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On Point Sets fixing a Convex Body from within

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Abstract: We study the concept of a set fixing a convex body from within that was recently proposed by V. Soltan. We prove his conjecture that a finite set which fixes a d-dimensional convex body from within contains a subset of at most $2d$ points with the same property.

1. Introduction

The classical definition [2],[4] of a set S fixing a convex body $K \subset \mathbb{R}^d$ is that $S \subset K$ and for each translation by a vector $t \in \mathbb{R}^d$, $t \neq 0$ there is a point $s \in S$ and an $\varepsilon > 0$ such that $s \notin \operatorname{interior} K$ but $s \in \operatorname{interior}(K + \varepsilon t)$. This models a solid object K which is held in its place (against translations) by some points S outside K which would have to move inside K if one tried to force a movement. At the 1995 Intuitive Geometry Conference, V. Soltan defined a new concept of a fixing set. He calls a set S fixing the convex body $K \subset \mathbb{R}^d$ from within if $S \subset K$ and for each translation $t \neq 0$ there is an $s \in S$ such that $s \notin K + t$. This models a rigid surface ∂K held in its place by some points S inside K which would have to penetrate that surface if one tried to force a movement of K.

Special interest found those fixing sets that are minimal with respect to inclusion, called primitive fixing systems. It was proved by Tomor [8] that in dimension 2 a primitive fixing system contains at most six points (see also [3]). For dimensions $d \geq 3$ Bollobás constructed arbitrary large (finite) primitive fixing systems [1].

For the new concept V. Soltan conjectured [6] that in $\mathbb{R}^d$ any finite set of points fixing a d-dimensional body from within has a subset of at most $2d$ points with the same property. We prove a slightly stronger theorem.

Theorem. *Let $K \subset \mathbb{R}^d$ be a convex body, $U \subset \mathbb{R}^d$ a linear subspace and $S \subset K$ a finite set with the property that for all $u \in U$, $u \neq 0$, there is an $s \in S$ such that $s \notin K + u$. Then there is a subset $T \subseteq S$ of at most $2 \dim U$ elements with the same property.*

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2. Proof of the Theorem

We first need the following lemmas, the first of which follows from Theorem 14.6 of [5].

Lemma 1. *If C is a closed cone which does not contain a nontrivial linear subspace, then there is a t such that $(t, c) < 0$ for all $c \in C$, $c \neq 0$.*

Lemma 2. *If C is a cone generated by X, and W is the maximum linear subspace contained in C, then W is generated as a cone by $W \cap X$.*

Proof of Lemma 2. Let V be the linear space that contains C, so $W \subseteq V$. The proof is by induction on $\dim(V) - \dim(W)$, the case $\dim(V) = \dim(W)$ being trivial. If $V \neq W$, then there is a supporting hyperplane H of C at its apex 0. Then $W \subset H \cap C$, and $H \cap C$ is a cone in H that is generated by $X \cap H$. Since $\dim(H) < \dim(V)$, we may apply the inductive assumption on $H \cap C$, $H \cap X$ and W.

To prove now the theorem, we may assume, since S is finite, that S is minimal with respect to inclusion. So S does not contain points $s \in S$ that lie in the relative interior of the cut $K \cap (U + s)$ generated by them. Therefore $N(s) := \{n \in U \mid (n, v) \geq 0$ for all $v \in (K - s) \cap U\}$ contains for each s at least one element (at least one normal to each supporting hyperplane (in $U + s$) of $K \cap (U + s)$ in s; if $K \cap (U + s)$ is fulldimensional in $U + s$, then the interior normal.).

The proof is by induction on $\dim U$. The case $\dim U = 1$ is trivial. Let $\dim U = i > 1$. Consider the cone $C \subseteq U$ generated by $\bigcup_{s \in S} N(s)$, i.e. the set of all nonnegative linear combinations of elements of $\bigcup_{s \in S} N(s)$. Suppose C does not contain a linear subspace. Since C is a closed cone (all the sets $N(s)$ are closed) by Lemma 1 there is a vector $t \in U$ such that $(t, v) < 0$ for all $v \in C$, $v \neq 0$. Then the movement by t cannot be blocked by S. Conversely, assume that $s \notin K + \epsilon t$ for all positive ϵ, then there is a supporting hyperplane (in $U + s$) of $K \cap (U + s)$ in s which (weakly) separates the halfline $s - \epsilon t$ ($\epsilon > 0$) from $K \cap (U + s)$. The interior normal of that hyperplane has nonnegative inner product with t and belongs to $N(s)$, contrary to the assumption.

So C does contain some linear subspace. Let $W \subseteq C \subseteq U$ be the maximal linear subspace contained in C. Then by Lemma 2 this subspace W is generated as a cone by $\left(\bigcup_{s \in S} N(s)\right) \cap W$. From this set $\left(\bigcup_{s \in S} N(s)\right) \cap W$ we select (using Steinitz's theorem) a subset of at most $2 \dim W$ elements that generates W (as a cone). Now this subset blocks all movement apart that in directions orthogonal to W. To see this, assume that t is a direction that has a nonzero inner product with some element of W; then ($W = -W$) it has positive inner product with some element of W. If a vector has a positive inner product with an element of a cone, it has a positive inner product with one of the generators of the cone, so t has positive inner product with one of the selected generators of W, which is an interior normal of a supporting hyperplane at some point $s \in S$, (one of the at most $2 \dim W$ points that correspond to the selected generators of W). This point s blocks the movement in direction t.

So we only have to block movement in the directions of the orthogonal complement of W in U. This is done by the inductive assumption. This completes the proof.

3. Infinite Counterexamples

This theorem fails for infinite S, as can be seen by taking $K := [0,1]$ and $S :=]0,1[$. In fact, for any K there are infinite S for which the theorem fails: We only have to take two strictly antipodal points a and b (so K lies between two parallel hyperplanes which touch K only at a and b). Then $\{a,b\}$ fixes K. If we now replace b by a sequence of points of K converging to b (either from within, or along the boundary following one direction), we obtain a set that fixes K but does not contain a finite subset which fixes K.

Surprisingly, it can even fail for a closed set S, as can be seen by the following 3-dimensional example of Soltan [7]: Let $p_n := (\sin\frac{\pi}{n}, \cos\frac{\pi}{n}, 0)$ and $q_n := (\sin\frac{\pi}{n}, \cos\frac{\pi}{n}, \frac{1}{n})$, and let $p_0 := (0,1,0), q_0 := (0,1,1)$. Then $S := \{p_0, p_1, p_2, \ldots\}$ is a closed set that fixes $K := \mathrm{conv}\left(\bigcup_{i=0}^{\infty}\{p_i, q_i\}\right)$ from within, but no finite subset of S does (each set containing p_0, p_1 and one further of the p_i blocks each movement apart that in direction $(0,0,-1)$, but no finite subset of S blocks movement in that direction).

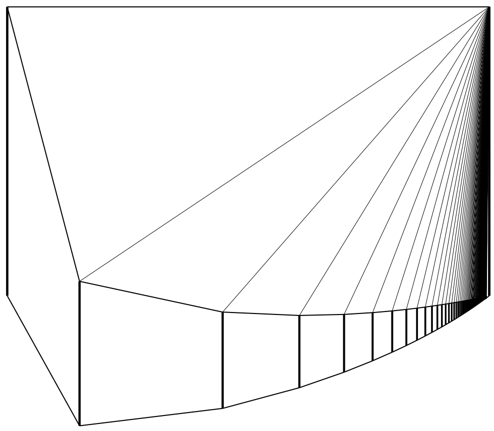

Figure 1

In none of these infinite counterexamples the fixing set S is inclusion-minimal.

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