Proceedings of ICTAMI
International Conference on Theory and Applications in Mathematics and Informatics
17th–20th of September, 2015, Alba-Iulia, Romania
Mathematics Section

pp. 135-140

SOME PROPERTIES OF AN INTEGRAL OPERATOR

D. Wainberg, M. Aldea, L. Căbulea

Abstract. In this paper we will prove, using Pescar's criterion, the univalence of an integral operator, considered for analitic functions in the open unit disk $\mathcal{U}$.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Integral operators, Univalence criterions.

1. Introduction and Preliminaries

Let the unit disk $\mathcal{U} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ containing univalent and regular functions. In 1996, V. Pescar has proved univalent condition:

Theorem 1. [4] *Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ and $c \in \mathbb{C}$ with $|c| < 1$. We consider also a function $f(z)$ of the form (1) which is analytic in $\mathcal{U}$. If:*

$$\left| c|z|^{2\alpha} + (1-|z|)^{2\alpha} \frac{z f_n''(z)}{\alpha f_n'(z)} \right| \leq 1,$$

for every $z \in \mathcal{U}$, then the function $F_\alpha(z)$ defined by:

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t)dt \right)^{\frac{1}{\alpha}} \tag{2}$$

is univalent in $\mathcal{U}$.

In [3] Ozaki and Nunokawa gave the following result:

Theorem 2. *If $f \in \mathcal{A}$ satisfyes the following inequality:*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1, \tag{3}$$

for every $z \in \mathcal{U}$, then f is univalent in $\mathcal{U}$.

Also, an important result that we will use in our paper is General Schwarz Lemma. We remind it here:

Lemma 3. [1] *Let the regular function f in the disk $\mathcal{U}_R = \{z \in \mathbb{C} \mid |z| < R\}$, with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m \ , \quad z \in \mathcal{U}_R. \tag{4}$$

It is obviously that for $R = m = 1$ the relation (4) becomes:

$$\left|\frac{f(z)}{z}\right| \leq M \ , \quad z \in \mathcal{U}. \tag{5}$$

The goal of our paper is to introduce an integral operator, to prove the univalence for it and present some properties obtained from here.

2. Main results

Theorem 4. *Let $f_i \in A$, $i = \overline{1,n}$, the functions that satisfy the inequality (3) α_i, γ, c be complex numbers with $\operatorname{Re}\gamma > 0$ and $M_i, N_i \in \mathbb{R}_+^*$, $M_i \geq 1$.*

If:

i) $|f_i(z)| \leq M_i$, $i = \overline{1,n}$;

ii) $\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq N_i$, $i = \overline{1,n}$;

iii) $|c| \leq 1 - \frac{1}{|\gamma|}\sum\limits_{i=1}^{n}|\alpha_i|\left(2M_i + N_i + 1\right)$,

then the function:

$$G_n(z) = \left(\gamma\int\limits_0^z u^{\gamma-1}\prod_{i=1}^{n}\left(\frac{u}{f_i(u)f_i'(u)}\right)^{\alpha_i}du\right)^{\frac{1}{\gamma}}$$

is univalent in $\mathcal{U}$.

Proof. Let the function g_n, regular in $\mathcal{U}$ and $g_n(0) = g_n'(0) - 1 = 0$, defined as:

$$g_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{u}{f_i(u) f_i'(u)} \right)^{\alpha_i} du.$$

For this function we have:

$$g_n'(z) = \prod_{i=1}^n \left(\frac{z}{f_i(z) f_i'(z)} \right)^{\alpha_i}$$

and:

$$\begin{aligned} g_n''(z) &= \sum_{i=1}^n \left\{ \left[\left(\frac{z}{f_i(z) f_i'(z)} \right)^{\alpha_i} \right]' \cdot \prod_{\substack{j=1 \\ j \neq i}}^n \left(\frac{z}{f_j(z) f_j'(z)} \right)^{\alpha_j} \right\} \\ &= \prod_{i=1}^n \left(\frac{z}{f_i(z) f_i'(z)} \right)^{\alpha_i} \cdot \sum_{i=1}^n \alpha_i \left(\frac{1}{z} - \frac{f_i'(z)}{f_i(z)} - \frac{f_i''(z)}{f_i'(z)} \right). \end{aligned}$$

So we have:

$$\frac{z g_n''(z)}{g_n'(z)} = \sum_{i=1}^n \alpha_i \left(1 - \frac{z f_i'(z)}{f_i(z)} - \frac{z f_i''(z)}{f_i'(z)} \right),$$

hence:

$$\begin{aligned} \left| \frac{z g_n''(z)}{g_n'(z)} \right| &\le \sum_{i=1}^n |\alpha_i| \left(1 + \left| \frac{z f_i'(z)}{f_i(z)} \right| + \left| \frac{z f_i''(z)}{f_i'(z)} \right| \right) \\ &\le \sum_{i=1}^n |\alpha_i| \left(1 + \left| \frac{z^2 f_i'(z)}{f_i^2(z)} \right| \cdot \left| \frac{f_i(z)}{z} \right| + \left| \frac{z f_i''(z)}{f_i'(z)} \right| \right). \end{aligned}$$

Because $|f_i(z)| \le M_i$ and using Schwarz Lemma, we obtain:

$$\begin{aligned} \left| \frac{z g_n''(z)}{g_n'(z)} \right| &\le \sum_{i=1}^n |\alpha_i| \left(1 + \left| \frac{z^2 f_i'(z)}{f_i^2(z)} \right| \cdot M_i + \left| \frac{z f_i''(z)}{f_i'(z)} \right| \right) \\ &\le \sum_{i=1}^n |\alpha_i| \left(1 + \left| \frac{z^2 f_i'(z)}{f_i^2(z)} - 1 \right| \cdot M_i + M_i + \left| \frac{z f_i''(z)}{f_i'(z)} \right| \right). \end{aligned}$$

Applying inequality (3) and ii), we have:

$$\left| \frac{z g_n''(z)}{g_n'(z)} \right| \le \sum_{i=1}^n |\alpha_i| \left(2M_i + N_i + 1 \right).$$

From this relation, we obtain:

$$\left| c|z|^{2\gamma} + (1-|z|)^{2\gamma} \frac{zg_n''(z)}{\gamma g_n'(z)} \right| \leq |c| + \frac{1}{|\gamma|} \sum_{i=1}^{n} |\alpha_i| \left(2M_i + N_i + 1\right).$$

So, because of iii), it results:

$$\left| c|z|^{2\gamma} + (1-|z|)^{2\gamma} \frac{zg_n''(z)}{\gamma g_n'(z)} \right| \leq 1$$

and according with Theorem 1, we obtain that the function G_n is in the class $\mathcal{S}$.

Corollary 5. *Let $f \in \mathcal{A}$ a function that satisfy the inequality (3) α, γ, c be complex numbers with $\operatorname{Re}\gamma > 0$ and $M, N \in \mathbb{R}_+^*, M \geq 1$.*

If:

i) $|f(z)| \leq M$;
ii) $\left| \frac{f''(z)}{f'(z)} \right| \leq N$;
iii) $|c| \leq 1 - \frac{|\alpha|}{|\gamma|} \sum\limits_{i=1}^{n} (2M + N + 1)$, *then the function:*

$$G_n(z) := \left(\gamma \int_0^z u^{\gamma - 1} \left(\frac{u}{f(u) f'(u)} \right)^{\alpha} du \right)^{\frac{1}{\gamma}}$$

is univalent in $\mathcal{U}$.

Proof. We consider $n = 1$ in Theorem 4.

Corollary 6. *Let $f_i \in \mathcal{A}$, $i = \overline{1, n}$, the functions that satisfy the inequality (3) α, γ, c be complex numbers with $\operatorname{Re}\gamma > 0$ and $M_i, N_i \in \mathbb{R}_+^*, M_i \geq 1$.*

If:

i) $|f_i(z)| \leq M_i$, $i = \overline{1, n}$;
ii) $\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq N_i$, $i = \overline{1, n}$;
iii) $|c| \leq 1 - \frac{|\alpha|}{|\gamma|} \sum\limits_{i=1}^{n} (2M_i + N_i + 1)$, *then the function:*

$$G_n(z) := \left(\gamma \int_0^z u^{\gamma - 1} \prod_{i=1}^{n} \left(\frac{u}{f_i(u) f_i'(u)} \right)^{\alpha} du \right)^{\frac{1}{\gamma}}$$

is univalent in $\mathcal{U}$.

Proof. We consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ in Theorem 4.

Corollary 7. *Let $f_i \in A$, $i = \overline{1,n}$, the functions that satisfy the inequality (3) α_i, c be complex numbers and $M_i, N_i \in \mathbb{R}_+^*, M_i \geq 1$.*

If:

i) $|f_i(z)| \leq M_i$, $i = \overline{1,n}$;

ii) $\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq N_i$, $i = \overline{1,n}$;

iii) $|c| \leq 1 - \sum\limits_{i=1}^{n} |\alpha_i| (2M_i + N_i + 1)$,

then the function:

$$G_n(z) := \int_0^z \prod_{i=1}^{n} \left(\frac{u}{f_i(u) f_i'(u)} \right)^{\alpha_i} du$$

is univalent in $\mathcal{U}$.

Proof. In Theorem 4, we consider $\gamma = 1$.

References

[1] T. Ceauşu, N. Suciu, *Funcţii complexe*, Ed. Mirton, Timişoara, (2001).

[2] S. Owa, H. M. Srivastava, *Some generalized convolution properties associated with certain subclasses of analytic functions*, Journal of Inequalities in Pure and Applied Mathematics, 3, 3, article 42 (2002), 1-27.

[3] S. Ozaki, M. Nunokawa, *The Schwartzian derivative and univalent functions*, Proc. Amer. Math. Soc. 33 (1972).

[4] V. Pescar, *A new generalization of Ahlfor's and Becker's criterion of univalence*, Bull. Malayisian Math. Soc. 19 (1996).

[5] N. Ularu, D. Breaz, *Univalence criterion for two integral operators*, Filomat, 25:3, University of Nis, Serbia (2011).

Dorin Wainberg
Department of Mathematics and Informatics,
"1 Decembrie 1918" University of Alba Iulia,
Romania
email: *wainbergdorin@yahoo.com*

Mihaela Aldea
Department of Mathematics and Informatics,

"1 Decembrie 1918" University of Alba Iulia,
Romania
email: *maldea7@yahoo.com*

Lucia Căbulea
Department of Mathematics and Informatics,
"1 Decembrie 1918" University of Alba Iulia,
Romania
email: *cabuleal@uab.ro*