Proceedings of ICTAMI
International Conference on Theory and Applications in Mathematics and Informatics
17^{th}–20^{th} of September, 2015, Alba-Iulia, Romania
Mathematics Section

pp. 121-134

STATIONARY DILATION OF A NONSTATIONARY Γ-CORRELATED PROCESS

I. Valuşescu

Abstract. Γ-correlated processes are analyzed, starting from the properties of a Γ-orthogonal projection and the algebraic embedding into an operator space. In the discrete case, for a periodically Γ-correlated process a stationary dilation Γ_1-correlated process is constructed and the linear predictor of the periodically Γ-correlated process is obtained from the linear predictor of its stationary dilation. Relations between periodicity and harmonizability are analyzed in discrete and continuous parameter cases, and it is proved that the positivity of the angle between the past and the future of a periodically Γ-correlated process is preserved by its stationary dilation Γ_1-correlated process.

2010 *Mathematics Subject Classification*: 47N30, 47A20, 60G25.

Keywords: Complete correlated actions, Γ-correlated processes, projection, periodically Γ-correlated processes, Γ-harmonizable processes, stationary dilation, angle between past and future.

1. Preliminaries

Let $\mathcal{E}$ be a separable Hilbert space. As usually, by $\mathcal{L}(\mathcal{E})$ is denoted the C^*-algebra of all linear bounded operators on a separable Hilbert space $\mathcal{E}$. A Γ-correlated process is a sequence $(f_t)_{t\in G}$ in a right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ endowed with a correlation of the action of $\mathcal{L}(\mathcal{E})$. The set G is $\mathbb{Z}$, $\mathbb{R}$, or more generally a locally compact abelian group.

By an *action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ we mean the map $\mathcal{L}(\mathcal{E})\times\mathcal{H}$ into $\mathcal{H}$ given by $Ah := hA$ in the sense of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$. We are writting Ah instead of hA to respect the classical notations from the scalar case. A *correlation* of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ is a map Γ from $\mathcal{H}\times\mathcal{H}$ into $\mathcal{L}(\mathcal{E})$ having the properties:

(i) $\Gamma[h,h]\geq 0,$ and $\Gamma[h,h]=0$ implies $h=0$;

(ii) $\Gamma[h,g]^*=\Gamma[g,h]$;

(iii) $\Gamma[h,Ag]=\Gamma[h,g]A$.

A triplet $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ defined as above was called [6, 7] a *correlated action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$.

An example of correlated action can be constructed as follows. Take as the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H} = \mathcal{L}(\mathcal{E}, \mathcal{K})$ – the space of the linear bounded operators from $\mathcal{E}$ into $\mathcal{K}$, where $\mathcal{E}$ and $\mathcal{K}$ are Hilbert spaces. An action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$ is given if we consider $AV := VA$ for each $A \in \mathcal{L}(\mathcal{E})$ and $V \in \mathcal{L}(\mathcal{E}, \mathcal{K})$. It is easy to see that $\Gamma[V_1, V_2] = V_1^* V_2$ is a correlation of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$, and the triplet $\{\mathcal{E}, \mathcal{L}(\mathcal{E}, \mathcal{K}), \Gamma\}$ is a correlated action (the *operator model*). It was proved [6] that any abstract correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ can be embedded into the operator model. Namely, there exists an algebraic embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$, where $\mathcal{K}$ is obtained as the Aronsjain reproducing kernel Hilbert space given by a positive definite kernel obtained from the correlation Γ. The generators of $\mathcal{K}$ are elements of the form $\gamma_{(a,h)} : \mathcal{E} \times \mathcal{H} \to \mathbb{C}$, where $\gamma_{(a,h)}(b, g) = \langle \Gamma[g, h]a, b\rangle_{\mathcal{E}}$ and the embedding $h \to V_h$ is given by $V_h a = \gamma_{(a,b)}$.

Due to such an embedding of any correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ into the operator model, prediction problems can be formulated and solved using operator techniques. In the particular case when the embedding $h \to V_h$ is onto, the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ is caled a *complete correlated action*. In this paper most of properties are analysed in the complete correlated case.

In the following the Hilbert space $\mathcal{K}$ uniquely attached to the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ will be called the *measuring space* of the correlated action. The name is justified by the fact that having a state h in the state space $\mathcal{H}$, what we can measure is the element $V_h a$ from the Hilbert space $\mathcal{K}$. In prediction problems we are inerested in measuring the closeness between two states, and this fact is not possible to be directely made in the state space $\mathcal{H}$ which is only a right $\mathcal{L}(\mathcal{E})$-module, but it is possible to be done in the Hilbert space $\mathcal{K}$, and must be interpreted in $\mathcal{H}$. So, we need to have the possibility to "interpret" each element from $\mathcal{K}$ in terms of the state space $\mathcal{H}$.

By the fact that generally in $\mathcal{H}$ we have no topology, the prediction subsets, such as past and present, future, etc., can not be seen as closed subspaces, therefore the powerful tool of the orthogonal projection can not be directly used. However, using the properties of the correlated action, a Γ-orthogonal projection "on" a right $\mathcal{L}(\mathcal{E})$-submodule can be constructed as follows.

Proposition 1.1. Let $\mathcal{H}_1$ be a submodule in the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ and

$$\mathcal{K}_1 = \bigvee_{x \in \mathcal{H}_1} V_x \mathcal{E} \subset \mathcal{K}. \tag{1.1}$$

For each $h \in \mathcal{H}$ there exists a unique element $h_1 \in \mathcal{H}$ such that for each $a \in \mathcal{E}$ we have

$$V_{h_1} a \in \mathcal{K}_1 \qquad \text{and} \qquad V_{h-h_1} a \in \mathcal{K}_1^{\perp}. \tag{1.2}$$

Moreover, we have

$$\Gamma[h-h_1,h-h_1]=\inf_{x\in\mathcal{H}_1}\Gamma[h-x,h-x], \tag{1.3}$$

where the infimum is taken in the set of all positive operators from $\mathcal{L}(\mathcal{E})$.

A complete proof can be found in [6]. This result assure that if we put

$$\mathcal{P}_{\mathcal{H}_1}h=h_1, \tag{1.4}$$

then we can interpret the endomorphism $\mathcal{P}_{\mathcal{H}_1}$ of $\mathcal{H}$ as a Γ-orthogonal projection "on" $\mathcal{H}_1$, since we have $\mathcal{P}_{\mathcal{H}_1}^2=\mathcal{P}_{\mathcal{H}_1}$ and $\Gamma[\mathcal{P}_{\mathcal{H}_1}h,g]=\Gamma[h,\mathcal{P}_{\mathcal{H}_1}g]$.

As a geometrical aspect, let us remark that the unique element h_1 obtained by the Γ-orthogonal projection of $h\in\mathcal{H}$ can belongs not necessary to $\mathcal{H}_1$, but, due to (1.3) it is close enough to be considered as the best estimation.

The previous result can be generalized to $\mathcal{H}^T$ - the cartesian product of T copies of $\mathcal{H}$, especially for the study of periodically correlated processes with the entier period T. To do this, appropriate correlations are constructed in $\mathcal{H}^T$.

Let $T\geq 2$ be a positive integer and

$$\mathcal{H}^T=\mathcal{H}\times\mathcal{H}\times\cdots\times\mathcal{H} \tag{1.5}$$

be the cartesian product of T copies of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$. An element X of $\mathcal{H}^T$ will be seen as a line vector $(h_1,\ldots,h_T)$. On $\mathcal{H}^T$ it is possible to have the action of $\mathcal{L}(\mathcal{E})$ on the components, with the same operator $A\in\mathcal{L}(\mathcal{E})$, or on each component with a different $A_i\in\mathcal{L}(\mathcal{E})$. Also we can consider the action of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$, taking for each matrix $A=(A_{ij})_{i,j=1}^T$from $\mathcal{L}(\mathcal{E})^{T\times T}$

$$A(h_1,\ldots,h_T):=(h_1,\ldots,h_T)A \tag{1.6}$$

in the sense of the right module. It is easy to see that $\mathcal{H}^T$ is an $\mathcal{L}(\mathcal{E})^{T\times T}$-right module and the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}^T$ is a particular case of the action of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$, taking the particular case of diagonal matrices with the same operator, or different operators on the diagonal.

Having the action of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$, various correlations of this action can be constructed. For our purposes we are interested in the following two operatorial correlations on $\mathcal{H}^T$, namely:

$$\Gamma_1[X,Y]=\sum_{k=0}^{T-1}\Gamma[x_k,y_k], \tag{1.7}$$

where $X = (x_0, x_1, \ldots x_{T-1})$, $\quad Y = (y_0, y_1, \ldots, y_{T-1})$, and

$$\Gamma_T[X,Y] = \Big(\Gamma[x_i, y_j]\Big)_{i,j\in\{0,1,\ldots,T-1\}}. \tag{1.8}$$

To make prediction about vector processes from $\mathcal{H}^T$ embedding of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ is necessary, and then an extended "orthogonal projection" on a submodule $\mathcal{M}$ of $\mathcal{H}^T$ with respect to an appropriate correlation.

Proposition 1.2. There exists a unique (up to a unitary equivalence) imbedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ such that

$$\Gamma_1[X,Y] = W_X^* W_Y = \sum_{i=1}^{T} V_{x_i}^* V_{y_i} \tag{1.9}$$

where $X = (x_1, \ldots, x_T)$, $Y = (y_1, \ldots, y_T)$.

The subset $\big\{W_X a; X \in \mathcal{H}^T, a \in \mathcal{E}\big\}$ is dense in $\mathcal{K}^T$.

The construction of a Γ_1-orthogonal projection on a submodule of $\mathcal{H}^T$ is given by the following result [10]

Proposition 1.3. Let $\mathcal{M}$ be a subset of $\mathcal{H}^T$. If we take

$$\mathcal{K}_1^T = \bigvee_{Z\in\mathcal{M}} W_Z \mathcal{E}, \tag{1.10}$$

then for each $X \in \mathcal{H}^T$ there exists a unique element X' in $\mathcal{H}^T$ such that for each $a \in \mathcal{E}$ we have

$$W_{X'} a \in \mathcal{K}_1^T \text{ and } W_{X-X'} a \in (\mathcal{K}_1^T)^{\perp}. \tag{1.11}$$

Moreover,

$$\Gamma_1[X - X', X - X'] = \inf_{Z\in\mathcal{M}} \Gamma_1[X - Z, X - Z], \tag{1.12}$$

where the infimum is taken in the set of all positive operators from $\mathcal{L}(\mathcal{E})$.

2. Γ-correlated processes

A Γ-correlated process $\{f_t\}$ is a sequence in the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$. The process is Γ-*stationary* if $\Gamma[f_s, f_t]$ depends only on $t - s$ and not by s and t separately.

For a Γ-correlated process (not necessary stationary) the *past-present* at the moment $t = n$ is the right $\mathcal{L}(\mathcal{E})$-submodule

$$\mathcal{H}_n^f = \Big\{\sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k \le n\Big\}, \tag{2.1}$$

the *future* is

$$\widetilde{\mathcal{H}}_n^f = \Big\{ \sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k > n \Big\}, \tag{2.2}$$

the *remote past*

$$\mathcal{H}_{-\infty}^f = \bigcap_n \mathcal{H}_n^f; \tag{2.3}$$

and the *time domain* is

$$\mathcal{H}_{\infty}^f = \Big\{ \sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k \in \mathbb{Z} \Big\}.$$

By the embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$, for the corresponding *past*, *remote past*, and the *future* from $\mathcal{H}$ will correspond the closed subspaces of the measuring space $\mathcal{K}$ given by

$$\mathcal{K}_n^f = \bigvee_{j \le n} V_{f_j}\mathcal{E}, \tag{2.4}$$

$$\mathcal{K}_{-\infty}^f = \bigcap_n \mathcal{K}_n^f, \tag{2.5}$$

$$\widetilde{\mathcal{K}}_n^f = \bigvee_{j > n} V_{f_j}\mathcal{E}, \tag{2.6}$$

respectively, and the *time domain* becomes

$$\mathcal{K}_{\infty}^f = \bigvee_{j \le n} V_{f_j}\mathcal{E}.$$

The function $\Gamma_f : \mathbb{Z} \to \mathcal{L}(\mathcal{E})$ given by

$$\Gamma_f(n) = \Gamma[f_0, f_n] \tag{2.7}$$

is called the *correlation function* of the process $\{f_n\}_{n\in\mathbb{Z}}$, and is a completely positive function on $\mathbb{Z}$.

Let $\{f_n\}$ and $\{g_n\}$ be two Γ-stationary processes in $\mathcal{H}$. We say that $\{f_n\}$ and $\{g_n\}$ are *cross-correlated* processes, if $\Gamma[f_n, g_m]$ depends only on the difference $m-n$ and not on m and n separately.

Proposition 2.1. For any stationary processes $\{f_n\}_{n\in\mathbb{Z}}$ in the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ there exists a unitary operator U_f on $\mathcal{K}_\infty^f$ such that

$$V_{f_n} = U_f^n V_{f_0}. \tag{2.8}$$

The Γ-stationary process $\{g_n\}_{n\in\mathbb{Z}}$ is stationary cross-correlated with $\{f_n\}_{n\in\mathbb{Z}}$ if and only if there exists a unitary operator U_{fg} on

$$\mathcal{K}_\infty^{fg} = \mathcal{K}_\infty^f \bigvee \mathcal{K}_\infty^g,$$

such that

$$U_f = U_{fg}|\mathcal{K}_\infty^f \qquad U_g = U_{fg}|\mathcal{K}_\infty^g.$$

A complete proof can be found in [6].

The unitary operator U_f is called the *shift operator* attached to the Γ-stationary process $\{f_n\}$, and U_{fg} is the *extended shift* of of the Γ-stationary cross-correlated processes $\{f_n\}$ and $\{g_n\}$.

In what follows, for a Γ-stationary process $\{f_n\}$ we will write V_f for the operator V_{f_0}, and by (2.8) the time domain can be written as

$$\mathcal{K}_\infty^f = \bigvee_{-\infty}^{\infty} U_f^n V_f \mathcal{E}. \tag{2.9}$$

Now, let us remember some special processes which was necessary in prediction procedure.

The Γ-stationary process $\{g_n\} \in \mathcal{H}$ is a *white noise* process, if for any $n \neq m$

$$\Gamma[g_n, g_m] = 0.$$

The Γ-stationary process $\{f_n\} \in \mathcal{H}$ *contains* the white noise process $\{g_n\}$ from $\mathcal{H}$ if:

$$\begin{cases} (i)\{f_n\}\text{and}\{g_n\}\text{are cross-correlated and}\Gamma[f_n, g_m] = 0 \text{ for } m \geq n; \\ (ii) V_g\mathcal{E} \subset \mathcal{K}_0^f; \\ (iii)\mathrm{Re}\Gamma[f_n - g_n, g_n] \geq 0. \end{cases} \tag{2.10}$$

The Γ-stationary process $\{f_n\} \in \mathcal{H}$ is *deterministic* if and only if does not contain a non-null white noise process.

The Γ-stationary process $\{f_n\} \in \mathcal{H}$ is a *moving average* of a white noise process $\{g_n\}$, if $\{f_n\}$ contains $\{g_n\}$ and the corresponding generated spaces are the same, i.e. $\mathcal{K}_\infty^g = \mathcal{K}_\infty^f$.

Using these facts, and an appropriate Wold decomposition theorem, for a Γ-stationary process from $\mathcal{H}$ the innovation part is determined as the maximal white noise contained in the considered process. Under a suplimentary condition of Harnack type, imposed to the spectral measure F attached to a stationary process, the predictable part and the prediction-error operator can be determined. (For detailles see [6], or [7]).

THEOREM 2.2. Let $\{f_n\}$ be a Γ-stationary process whose spectral distribution F verifies the boundedness condition

$$\frac{1}{2\pi} c \,\mathrm{d}t \leq F \leq \frac{1}{2\pi} c^{-1} \mathrm{d}t, \tag{2.11}$$

$\{\mathcal{E}, \mathcal{E}, \Theta(\lambda)\}$ be the maximal function of $\{f_n\}$ and $\{\mathcal{E}, \mathcal{E}, \Omega(\lambda)\}$ be its inverse. Then the predictible part $\hat{f}_n$ of f_n is given by

$$\hat{f}_n = \sum_{j=0}^{\infty} E_j f_{(n-1)-j}, \tag{2.12}$$

where E_j are given by

$$E_j = \sum_{p=0}^{j} \Omega_{j-p} \Theta_{p+1}, \tag{2.13}$$

and the predicton-error operator $\Delta[f]$ is

$$\Delta[f] = \Theta(0)^* \Theta(0). \tag{2.14}$$

In the remaining of this section, some results about periodically Γ-correlated processes will be presented. Similarly with the stationary case, but with a specific procedure, in [10] a way to obtain the predictable part and the prediction-error was given. To be mentioned that the prediction error which is a linear operator in the stationary case, in the case of a periodically correlated process will be an operator valued periodic function.

A process $\{f_t\}$ is periodically Γ-correlated if there exists a positive T such that $\Gamma[f_{s+T}, f_{t+T}] = \Gamma[f_s, f_t]$.

For a Γ-correlated process $\{f_t\}$, if we take sequences of consecutive T terms

$$X_n = \big(f_n, f_{n+1}, \ldots, f_{n+T-1}\big), \tag{2.15}$$

then $\{X_n\}$ is a stationary Γ_1-correlated process in $\mathcal{H}^T$. Taking consecutive blocks of length T

$$X_n^T = \big(f_{nT}, f_{nT+1}, \ldots, f_{nT+T-1}\big), \tag{2.16}$$

then $\{X_n^T\}$ is a stationary Γ_T-correlated process in $\mathcal{H}^T$.

From prediction point of view and the study of periodically Γ-correlated processes, the following result [10] was proved.

PROPOSITION 2.3. Let $\{f_n\}_{n\in\mathbb{Z}}$ be a Γ-correlated process in $\mathcal{H}$, $T \geq 2$, $\{X_n\}$ and $\{X_n^T\}$ defined by (2.15) and (2.16). The following are equivalent:

(i) $\{f_n\}$ is periodically Γ-correlated in $\mathcal{H}$, with the period T;

(ii) $\{X_n\}$ is stationary Γ_1-correlated in $\mathcal{H}^T$;

(iii) $\{X_n^T\}$ is stationary Γ_T-correlated in $\mathcal{H}^T$.

To obtain the linear predictor and the prediction error, a theorem of Gladyshev was generalized to Γ-correlated case. Using the embedding given by Proposition 1.2 a Γ_T-stationary process$\{\mathbf{Z}_n\}$ in the right $\mathcal{L}(\mathcal{E})^{T\times T}$ -module $\left[\mathcal{L}(\mathcal{E},\mathcal{K})^T\right]^T$ is attached to the periodically Γ-correlated process $\{f_n\}$ from $\mathcal{H}$. Under a Harnack type condition as in (2.11), a Wiener filter for prediction can be obtained. Then using the Γ_1-orthogonal projection given by Proposition 1.3 and the Γ-orthogonal projection given by Proposition 1.1, the following theorem was obtained [10].

Theorem 2.4. If $\{f_n\}_{n\in\mathbb{Z}}$ is a periodically Γ-correlated process, then its predictable part can be found as

$$\hat{f}_{n+1} = \sum_{k=0}^{\infty} C_k f_{n-k}, \quad \text{where} \quad C_k = \sum_{j=0}^{T-1} \tfrac{1}{\sqrt{T}} A_k^{j0} E^{j(n-k)}, \tag{2.17}$$

A_k are the coefficients of the predictable part of the attached stationary Γ_T-correlated process $\{\mathbf{Z}_n\}_{n\in\mathbb{Z}}$, and E is the operator of multiplying by $e^{-2\pi i/T}$.

Also, for the prediction error we have

Theorem 2.5. The prediction error $\Delta(n)$ of a periodically Γ-correlated process $\{f_n\}$ has the form

$$\Delta(n) = \sum_{k=0}^{T-1} D_k E^{-k(n+1)}, \tag{2.18}$$

where the operator coefficients $D_k \in \mathcal{L}(\mathcal{E})$ are the elements from the zero line of the prediction error matrix of the attached stationary process $\{\mathbf{Z}_n\}$, namely

$$D_k = \sum_{s=0}^{T-1} \Theta_{s0}^* \Theta_{sk}, \tag{2.19}$$

where $\Theta_{ij} = \Theta_{ij}(0)$ from the maximal function of the process $\{\mathbf{Z}_n\}$.

3. Stationary dilations

Let $\{f_t\}$ be a nonstationary process in $\mathcal{H}$. If there exists a stationary process $\{g_t\}$ in a larger right $\mathcal{L}(\mathcal{E})$-module $H \supset \mathcal{H}$ such that $f_t = \mathcal{P}_{\mathcal{H}}^H g_t$, then we say that $\{g_t\}$ is a *stationary dilation* of $\{f_n\}$.

Proposition 3.1. If $\{f_n\}$ is a periodically Γ-correlated process in $\mathcal{H}$, then the stationary Γ_1-correlated process $\{X_n\}$ in $\mathcal{H}^T$ given by (2.15) is a stationary dilation of $\{f_n\}$. Moreover, the predictable part of $\{f_n\}$ can be obtained from the predictable part of $\{X_n\}$ using the Γ_1-orthogonal projection $\mathcal{P}$ from $\mathcal{H}^T$ to $\mathcal{H}$, i.e. $\hat{f}_n = \mathcal{P}\hat{X}_n$.

Proof. By Proposition 2.3, the process $\{X_n\}$ attached to $\{f_n\}$ by (2.15) is a stationary Γ_1-correlated process. If we take the submodule of $\mathcal{H}^T$ given by $\mathcal{N} = \mathcal{H} \times \{0\} \times \cdots \times \{0\}$, then by the algebraic embedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$, to $\mathcal{N}$ it corresponds the subspace

$$\mathcal{K}_{\mathcal{N}} = \bigvee_{Z \in \mathcal{N}} W_Z \mathcal{E} = \mathcal{K} \times \{0\} \times \cdots \times \{0\} \subset \mathcal{K}^T,$$

and by the Γ_1-orthogonal projection from $\mathcal{H}^T$ to the submodule $\mathcal{N}$ we have

$$P_{\mathcal{K}_{\mathcal{N}}} W_{X_n} = P_{\mathcal{K}_{\mathcal{N}}}(V_{f_n}, V_{f_{n+1}}, \ldots, V_{f_{n+T-1}}) = (V_{f_n}, 0, \ldots, 0).$$

Therefore, by the natural identification of $\mathcal{H}$ with $\mathcal{N}$ we have that $\mathcal{P}X_n = f_n$, and it follows that $\{X_n\}$ is a stationary Γ_1-correlated dilation of $\{f_n\}$.

Similarly with (2.1), (2.2), (2.3), the past-present H_n^X, the future $\tilde{H}_n^X$ at the moment $t = n$ and the remote space $H_{-\infty}^X$ of the process $\{X_n\}$ as submodules in $\mathcal{H}^T$ can be constructed.

Let $\hat{X}_n$ be the predictable part of X_n, i.e. the best estimation of X_n in the sense that the prediction error is minimal

$$\Gamma_1[X_n - \hat{X}_n, X_n - \hat{X}_n] = \inf_{Z \in H_{n-1}^X} \Gamma_1[X_n - Z, X_n - Z].$$

Taking account by the Wold decomposition of a stationary process into its deterministic and purely nondeterministic parts, it is sufficient to consider the stationary Γ_1-correlated process $\{X_n\}$ to be purely nondeterministic and we have $\mathcal{P}H_{n-1}^X = \mathcal{H}_{n-1}^f$. Also

$$\Gamma_1[\mathcal{P}X_n - \mathcal{P}\hat{X}_n, \mathcal{P}X_n - \mathcal{P}\hat{X}_n] = \inf_{Z \in H_{n-1}^X} \Gamma_1[\mathcal{P}X_n - \mathcal{P}Z, \mathcal{P}X_n - \mathcal{P}Z],$$

or equivalently

$$\Gamma_1[f_n - \mathcal{P}\hat{X}_n, f_n - \mathcal{P}\hat{X}_n] = \inf_{Y \in \mathcal{H}_{n-1}^f} \Gamma[f_n - Y, f_n - Y],$$

and it follows that $\mathcal{P}\hat{X}_n = \hat{f}_n$.

Here we used the fact that the predictable part is the innovation process $\{\hat{X}_n\}$, i.e. the maximal white noise contained in $\{X_n\}$.

As a remark, to find the predictor for a periodically Γ-correlated process $\{f_n\}$ from $\mathcal{H}$, it is sufficient to know the predictor for its stationary dilation process. There exist methods to obtain such a predictor in correlated, or complete correlated cases. One of this was shortly presented in the previous section, where, based on the generalized Lowdenslager–Sz.-Nagy–Foias factorization theorem [5], the linear Wiener filter for prediction is given by the operator coefficients of the attached maximal function of the process. This can lead to a concrete expression of the predictor and prediction error corresponding to a periodically Γ-correlated process $\{f_n\}$ from $\mathcal{H}$.

Another class of nonstationary processes which can admit a stationary dilation is the class of harmonizable processes. A process $\{f_t\}_{t\in\mathbb{R}}$ is *strongly* Γ-*harmonizable* if the *correlation function* $\Gamma_f(s,t)$ can be expressed as

$$\Gamma_f(s,t) = \iint_{\mathbb{R}^2} \mathrm{e}^{i(su-tv)} K(\mathrm{d}u, \mathrm{d}v) \qquad (s,t\in\mathbb{R}), \tag{3.1}$$

for some positive definite $\mathcal{L}(\mathcal{E})$-valued semispectral bimeasure K of bounded variation.

A process $\{f_t\}_{t\in\mathbb{R}}$ is *weakly* Γ-*harmonizable* if its correlation function can be expressed in the form (3.1) for some $\mathcal{L}(\mathcal{E})$-valued semispectral bimeasure K of finite variation.

It is well known that in the discrete case ($G = \mathbb{Z}$) each periodically Γ-correlated process with the period T is Γ-harmonizable and the support of the $\mathcal{L}(\mathcal{E})$-valued semispectral bimeasure attached to a discrete periodically Γ-correlated process with the period $T \geq 1$ is concentrated on $2T-1$ equidistant stright line segments $v = u - 2k\pi k/T,\ k \in \{0, \pm 1, \ldots. \pm (T-1)\}$ parallel to the diagonal of the square $[0,2\pi]\times[0,2\pi]$. Obvious if $T = 1$ then the Γ-harmonizable process $\{f_n\}_{n\in\mathbb{Z}}$ is stationary Γ-correlated and the support is concentrated only on the diagonal of the square.

In the continuous parameter case, this nice property is no longer valid even in the scalar case. Only on supplementary conditions, some particular periodically Γ-correlated processes with continuous time will become Γ-harmonizable, and similarly, the support of the bimeasure will be on parallel equidistant stright lines in the plane. Concerning the stationary dilatability of harmonizable processes, even if there exist continuous time weakly harmonizable processes with stationary dilations, there exist operator weakly harmonizable processes which does not have operator stationary dilations.

Another tool in prediction theory is the notion of the angle between the past and the future of a process. Actually the notion of the angle between two subspaces of a Hilbert space arise in [2], starting from the general definition of the scalar product of two vectors into the form $\langle h, g\rangle = \|h\|\,\|g\| \cdot \cos\alpha$. The *angle* (sometimes called the Dixmier angle) between two subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert space $\mathcal{K}$ is given by its cosine

$$\rho(\mathcal{M},\mathcal{N}) := \sup\big\{\, |\langle h, g\rangle|\,;\ h\in\mathcal{M}\cap B_{\mathcal{K}}, g\in\mathcal{N}\cap B_{\mathcal{K}}\big\}. \tag{3.2}$$

where $B_{\mathcal{K}}$ is the unit ball of $\mathcal{K}$.

In the context of a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ the cosine between the submodules $\mathcal{M}$ and $\mathcal{N}$ of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ is given by

$$\rho(\mathcal{M}, \mathcal{N}) = \sup\left\{ |\langle \Gamma[g,h]a, b\rangle| \,;\, \|\Gamma[h,h]a\| \leq 1, \|\Gamma[g,g]b\| \leq 1 \right\},$$

where $h \in \mathcal{M}, g \in \mathcal{N},\ a, b \in \mathcal{E}$.

We say that $\mathcal{M}$ and $\mathcal{N}$ have a *positive angle* if $\rho(\mathcal{M}, \mathcal{N}) < 1$, or equivalently, if there exists $\rho < 1$ such that for any $h \in \mathcal{M}, g \in \mathcal{N},\ a, b \in \mathcal{E}$

$$|\langle \Gamma[g,h]a, b\rangle_{\mathcal{E}}| \leq \rho\, \|V_h a\|\, \|V_g b\| \,. \tag{3.3}$$

In the study of prediction problems we are interested in the case when the angle between past and future is positive, i.e., when $\rho(n) = \rho(\mathcal{H}_n^f, \widetilde{\mathcal{H}}_n^f) < 1$, which will give the possibility of finding the predictor. A nice geometrical aspect of stationary Γ-correlated process is the fact that the angle between the past and future not depends on the choosing of the fixed time $t = n$.

In this paper only the one step ahead future is considered (2.2), but analogously the p-step ahead future can be constructed as

$$\widetilde{\mathcal{H}}_{n,p}^f = \Big\{ \sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k \geq n+p \Big\},$$

the corresponding subspace from the time domain $\mathcal{K}_\infty^f \subset \mathcal{K}$ being

$$\widetilde{\mathcal{K}}_{n,p}^f = \bigvee_{j \geq n+p} V_{f_j}\mathcal{E},$$

and the p-step prediction is done using informations from the past $\mathcal{H}_n^f$, obtained with the action of $\mathcal{L}(\mathcal{E})$ on (f_t) from $\mathcal{H}$ till the moment $t = n$.

Similarly the angle $\rho(n,p)$ between the past $\mathcal{H}_n^f$ and the p-step ahead future $\widetilde{\mathcal{H}}_{n,p}^f$ can be considered and the fact that $\rho(n) < 1$ is equivalent with $\rho(n,p) < 1$ can be proved, giving the possibility to find the p-step ahead predictor.

We have seen that each periodically Γ-correlated process $\{f_n\}_{n\in\mathbb{Z}}$ from $\mathcal{H}$ has a stationary Γ_1-correlated dilation $\{X_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}^T$. The positivity angle property of a periodically Γ-correlated process is preserved by its stationary dilation.

Proposition 3.2. If $\{f_n\}$ from $\mathcal{H}$ is a periodically Γ-correlated process with a positive angle between its past and future, then the angle between the past and the future of its stationary Γ_1-correlated dilation $\{X_n\}$ from $\mathcal{H}^T$ it is also positive.

Proof. Analogously as in (2.1) and (2.2), in $\mathcal{H}^T$ the past H_n^X and the future $\tilde{H}_n^X$ for a process $\{X_n\} \subset \mathcal{H}^T$ is constructed as linear combinations of finite actions of $\mathcal{L}(\mathcal{E})$ on $(X_n) \subset \mathcal{H}^T$. If $\{f_n\}$ from $\mathcal{H}$ is a periodically Γ-correlated process having a positive angle between its past and future, then at each time $t = n$ there exists $\rho(n) < 1$ such that

$$|\langle \Gamma[g,h]a,b\rangle_{\mathcal{E}}| \leq \rho(n)\,\|V_h a\|\,\|V_g b\|$$

for each $h \in \mathcal{H}_n^f$ and $g \in \tilde{\mathcal{H}}_n^f$. For each element $X = \sum\limits_{k\leq n} A_k X_k$ from the past H_n^X and $Y = \sum\limits_{p>n} B_p X_p$ from the future $\tilde{H}_n^X$ of the Γ_1-correlated process $\{X_n\}$ given by (2.15), and for any $a, b \in \mathcal{E}$ we have

$$|\langle \Gamma_1[X,Y]a,b\rangle_{\mathcal{E}}| = \left|\left\langle \Gamma_1\Big[\sum_{p>n} B_p X_p, \sum_{k\leq n} A_k X_k\Big]a,b\right\rangle_{\mathcal{E}}\right| =$$

$$= \left|\sum_{p>n}\sum_{k\leq n}\langle \Gamma_1[B_p X_p, A_k X_k]a,b\rangle_{\mathcal{E}}\right| =$$

$$= \left|\sum_{p>n}\sum_{k\leq n}\sum_{i=0}^{T-1}\langle \Gamma[B_p f_{p+i}, A_k f_{k+i}]a,b\rangle_{\mathcal{E}}\right| =$$

$$= \left|\sum_{p>n}\sum_{k\leq n}\sum_{i=0}^{T-1}\left\langle B_p^*\Gamma[f_{p+i}, f_{k+i}]A_k a,b\right\rangle_{\mathcal{E}}\right| =$$

$$= \left|\sum_{i=0}^{T-1}\left\langle \Gamma\Big[\sum_{p>n} B_p f_{p+i}, \sum_{k\leq n} A_k f_{k+i}\Big]a,b\right\rangle_{\mathcal{E}}\right| \leq$$

$$\leq \sum_{i=0}^{T-1}\rho_i(n)\left\|\sum_{k\leq n} A_k f_{k+i}a\right\|\left\|\sum_{p>n} B_p f_{p+i}b\right\| \leq$$

$$\leq \rho(n)\sum_{i=0}^{T-1}\left\|\sum_{k\leq n} A_k f_{k+i}a\right\|\left\|\sum_{p>n} B_p f_{p+i}b\right\| \leq$$

$$\leq \rho(n)\Big(\sum_{i=0}^{T-1}\left\|\sum_{k\leq n} A_k f_{k+i}a\right\|^2\Big)^{1/2}\Big(\sum_{i=0}^{T-1}\left\|\sum_{p>n} B_p f_{p+i}b\right\|^2\Big)^{1/2} =$$

$$= \rho \left\| \sum_{k \leq n} A_k W_{X_k} a \right\| \left\| \sum_{p > n} B_p W_{X_p} b \right\| = \rho \left\| W_X a \right\| \left\| W_Y b \right\|,$$

where $\rho(n)$ is the maximum of $\rho_i(n) < 1$; $i = 0, 1, \ldots, T-1$, and we used the embedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ and the fact that $\rho(n) = \rho$ for stationary Γ_1-correlated proces $\{X_n\}$. Therefore $|\langle \Gamma_1[X,Y]a, b\rangle_{\mathcal{E}}| \leq \rho \|W_X a\| \|W_Y b\|$ for each $X \in H_n^X$, $Y \in \tilde{H}_n^X$, and the angle between the past and the future of the stationary Γ_1-correlated dilation $\{X_n\}$ is positive.

Acknowledgements. Supported by UEFISCDI Grant PN-II-ID-PCE-2011-3-0119

References

[1] F. Deutsch, *The angle between subspaces of a Hilbert space*, Approximation theory, wavelets and applications (Maratea, 1994), 10730, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 454, Kluwer Acad. Publ., Dordrecht, 1995.

[2] J. Dixmier, *Étude sur les variétés et les opérateurs de Julia avec quelques applications*. Bull. Soc. Math. France, **77**(1949), 11-101.

[3] H. Helson and G. Szegő, *A problem in prediction theory*. Ann. Mat. Pura. Appl. **51** (1960), 107-138.

[4] H. Helson and D. Sarason, *Past and future*. Math. Scand. **21**(1967), 5-16.

[5] I. Suciu and I. Valusescu, *Factorization of semispectral measures*, Rev. Roumaine Math. Pures et Appl., **21**, *6*(1976), 773-793.

[6] I. Suciu and I. Valusescu, *Factorization theorems and prediction theory*. Rev. Roumaine Math. Pures et Appl. **23**, *9*(1978), 1393-1423.

[7] I. Suciu and I. Valusescu, *A linear filtering problem in complete correlated actions*. Journal of Multivariate Analysis, **9**, *4*(1979), 559-613.

[8] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, Acad Kiadó, Budapest, North Holland Co., 1970.

[9] D. Timotin, *Prediction theory and choice sequences: an alternate approach*. Advances in invariant subspaces and other results of operator theory, Birkhăuser Verlag, Basel, 1986, 341-352.

[10] I. Valusescu, *A linear filter for the operatorial prediction of a periodically correlated process*. Rev. Roumaine Math. Pures et Appl. **54**, *1*(2009), 53-67.

[11] I. Valusescu, Notes on continuous parameter periodically Γ-correlated processes, An. Univ. Timişoara Vol. **50** nr.1 (2012), pag. 115 – 125.

[12] I. Valusescu, Some geometrical aspects of the Γ-correlated processes. Analele Universitatii Oradea, Fasc. Matematica, Tom XXI (2014), Issue No.2, 113-123.

Ilie Valuşescu
Institute of Mathematics "Simion Stoilow" of the Romanian Academy,
Calea Griviţei nr. 21, Bucharest, Romania. Research unit nr.1
email: *Ilie.Valusescu@imar.ro*