parallel class, one gets the group order 36 and the original latin square has an automorphism group of order $36 \cdot 16 = 576$. Note that the two linear spaces in [1, p.18] are isomorphic to the two latin squares of order 4 by the procedure described above.

Next we point out that the block derivations of the 12 latin squares of order 6 with parameters $(18|0,36,0,0,3)$ correspond to the 40 regular linear spaces with parameters $(15|15,20,0,3)$. For this we remove one 3-block from a latin square of order 6 and also the 15 3-blocks which have the same row or column or digit as the first one. Then we get a regular geometry on 15 points having three 5-blocks and 20 3-blocks. Conversely, it is easy to see that the extension from the small geometry to the greater one is uniquely possible. In other words, the 40 geometries correspond bijectively to the block orbits of the automorphism groups of all latin squares of order six.

Similarly, the automorphism groups of the two latin squares of order 5 (parameters as linear space: $(15|0,25,0,3)$) have four block orbits and block derivation leads to the 4 linear spaces with parameters $(12|12,12,3)$ having automorphism groups of order $6,6,24,72$. One of the squares is "cyclic" and the group is transitive on the block set. Derivation with respect to one block defines the space with automorphism group 24 which can be described as follows: Take on the point set $\{1,2,...,12\}$ the $3-$block $\{1,5,9\}$ and all images under the group $Z_{12}$. The full automorphism group is generated by $Z_{12}$ and one element of order 2. It follows that the automorphism group of the cyclic latin square has order $25 \cdot 24 = 600$. The second square has one $3-$block which is fixed under all automorphisms. Derivation with respect to this fixed block leads to the space with automorphism group of order 72, hence the automorphism group of the second latin square has also the order 72 (compare figure 2).

cyclic

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 1 |
| 3 | 4 | 5 | 1 | 2 |
| 4 | 5 | 1 | 2 | 3 |
| 5 | 1 | 2 | 3 | 4 |

$|\mathrm{Aut}| = 600$

fixed block

| | | | | |
|---|---|---|---|---|
| $\underline{1}$ | 2 | 3 | 4 | 5 |
| 2 | 1 | 5 | 3 | 4 |
| 3 | 4 | 1 | 5 | 2 |
| 4 | 5 | 2 | 1 | 3 |
| 5 | 3 | 4 | 2 | 1 |

$|\mathrm{Aut}| = 72$

Figure 2: the two LSQ(5)

### 4.2. Linear spaces related to Steiner triple systems

The list contains the parameters $(13|0,26)$ which describe the two Steiner triple systems on 13 points having automorphim groups of order 39 and 6, respectively. Figure 3 is a picture of the second one; one can see the fixed point, the orbit of length 6 and the two orbits of length 3. (Note that this is not an embedding into the plane since we have used double points).

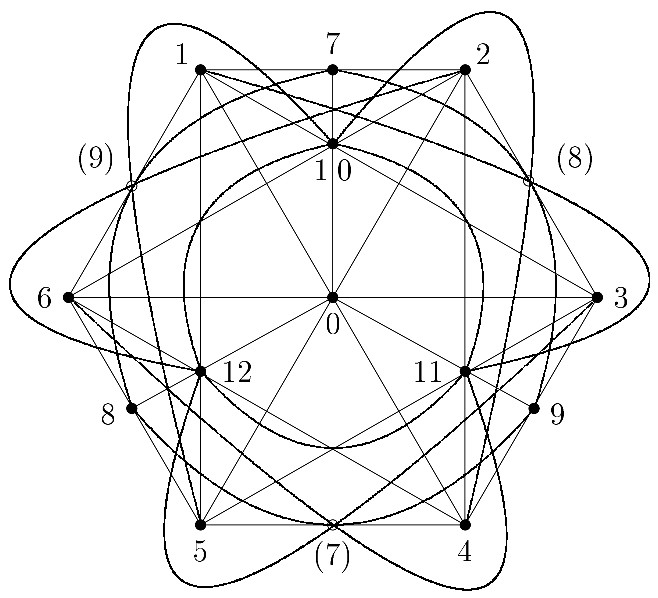



Figure 3: Steiner tripel system on 13 points

The two Steiner systems have together 5 point orbits and by taking point derivations with respect to a point in each of these orbits we get the five geometries of type $(12|6,20)$, [9]. The list contains also the 80 Steiner triple systems on 15 points, type $(15|0,35)$, see also [13], and their 787 point derivations $(14|7,28)$, see also [9]. These point derivations correspond to the point orbits of the automorphism groups of all 80 Steiner systems.

### 4.3. Linear spaces related to the symplectic generalized quadrangle $W_3(2)$

The linear space of type $(15|0,15,0,6)$ may be described as follows: Take the complete graph on 6 vertices and dualize it. We have thus 15 "points" and six 5-blocks on it. A parallel class consists of three points (i.e., pairs of vertices) which cover all six vertices of the graph. There exist 15 parallel classes. We add them as 3-blocks to the six 5-blocks and get the regular space $(15|0,15,0,6)$. An incidence matrix for this geometry is shown by the first 15 lines of figure 4.

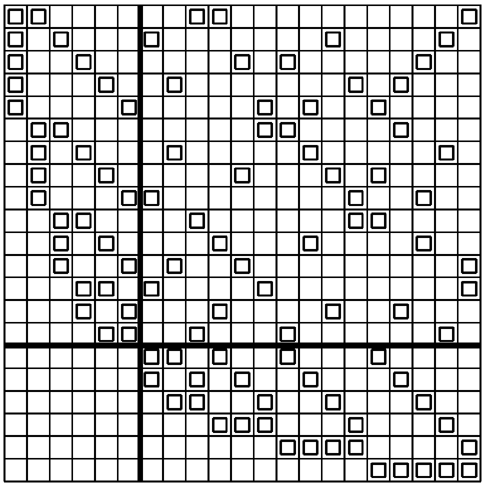


Figure 4: the projective plane of order 4

Note that the $15 \times 15$-square corresponding to the 15 3-blocks is the incidence matrix of the symplectic generalized quadrangle $W_3(2)$ [15], see figure 5.
In the space $(15|0, 15, 0, 6)$ we now specify one parallelism, that is a 5-tupel of parallel classes which covers all 15 pairs of points (compare Figure 6).
After mutating these five 3-blocks to five triangles (which does not disturb the other incidences) we get the unique space of type $(15|15, 10, 0, 6)$. If we retain these five 3-blocks but change the remaining 10 3-blocks to triangles, then we arrive at the unique space with parameters $(15|30, 5, 0, 6)$.

**Remark.** There are altogether 6 parallelisms of the complete 6-graph (all equivalent) defining line 16 to 21 in the figure 4. The whole incidence matrix decribes the projective plane of order 4 and displays the outer automorphism of the symmetric group $S_6$: We have 6 vertices (upper left) and six parallelisms (lower right). Now each permutation of the six vertices induces a permutation of the 15 2-subsets. This in turn produces a permutation of the 15 3-subsets of these 15 elements. Since the 6 parallelisms are subsets of this set of cardinality 15, the last permutation induces a permutation on the set of six parallelisms. We have thus a natural map from the symmetric group on the six vertices to the symmetric group on the six parallelisms, and this map turns out to be the outer automorphism of $S_6$. We have a diagram shown in figure 7.

### 4.4. Symmetric configurations

A symmetric configuration $n_3$ is defined by $v = b = n$, $r = k = 3$ (and we always assume that each pair of points is on at most one block). The numbers $N(n_3)$ of these configurations with $7 \leq n \leq 15$ are:

| $n$: | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| $N(n_3)$: | 1 | 1 | 3 | 10 | 31 | 229 | 2036 | 21399 | 245342 |

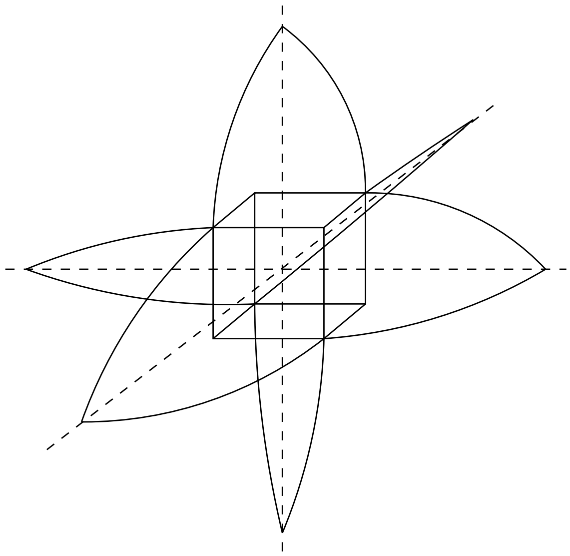


Figure 5: the symplectic generalized quadrangle $W_3(2)$

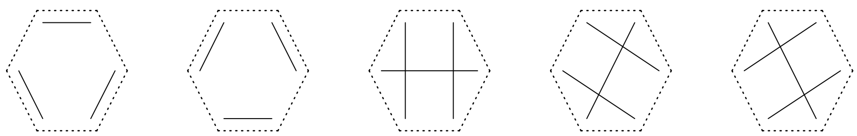


Figure 6: one of the 6 equivalent parallelisms of $(15|0,15,0,6)$

see [8], where the number of configurations $14_3$ is calculated for the first time and literature for the smaller cases is given. In our list we have calculated the number of configurations $15_3 = 245342$. After adding the missing 2-blocks these are regular linear spaces of type $(15|60,15)$. Most of these geometries are "rigid" (have automorphism group $= 1$) but there are considerably many species having not too few collineations and they might be interesting. The distribution of the automorphism group size over all geometries is:

$$\begin{array}{rrrrrr} 241240 \times 1, & 3709 \times 2, & 69 \times 3, & 180 \times 4, & 5 \times 5, & 59 \times 6, \\ 34 \times 8, & 3 \times 10, & 11 \times 12, & 2 \times 15, & 10 \times 16, & 1 \times 18, \\ 2 \times 20, & 2 \times 24, & 2 \times 30, & 1 \times 32, & 6 \times 48, & 1 \times 72, \\ 1 \times 128, & 2 \times 192, & 1 \times 720, & 1 \times 8064. & & \end{array}$$

Note that the configuration with the largest group size 8064 is the union of $8_3$ and $7_3$. For $8_3$ is the punctured affine plane of order 3 with automorphism group order $8 \cdot 6$ and $7_3$ is the projective plane of order 2 having automorphism group order $7 \cdot 6 \cdot 4$. The product of these two orders is just 8064.

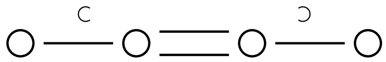


Figure 7: Diagram associated to the outer automorphism of $S_6$

The numbers $N(n_4)$ of symmetric configurations $n_4$, $13 \leq n \leq 16$, are:

| $n$ : | 13 | 14 | 15 | 16 |
|---|---|---|---|---|
| $N(n_4)$ : | 1 | 1 | 4 | 19 |

The first case describes the projective plane of order 3: $(13|0,0,13)$. An incidence matrix for $14_4$ is drawn in figure 8. The four geometries $15_4$ define linear spaces with parameters $(15|15,0,15)$ and have automorphism groups of order 24, 15, 30 and 360. Three of these geometries are already in [14]. The 19 configurations $16_4$ have the following distribution of automorphism group sizes:

$$1^2 2^6 3^3 4^1 6^2 12^1 16^1 18^1 32^1 1152^1.$$

One would like to have figures for these geometries reflecting their automorphism groups, see for instance the figures for the 10 configurations $10_3$ in [4].

### 4.5. Some large classes of regular spaces

An invariant for the geometries of parameter type $(15|15,30)$ is the structure of the set of 2-blocks, which is also called the configuration graph of the corresponding configuration $(15_6, 30_3)$ This substructure is a configuration with $v = b = 15$, $k = r = 2$ and can be described by the cycles formed by the 2-blocks. This cycle scheme is a partition of 15 into summands $\geq 3$, and there are 17 partitions of this kind. The numbers of regular spaces $(15|15,30)$ in the first three cases are

| | | |
|---|---|---|
| 1. : | 3, 3, 3, 3, 3 | 146 |
| 2. : | 3, 3, 3, 6 | 7891 |
| 3. : | 3, 3, 4, 5 | 22816 |

In the first case we have five disjoint triangles. We may mutate them to five 3-blocks and get a parallel class of a Steiner triple system on 15 points. Hence there are – up to equivalence – 146 parallel classes in the 80 Steiner triple systems on 15 points.

The regular linear spaces with parameters $(16|24,16,8)$ have a subgeometry consisting of 8 4-blocks on 16 points, i.e., the dual of a regular 4-graph on 8 points. There are 6 geometries of this type, compare the parameter case $(16|72,0,8)$. We begin with constructing these 6 geometries and then generate the 16 3-blocks. The 24 2-blocks then follow, of course. In this way we have 6 disjoint cases, and we list in the following the number of geometries for these cases:

| | | | |
|---|---|---|---|
| 1.(48) : | 2791 | 4.(4) : | 170399 |
| 2.(12) : | 39103 | 5.(16) : | 64595 |
| 3.(16) : | 22561 | 6.(1152) : | 1431 |

In brackets are the orders of the automorphism groups of the related 4-graphs which nearly characterize them. Note that the distribution of the geometries over the cases is rather irregular, case 4 being larger than the other five cases together.

### 4.6. Some more remarks

The parameter case $(16|0,32,4)$ describes the group divisible designs of [6]. The list contains some regular graphs (in dualized form), for instance the parameter case $(15|75,10)$ describes the 21 cubic graphs on 10 points, [12] and [10]. Some classes contain only few geometries (or exactly one geometry) and one would like to know these species explicitly. Let us describe some of these "sporadic" regular spaces.

$(12|30,0,6)$: We give a model using the cube. There are 12 pairs of faces which have a common edge, and we take these 12 pairs as points. We define the 6 faces as blocks. Then each block is incident with 4 points since each face of the cube has four neighbouring faces. Furthermore each point is a pair of faces and contains these two faces. Therefore we have six 4-blocks on 12 points which are regular and by adding the 30 missing pairs of points we get the unique regular space of type $(12|30,0,6)$.

$(12|6,8,6)$: In the situation just stated we define as parallel class each triple of points (=pairs of neighboured faces) such that all 6 faces of the cube are covered. There are exactly 8 parallel classes and we add them as 3-blocks to the six 4-blocks. This leads to the geometry $(12|6,8,6)$.

$(12|18,4,6)$: If we add not all 8 parallel classes but only four of them which form a parallelism, then we get the geometry $(12|18,4,6)$.

It is clear from the interpretations that the geometries $(12|30,0,6)$ and $(12|6,8,6)$ have automorphism group of order 48 (the full group of the cube). The orientation reversing maps interchange the two parallelisms. Since in the construction of $(12|18,4,6)$ one of the two parallelisms is distinguished, this last linear space has automorphism group of order 24 only (the orientation preserving maps of the cube), isomorphic to $S_4$.

The first 12 rows in figure 8 show the incidence matrix for the linear space $(12|6,8,6)$ (without the 2-blocks). The last two rows define the two parallelisms. Note that the whole incidence matrix is a model for the symmetric configuration $14_4$, i.e. for the linear space $(14|7,0,14)$.

$(14|49,0,7)$: One can at once figure out a model based on the regular 7-gon, but note that there is a second geometry, see figure 9 for both. These two linear spaces may also be described as the duals of the 4-regular graphs on 7 vertices. These correspond to the 2-regular graphs on 7 vertices, the graphs $C_7$ and $C_4 \cup C_3$.

$(16|0,16,12)$: We take the cyclic latin square of order 4 and call its 16 elements points. Define as 4-blocks the 4 rows, the 4 columns and the 4 digits in order to get 12 4-blocks. Now choose a partial transversal of length 3 as a 3-block and shift it cyclically in horizontal and vertical direction. In this way we get 16 3-blocks, figure 10.

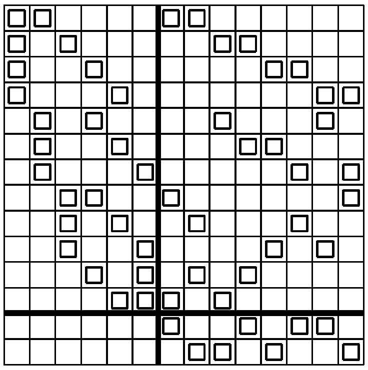


Figure 8: the symmetric configuration $14_4$

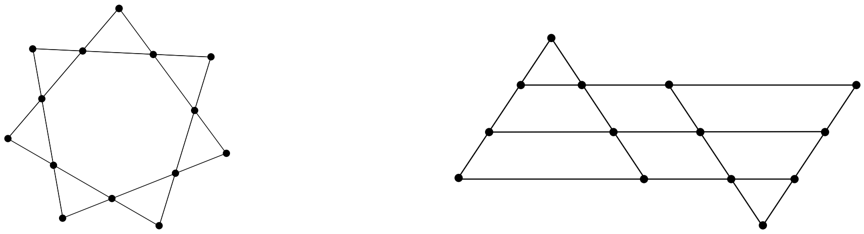


Figure 9: the two $(14|49,0,7)$

| | | | |
|---|---|---|---|
| $\underline{1}$ | 2 | 3 | 4 |
| 2 | 3 | $\underline{4}$ | 1 |
| 3 | 4 | 1 | 2 |
| 4 | 1 | 2 | $\underline{3}$ |

Figure 10: the unique geometry $(16|0,16,12)$

Note: Our (universal) computer program makes extensive use of the TDO-method introduced in [3]. We also use the isomorphism algorithm [7], in an improved form.

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