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An Example of a Flexible Polyhedron with Nonconstant Volume in the Spherical Space

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1. Introduction

Let K be a two-dimensional closed simplicial complex; E^n be the n-dimensional Euclidean space; S^n be the n-dimensional unit sphere in E^{n+1}, and let S^n_+ be an open half sphere of S^n (i.e. a connected component of the difference $S^n \setminus \tau$, where τ is some hyperplane in E^{n+1} passing through the center of the sphere S^n).

A continuous mapping $P : K \to E^3$ affine over each simplex of K is said to be a *polyhedron* in E^3. A polyhedron $P : K \to S^3$ in S^3 is defined similarly. We say that a polyhedron P is *immersed* or *embedded* in the corresponding space if the mapping P is a local or global homeomorphism. However it is more convenient to consider the image $P(K)$ of the simplicial complex K under the mapping P as the polyhedron P itself.

A continuous (with respect to a parameter) family of polyhedrons $P_t : K \to E^3$ (resp. $P_t : K \to S^3$) such that all $P_t(K)$ are mutually isometric in the intrinsic metric induced from E^3 (resp. S^3) is said to be a *flex* of the polyhedron $P = P_0$. A polyhedron P is said to be *flexible* if there exists a nontrivial flex of it, i.e. a flex other than a motion of $P(K)$ as a solid body in the space E^3 (resp. S^3).

The Bellows Conjecture formulated [1] by R. Connelly and D. Sullivan asserts that *each (not necessarily embedded or immersed) oriented flexible polyhedron in E^3 conserves its (oriented) volume during a flex.*

One can find some details about the Bellows Conjecture in the fundamental survey [2]. Recently the affirmative answer to the Bellows Conjecture was obtained for all flexible polyhedra in E^3 [3, 4].

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On the other hand it is known that each flexible polyhedron in E^3 conserves its mean curvature during the flex [5].

In the present paper we construct a flexible polyhedron in an open halfsphere S^3_+ which conserves neither volume nor mean curvature during flex.[2]

2. Preliminaries

The intersection of S^3 and some two-dimensional (resp. three-dimensional) plane of E^4 passing through the center of S^3 is said to be a *line* (resp. a two-dimensional *plane*) in S^3. Similarly, notions of angle between lines or planes on the sphere as well as motion of the sphere we reduce to the corresponding notions of Euclidean space. Properties of lines and planes on the sphere that we need below follow from the corresponding properties of their Euclidean "doubles" in so clear manner that we give neither proof nor reference.

A part of two-dimensional plane in S^3 bounded by two line segments that join two antipodal points on S^3 is said to be a *spherical digon*; the two line segments are said to be *sides* of the spherical digon and the common points of the sides are said to be its *vertices*.

3. An auxiliary polyhedron Q

Take four congruent spherical digons and glue them pairwise in cyclic order as it is schematically depicted in Fig. 1. Denote the resulting polyhedron by Q.

Obviously, Q is a flexible polyhedron.

It is clear that Q does not conserve its volume during the flex because on the one hand it can be deformed into a twice covered spherical digon with zero volume and on the other hand it can be deformed into a polyhedron with positive volume.

Similarly we can verify that Q does not conserve mean curvature during the flex. To do so, denote the vertices of Q by N and S and construct a three-dimensional plane $\tau \in E^4$ which is orthogonal to the straight line segment NS in Euclidean space E^4 containing the sphere S^3. Denote the points of intersection of τ and the edges of Q by A_1, A_2, A_3, and A_4 (see Fig. 1).

The mean curvature H_Q of the polyhedron Q equals the sum (calculated over all edges of Q) of the products of the edge lengths l_j and the values α_j of "inner" dihedral angles between the contiguous faces of Q, i.e.

$$H_Q = \sum_{j=1}^{4} l_j \alpha_j .$$

For each $j = 1, \dots, 4$ we have $l_j = \pi$ (because the radius of S^3 equals 1). The angle α_j equals the corresponding angle of the quadrangle $A_1A_2A_3A_4$ that lies on a two-dimensional unit sphere. It is known from spherical trigonometry that the area of a quadrangle on the

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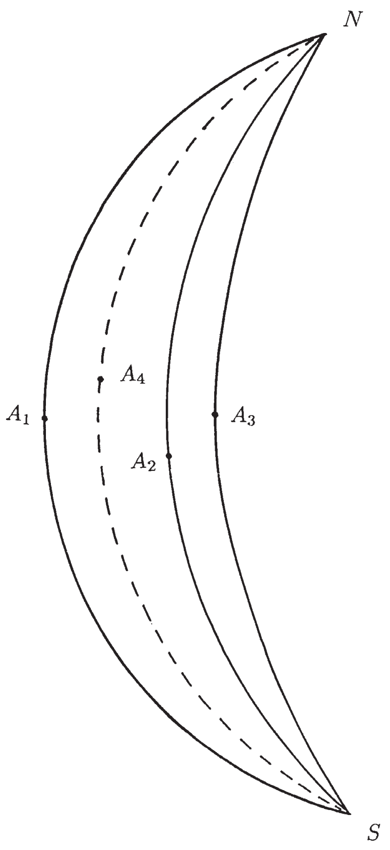

Fig. 1

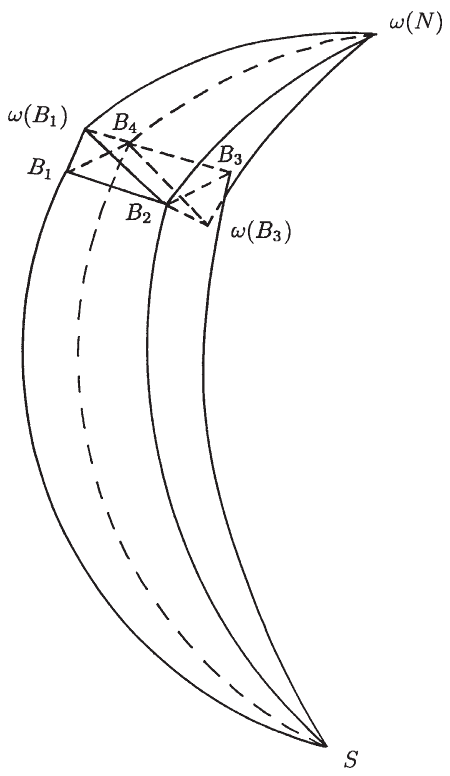

Fig. 2

unit sphere equals the excess of the sum of the inner angles of the quadrangle against 2π. Finally we obtain

$$H_Q = \sum_{j=1}^{4} l_j \alpha_j = \pi \sum_{j=1}^{4} \alpha_j = \pi(2\pi + \sigma), \tag{1}$$

where σ denotes the area of the quadrangle $A_1A_2A_3A_4$.

During the flex of Q, the area σ takes zero (if $A_1 = A_3$ or $A_2 = A_4$) as well as positive values (if all the points A_1, A_2, A_3, and A_4 are pairwise distinct). Hence σ (as well as H_Q) is not conserved during the flex.

In some sense the polyhedron Q constructed above is an "unpersuasive" example because Q contains two antipodal points of S^3 (namely, S and N). In the next item we shall modify Q to free it from this defect.

4. Construction of the polyhedron *R*

On each edge of Q choose a single point and denote it by B_1, B_2, B_3, or B_4 in such a manner that

$$\rho(B_1, B_2) = \rho(B_2, B_3) = \rho(B_3, B_4) = \rho(B_4, B_1), \tag{2}$$

where ρ is the intrinsic distance between points on the sphere S^3, i.e. the infimum of lengths of curves that lie on S^3 and join the points. The number $t = \rho(B_2, B_4)$ will be referred to as the *flex parameter*. Denote the constant $\rho(B_1, B_2)$ by a.

Cut the polyhedron Q by the line segments B_1B_2, B_2B_3, B_3B_4 and B_4B_1 onto 2 polyhedrons with boundary, denoted by Q_1 and Q_2.

Fix a (sufficiently small) positive number b.

Choose a (spherical) rotation $\omega : S^3 \to S^3$ that fixes the (spherical) line B_2B_4 immovable and is such that the (spherical) distance between the points B_1 and $\omega(B_1)$ equals b. Note that ω depends on the current value of the flex parameter t.

Since the (spherical) distance between the point B_3 and the line B_2B_4 equals the (spherical) distance between the point B_1 and the line B_2B_4, the distance between the points B_3 and $\omega(B_3)$ equals b.

Denote by R a polyhedron (without boundary) that comprises a) the polyhedron Q_2, b) the polyhedron $\omega(Q_1)$, and c) the four (spherical) triangles $B_1B_2\omega(B_1)$, $B_1B_4\omega(B_1)$, $B_2B_3\omega(B_3)$, and $B_4B_3\omega(B_3)$ (see Fig. 2).

From the above construction it follows that R is a flexible polyhedron for arbitrary points B_1, B_2, B_3, and B_4 satisfying the condition (2) and for each sufficiently small positive number b.

5. On the volume of the polyhedron R

We show that for arbitrary points B_1, B_2, B_3, and B_4 satisfying the condition (2) and for each sufficiently small positive number b, *the polyhedron R does not conserve its volume during a nontrivial flex.*

For each value of the parameter t the polyhedron R can be regarded as a union of four following polyhedra (here a polyhedron is seen as a solid bounded by the corresponding polyhedral surface):

A) the polyhedron Q_2 with the triangles $B_1B_2B_4$ and $B_2B_3B_4$ attached to it;
B) the polyhedron $\omega(Q_1)$ with the triangles $B_2B_4\omega(B_1)$ and $B_2B_4\omega(B_3)$ attached to it;
C) the tetrahedron $B_1B_2B_4\omega(B_1)$;
D) the tetrahedron $B_2B_3B_4\omega(B_3)$.

Any orientation of R induces mutually opposite orientations of the tetrahedra C) and D) (i.e. if one of them is oriented by the outward normal vector then the other is oriented by the inward normal vector). On the other hand, the corresponding sides of the tetrahedra C) and D) are equal to each other. Hence the oriented volumes of these tetrahedra are equal in magnitude, but opposite in sign. Thus the volume of R equals the sum of the volumes of the polyhedra A) and B). Hence the volume of R equals the volume of Q; for this reason it is not conserved during a nontrivial flex.

6. On the mean curvature of the polyhedron R

Here we verify that also *the mean curvature of R is not conserved during a nontrivial flex.*

The tetrahedron C) is symmetric with respect to the following two-dimensional planes. One of them passes through the points B_2, B_4 and through the midpoint of the straight line segment $B_1\omega(B_1)$ while the other passes through the points B_1, $\omega(B_1)$ and through the midpoint of the straight line segment B_2B_4. Thus the inner dihedral angles of the tetrahedron C) contiguous to the edges B_1B_2, $B_2\omega(B_1)$, B_1B_4 and $B_4\omega(B_1)$ are congruent. Denote the common value of the above-mentioned dihedral angles by α. Note that α depends on the current value of the flex parameter t.

Similar arguments show that the inner dihedral angles of the tetrahedron D) contiguous to the edges B_2B_3, $B_2\omega(B_3)$, B_3B_4, and $B_4\omega(B_3)$ are congruent. Moreover their common value equals to α because the tetrahedra C) and D) have respectively equal edges and thus are congruent. The last argument shows also that the values of the dihedral angles of the tetrahedra C) and D) contiguous to the edges $B_1\omega(B_1)$ and $B_3\omega(B_3)$ are congruent. Denote the common value of these angles by $\beta(t)$.

Given a polyhedron T and an edge CD of T, we denote the value of the (inner) dihedral angle contiguous to CD by $\angle_T CD$.

It follows from Fig. 2 that

$$\begin{aligned}
\angle_R B_1B_2 &= \angle_{\,\mathrm{A})} B_1B_2 + \angle_{\,\mathrm{C})} B_1B_2 &&= \angle_{\,\mathrm{A})} B_1B_2 + \alpha;\\
\angle_R B_2\omega(B_1) &= \angle_{\,\mathrm{B})} B_2\omega(B_1) + \angle_{\,\mathrm{C})} B_2\omega(B_1) &&= \angle_{\,\mathrm{B})} B_2\omega(B_1) + \alpha;\\
\angle_R B_1B_4 &= \angle_{\,\mathrm{A})} B_1B_4 + \angle_{\,\mathrm{C})} B_1B_4 &&= \angle_{\,\mathrm{A})} B_1B_4 + \alpha;\\
\angle_R B_4\omega(B_1) &= \angle_{\,\mathrm{B})} B_4\omega(B_1) + \angle_{\,\mathrm{C})} B_4\omega(B_1) &&= \angle_{\,\mathrm{B})} B_4\omega(B_1) + \alpha;\\
\angle_R B_2B_3 &= \angle_{\,\mathrm{A})} B_2B_3 - \angle_{\,\mathrm{D})} B_2B_3 &&= \angle_{\,\mathrm{A})} B_2B_3 - \alpha;\\
\angle_R B_2\omega(B_3) &= \angle_{\,\mathrm{B})} B_2\omega(B_3) - \angle_{\,\mathrm{D})} B_2\omega(B_3) &&= \angle_{\,\mathrm{B})} B_2\omega(B_3) - \alpha;\\
\angle_R B_4B_3 &= \angle_{\,\mathrm{A})} B_4B_3 - \angle_{\,\mathrm{D})} B_4B_3 &&= \angle_{\,\mathrm{A})} B_4B_3 - \alpha;\\
\angle_R B_4\omega(B_3) &= \angle_{\,\mathrm{B})} B_4\omega(B_3) - \angle_{\,\mathrm{D})} B_4\omega(B_3) &&= \angle_{\,\mathrm{B})} B_4\omega(B_3) - \alpha;
\end{aligned}$$

$$\begin{aligned}
&\angle_{\,\mathrm{A})} B_1B_2 + \angle_{\,\mathrm{B})} B_2\omega(B_1) = \pi;\\
&\angle_{\,\mathrm{A})} B_1B_4 + \angle_{\,\mathrm{B})} \omega(B_1)B_4 = \pi;\\
&\angle_{\,\mathrm{A})} B_2B_3 + \angle_{\,\mathrm{B})} B_2\omega(B_3) = \pi;\\
&\angle_{\,\mathrm{A})} B_3B_4 + \angle_{\,\mathrm{B})} \omega(B_3)B_4 = \pi.
\end{aligned}$$

Using these formulas we infer the following relation between the mean curvature H_R of R and the mean curvature H_Q of Q:

$$H_R(t) = H_Q(t) + 4\pi\rho(B_1, B_2) + 2b\beta(t).$$

Using the notation $a = \rho(B_1, B_2)$ and the formula (1), we can rewrite the preceding relation as

$$H_R(t) = \pi\sigma(t) + 2b\beta(t) + 2\pi^2 + 4\pi a. \tag{3}$$

Suppose H_R is constant on some interval I where the parameter t varies. Let f stand for the function identically equal to 1. Then (3) implies that the functions f, σ, and

β are linearly dependent over the interval I. Consider the analytic continuations of the functions f, σ, and β from the interval I to the complex plane $\mathbb{C}$. Then these functions will be defined and will be linearly dependent over the interval J where the functions σ and β have a geometric meaning.

Since $0 \in J$ and $\sigma(0) = \beta(0) = 0$, (3) implies that

$$H_R(t) \equiv \pi\sigma(t) + 2b\beta(t) + 2\pi^2 + 4\pi a = 2\pi^2 + 4\pi a,$$

or

$$\pi\sigma(t) + 2b\beta(t) = 0$$

for all $t \in J$. The latter formula and the fact that both the functions σ and β are nonnegative over J imply that σ and β are identically equal to zero. Thus the supposition that H_R is constant leads to a contradiction.

7. Theorem

In an open halfsphere S^3_+ of the three-dimensional unit sphere S^3 there exists a flexible polyhedron that conserves neither its (oriented) volume nor its mean curvature during a nontrivial flex.

Proof. Choose two antipodal points on the sphere S^3. Denote them by N and S. Join N and S by a line segment and denote its midpoint by M. Specify a rotation $\nu : S^3 \to S^3$ different from the identity that fixes point M.

Obviously, the broken line composed of the line segments SM and $M\nu(N)$ lies in some open halfsphere S^3_+. Choose neighborhoods V and W of the line segments SM and $M\nu(N)$ in such a manner that $V \cup W \subset S^3_+$.

Starting from some very narrow digons, construct the auxiliary polyhedron Q (see section 3) that is contained in the set $V \cup \nu^{-1}(W)$ and has the points N and S as its vertexes.

Transform Q into the polyhedron R according to section 4. To this end, let the points B_1, B_2, B_3, and B_4 be the midpoints of the edges of Q and let the rotation ω be such that R is still included in the set $V \cup W$.

The resulting polyhedron R is the one desired because it is contained in a halfsphere, it is flexible and it conserves neither volume nor mean curvature.

Remark that R is neither immersed nor embedded and in this sense it is similar to the well-known Bricard octahedra.

8. Generalization

Roughly speaking, we can say that in the proof of the Theorem in section 7 we have constructed a polyhedron Q that is very close to the meridian NS and then we have broken it at the point M. Obviously, we can roll up the meridian NS in a curve that is included in a ball of arbitrary small radius by repeatedly breaking it. Thus the following generalization of this theorem holds: *For every* $\varepsilon > 0$ *there exists a flexible polyhedron that is contained in a ball of radius* $\varepsilon > 0$ *in* S^3 *and conserves neither volume nor mean curvature during a nontrivial flex.*

9. Remarks

Remark 1. A ball of radius ε in the three-dimensional unit sphere S^3 "tends" to a ball of radius ε in Euclidean space as $\varepsilon \to 0$. Thus the generalization in section 8 reveals instability of the affirmative answer to the Bellows Conjecture in the Euclidean space [3, 4] and so testifies to nontriviality of the Conjecture. The fact that the infinitesimal version of the Bellows Conjecture in Euclidean space is wrong [6] can be considered as another illustration of nontriviality of the Conjecture.

Remark 2. To compare our results with some previously known theorems, recall that A. V. Pogorelov [7] constructed a geodesic mapping $\Omega : S^3_+ \to E^3$ that associates a pair of isometric (in intrinsic metrics) closed surfaces in Euclidean space E^3 with each pair of isometric (in intrinsic metrics) closed surfaces in the halfsphere S^3_+.

One might think that it is possible to obtain a family of flexible suspensions in the Euclidean space by using the mapping Ω and starting from the family R_t of flexible suspensions in S^3_+ constructed in the proof of the theorem in Section 7. The latter fact would contradict to R. Connelly's theorem [8] according to which a suspension in Euclidean space is not flexible with the exception of few trivial cases.

In fact no contradiction arises here because it is impossible to use Ω to construct a family of flexible suspensions in the Euclidean space. Indeed, for $t_1 \neq t_2$ the mapping Ω assigns a pair of isometric polyhedra $X(0,t_1), Y(0,t_1)$ in Euclidean space to a pair of isometric polyhedra P_0, P_{t_1} in a halfsphere. Similarly, it assigns a pair $X(0,t_2), Y(0,t_2)$ to a pair P_0, P_{t_2}. It is considered to mean that the polyhedron $Y(0,t_i)$ is obtained from the polyhedron $X(0,t_i)$ by flex. However there are no reasons for the "initial" polyhedra $X(0,t_1)$ and $X(0,t_2)$ to be isometric to each other in intrinsic metrics.

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