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LYAPUNOV FUNCTIONS FOR TRICHOTOMY WITH DIFFERENT GROWTH RATES

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Abstract. The paper considers a general property of trichotomy with different growth rates for dynamical systems in Banach spaces. The main aim is to give necessary and sufficient conditions of this property in terms of Lyapunov functions.

As particular cases, criteria for exponential and respectively polynomial trichotomy are obtained.

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1. Introduction and Basic Concepts

The study of the asymptotic properties for dynamical systems has a long and impressive history. Of the most important contributions from the stability theory, we emphasize the results of E. A. Barbashin ([5]), R. Datko ([13]), A. M. Lyapunov ([21]) and O. Perron ([32]), that represented important directions for the development of the area.

The approach of the stability property for differential equations and difference equations was extended to various concepts of dichotomy, treated in [1], [10], [12], [15], [26], [33], [40] by different methods.

The property of trichotomy comes naturally, as a generalization of the dichotomy and is studied in [14], [28]. The classical issues of exponential and polynomial trichotomy approached in [19], [23], [29], [39] was extended to general forms (see for instance [18], [20], [35]).

In what follows we recall the main concepts of dichotomy and trichotomy studied for linear discrete-time systems.

Let X be a real or complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. We define the set

$$\Delta = \{(m,n) \in \mathbb{N}^2 : m \geq n\},$$

where $\mathbb{N}$ is the set of natural numbers.
The norms on X and on $\mathcal{B}(X)$ will be denoted by $||\cdot||$ and I represents the identity operator on X.
We consider the linear discrete-time system

$$(\mathcal{A}) \qquad x_{n+1} = A_n x_n, \quad n \in \mathbb{N},$$

where $A : \mathbb{N} \to \mathcal{B}(X),\ A(n) = A_n$.

Every solution of $(\mathcal{A})$ is defined by

$$x_m = A_m^n x_n,$$

for all $(m, n) \in \Delta$, where

$$A_m^n := \begin{cases} A_{m-1}...A_n, & \text{if } m > n \\ \\ I, & \text{if } m = n. \end{cases}$$

It results that

$$A_m^n A_n^p = A_m^p, \quad for\ all \quad (m,n),(n,p) \in \Delta.$$

Definition 1. *A sequence $P : \mathbb{N} \to \mathcal{B}(X),\ P(n) = P_n$ is called*

(i) projections sequence on X if $P_n^2 = P_n, \quad for\ all \quad n \in \mathbb{N}$;

(ii) invariant for $(\mathcal{A})$ if $A_n P_n = P_{n+1} A_n, \quad for\ all \quad n \in \mathbb{N}$.

Remark 1. *If $(P_n^1)_n$ is invariant for $(\mathcal{A})$ then $P_n^2 = I - P_n^1$ is also a projections sequence invariant for $(\mathcal{A})$ (called the complementary of $(P_n^1)_n$).*

Remark 2. *It is easy to see that $(P_n^1)_n$ is invariant for $(\mathcal{A})$ if and only if*

$$A_m^n P_n^1 = P_m^1 A_m^n, \quad for\ all \quad (m,n) \in \Delta.$$

Definition 2. *Three projections sequence $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ are said to be supplementary if*

(s_1) $P_n^1 + P_n^2 + P_n^3 = I, \quad for\ all \quad n \in \mathbb{N}$;

(s_2) $P_n^i P_n^j = 0, \quad for\ all \quad i \neq j,\ i,j \in \{1,2,3\}$.

Definition 3. *An increasing sequence $h : \mathbb{N} \to [1, +\infty),\ h(n) = h_n$ is called a growth rate if $\lim\limits_{n\to\infty} h_n = +\infty$.*

An important concept of dichotomy that generalizes the exponential and the polynomial dichotomy is treated in [2], [11] for the discrete case and in [17], [22] for the continuous case. Also, we mention in this context the contributions from [3] and [4].
Let $h, k : \mathbb{N} \to [1, +\infty)$ two growth rates and $P = (P_n^1)_n$ a projections sequence invariant for $(\mathcal{A})$, $P_n^2 = I - P_n^1$.

Definition 4. *The pair* $(\mathcal{A}, P)$ *is called (h,k)-dichotomic if there exists a nondecreasing sequence* $(s_n)_n,\ s_n \geq 1$ *with*

$$h_m||A_m^n P_n^1 x|| + k_m||P_n^2 x|| \leq h_n s_n||P_n^1 x|| + k_n s_m||A_m^n P_n^2 x||,$$

for all $(m, n, x) \in \Delta \times X$.

As particular cases of (h,k)-dichotomy we have

(i) if $(s_n)_n$ is a constant sequence, then we have the property of *uniform (h,k)-dichotomy*;

(ii) if $h_n = e^{n\alpha},\ k_n = e^{n\beta},\ \alpha, \beta > 0$ we recover the notion of *(nonuniform) exponential dichotomy* and if $(s_n)_n$ is constant it results the property of *uniform exponential dichotomy*;

(iii) if $h_n = (n+1)^{\alpha},\ k_n = (n+1)^{\beta},\ \alpha, \beta > 1$ we obtain the property of *(nonuniform) polynomial dichotomy* and if $(s_n)_n$ is constant it results the notion of *uniform polynomial dichotomy;*

(iv) for $P_n^2 = 0$ for all $n \in \mathbb{N}$, we have the *h-stability*.

We present a general example of a pair $(\mathcal{A}, P)$ that is (h,k)-dichotomic.

Example 1. *Let* $X = \mathbb{R}^2$ *with the norm* $||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$.
Also, $(h_n)_n,\ (k_n)_n$ *are growth rates and* $(s_n)_n$ *is a nondecreasing sequence.*
We consider the canonical projections sequence P_n^1, P_n^2 *and the linear discrete-time system* $(\mathcal{A})$ *is given by*

$$A_n = \frac{h_n}{h_{n+1}} P_n^1 + \frac{k_{n+1}}{k_n} P_{n+1}^2, \quad n \in \mathbb{N}.$$

Thus,

$$A_m^n = \frac{h_n}{h_m} P_n^1 + \frac{k_m}{k_n} P_m^2, \quad for\ all\ (m.n) \in \Delta.$$

A simple verification shows that the pair $(\mathcal{A}, P)$ *is* (h, k)*-dichotomic.*

Concerning this direction of study, we observe a large number of papers where are obtained results of Datko type ([37], [43]), Lyapunov type ([8], [9], [16], [24], [25], [36], [38]) and of Perron type ([27], [41]).

In this paper, we focus on the following concept of trichotomy

Let $h, k, \mu : \mathbb{N} \to [1, +\infty)$ be growth rates and $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ three projections sequence invariant for $(\mathcal{A})$.

Definition 5. *We say that pair $(\mathcal{A}, \mathcal{P})$ is (h, k, μ)-trichotomic if there exists a nondecreasing sequence $(s_n)_n,\ s_n \geq 1,$ with*

(t_1)
$$h_m||A_m^n P_n^1 x|| + k_m||P_n^2 x|| \leq h_n s_n||P_n^1 x|| + k_n s_m||A_m^n P_n^2 x||;$$

(t_2)
$$\frac{1}{s_m}\frac{\mu_n}{\mu_m}||P_n^3 x|| \leq ||A_m^n P_n^3 x|| \leq \frac{\mu_m}{\mu_n} s_n||A_m^n P_n^3 x||,$$

for all $(m, n, x) \in \Delta \times X$.

As particular cases, we recall the following:

(i) if $(s_n)_n$ is a constant sequence, then we obtain *uniform (h, k, μ)-trichotomy*;

(ii) if $h_n = e^{n\alpha},\ k_n = e^{n\beta},\ \mu_n = e^{n\gamma},\ \alpha, \beta, \gamma > 0$ we have the notion of *(nonuniform) exponential trichotomy* and if $(s_n)_n$ is constant it results the concept of *uniform exponential trichotomy*;

(iii) if $h_n = (n+1)^\alpha,\ k_n = (n+1)^\beta,\ \mu_n = (n+1)^\gamma,\ \alpha, \beta, \gamma > 1$ it results the notion of *(nonuniform) polynomial trichotomy* and if $(s_n)_n$ is constant we have a *uniform polynomial trichotomy;*

(iv) for $P_n^3 = 0$ for all $n \in \mathbb{N}$, we recover the notion of *(h,k)-dichotomy.*

Example 2. *On the Banach space $X = \mathbb{R}^3$ with the norm*

$$||(x_1, x_2, x_3)|| = \max\{|x_1|, |x_2|, |x_3|\},$$

we consider the linear discrete-time system

$$(\mathcal{A})\begin{cases} x_{n+1}^1 &= \dfrac{h_n}{h_{n+1}} e^{\tau_n - \tau_{n+1}} x_n^1 \\ x_{n+1}^2 &= \dfrac{k_{n+1}}{k_n} e^{\tau_{n+1} - \tau_n} x_n^2 \\ x_{n+1}^3 &= \dfrac{\mu_{n+1}}{\mu_n} e^{\tau_n - \tau_{n+1}} x_n^3, \end{cases}$$

with $(h_n)_n,\ (k_n)_n,\ (\mu_n)_n$ arbitrary growth rates and $\tau_n = (n+1)\cos\ln(n+1),\ n \in \mathbb{N}$. We have that

$$A_m^n = \frac{h_n}{h_m} e^{\tau_n - \tau_m} P_n^1 + \frac{k_m}{k_n} e^{\tau_m - \tau_n} P_m^2 + \frac{\mu_m}{\mu_n} e^{\tau_n - \tau_m} P_n^3,$$

for all $(m,n) \in \Delta$, where $P_n^1,\ P_n^2,\ P_n^3$ are the canonical projections.
It it easy to check that the pair $(\mathcal{A}, \mathcal{P})$ is (h,k,μ)-trichotomic, with $s_n = e^{n+1},\ n \in \mathbb{N}$.

On this direction we mention the papers [6], [7], [31] (with results of Lyapunov type), [19], [30], [42] (with results of Datko type) and [39] (with results of Perron type).

In the last period we remark a particular interest for the different notions of nonuniform dichotomy and trichotomy for the continuous case and also for the discrete case (see [34], [44]-[46]).

In this paper, we approach the property of (h,k,μ)-trichotomy, introduced in Definition 5 for linear discrete-time systems in Banach spaces as a generalization of the (h,k)-dichotomy. The main aim is to obtain necessary and sufficient conditions of Datko and Lyapunov type for this notion.

Also, as consequences, are illustrated the results for the nonuniform exponential trichotomy and nonuniform polynomial trichotomy.

2. Necessary conditions for (h,k,μ)-trichotomy

We denote by $\mathcal{H}$ the set of the growth rates $h : \mathbb{N} \to [1,+\infty)$ with the property that there exists $(\varphi_n)_n \in \mathcal{H}$ and a constant $B > 1$ such that

$$\sum_{j=0}^{+\infty} \frac{\varphi_j}{h_j} \leq B.$$

Also, $\mathcal{K}$ represents the set of the growth rates $k : \mathbb{N} \to [1,+\infty)$ with the property that there exists $(\psi_n)_n \in \mathcal{K},\ (r_n)_n \in \mathcal{K}$ and a constant $B > 1$ with

$$\sum_{j=0}^{+\infty} \frac{\psi_j}{k_j} \leq B$$

and

$$\sum_{j=n}^{m} \frac{k_j}{\psi_j} \leq B r_m \frac{k_m}{\psi_m}, \quad \text{for all} \quad (m,n) \in \Delta.$$

Remark 3. *Considering $\mathcal{E}$ the set of all sequences $h, k : \mathbb{N} \to [1, +\infty)$, $h_n = e^{n\alpha}$, $k_n = e^{n\beta}$ with $\alpha, \beta > 0$, it is immediate that $\mathcal{E} \subset \mathcal{H}$ and $\mathcal{E} \subset \mathcal{K}$.*

Indeed, let $h_n = e^{n\alpha}$, $k_n = e^{n\beta}$ with $\alpha, \beta > 0$. Then $\varphi_n = e^{n\gamma}$, $\psi_n = e^{n\delta}$, with $\gamma \in (0, \alpha)$, $\delta \in (0, \beta)$ and $r_n = e^{n\varepsilon}$, $\varepsilon > 0$. We obtain

$$\sum_{j=0}^{+\infty} \frac{\varphi_j}{h_j} = \sum_{j=0}^{+\infty} e^{j(\gamma-\alpha)} = \frac{e^{\alpha}}{e^{\alpha} - e^{\gamma}} \leq B;$$

$$\sum_{j=0}^{+\infty} \frac{\psi_j}{k_j} = \sum_{j=0}^{+\infty} e^{j(\delta-\beta)} = \frac{e^{\beta}}{e^{\beta} - e^{\delta}} \leq B,$$

where $B = \max\left\{\frac{e^{\alpha}}{e^{\alpha} - e^{\gamma}}, \frac{e^{\beta}}{e^{\beta} - e^{\delta}}\right\}$;

$$\sum_{j=n}^{m} \frac{k_j}{\psi_j} = \sum_{j=n}^{m} e^{j(\beta-\delta)} \leq \frac{e^{(\beta-\delta)(m+1)} - e^{n(\beta-\delta)}}{e^{\beta-\delta} - 1} \leq$$

$$\leq \frac{e^{\beta}}{e^{\beta} - e^{\delta}} e^{(\beta-\delta)m} \leq B r_m \frac{k_m}{\psi_m},$$

for all $(m, n) \in \Delta$.

Remark 4. *We denote by $\mathcal{P}$ the set of all sequences $h, k : \mathbb{N} \to [1, +\infty)$, $h_n = (n+1)^{\alpha}$, $k_n = (n+1)^{\beta}$, $\alpha, \beta > 1$. It results that $\mathcal{P} \subset \mathcal{H}$ and $\mathcal{P} \subset \mathcal{K}$.*

Similar with the previous remark, let $h_n = (n+1)^{\alpha}$, $k_n = (n+1)^{\beta}$, with $\alpha, \beta > 1$. Then $\varphi_n = (n+1)^{\gamma-1}, \gamma \in (1, \alpha)$, $\psi_n = (n+1)^{\delta-1}, \delta \in (1, \beta)$, $r_n = (n+1)^{\varepsilon}$, $\varepsilon > 1$. Thus,

$$\sum_{j=0}^{+\infty} \frac{\varphi_j}{h_j} = \sum_{j=0}^{+\infty} \frac{1}{(j+1)^{\alpha-\gamma+1}} \leq B;$$

$$\sum_{j=0}^{+\infty} \frac{\psi_j}{k_j} = \sum_{j=0}^{+\infty} \frac{1}{(j+1)^{\beta-\delta+1}} \leq B,$$

where $B = \max\left\{\sum\limits_{j=0}^{+\infty} \frac{1}{(j+1)^{\alpha-\gamma+1}}, \sum\limits_{j=0}^{+\infty} \frac{1}{(j+1)^{\beta-\delta+1}}\right\}$;

$$\sum_{j=n}^{m} \frac{k_j}{\psi_j} = \sum_{j=n}^{m} (j+1)^{\beta-\delta+1} \leq (m-n+1)(m+1)^{\beta-\delta+1} \leq B r_m \frac{k_m}{\psi_m},$$

for all $(m,n)\in\Delta$.
In the following, we consider a pair $(\mathcal{A},\mathcal{P})$, where $\mathcal{P}=\{P_n^1,P_n^2,P_n^3\}$ are invariant projections sequence for $(\mathcal{A})$.
We illustrate a necessary condition for (h,k,μ)-trichotomy

Theorem 1. *If $(\mathcal{A},\mathcal{P})$ is (h,k,μ)-trichotomic and $(h_n)_n\in\mathcal{H}$, $(k_n)_n\in\mathcal{K}$ then there exists a growth rate $(\nu_n)_n$ and a nondecreasing sequence $(d_n)_n$, $d_n\geq 1$ with the properties:*

$$(D_1)\quad \sum_{j=n}^{+\infty}\varphi_j||A_j^nP_n^1x||+\sum_{j=n}^{m}\frac{||A_j^nP_n^2x||}{\psi_j}\leq$$

$$\leq\varphi_nd_n||P_n^1x||+\frac{d_m}{\psi_m}||A_m^nP_n^2x||,\quad \textit{for all}\quad (m,n,x)\in\Delta\times X;$$

$$(D_2)\quad \sum_{j=n}^{+\infty}\frac{1}{\nu_j}||A_j^nP_n^3x||\leq\frac{d_n}{\nu_n}||P_n^3x||,\quad \textit{for all}\quad (n,x)\in\mathbb{N}\times X;$$

$$(D_3)\quad \sum_{j=n}^{m}\nu_j||A_j^nP_n^3x||\leq\nu_md_m||A_m^nP_n^3x||,\quad \textit{for all}\quad (m,n,x)\in\Delta\times X.$$

Proof. Let $d_n=Bs_n\max\{h_n,k_n,r_n\}$ and $\nu_n=\dfrac{\mu_nk_n}{\psi_n}$. Then,
(D_1)

$$\sum_{j=n}^{+\infty}\varphi_j||A_j^nP_n^1x||+\sum_{j=n}^{m}\frac{||A_j^nP_n^2x||}{\psi_j}\leq$$

$$\leq\sum_{j=n}^{+\infty}\frac{\varphi_j}{h_j}h_ns_n||P_n^1x||+\sum_{j=n}^{m}\frac{||P_j^2A_j^nx||}{\psi_j}\leq$$

$$\leq Bh_ns_n||P_n^1x||+\sum_{j=n}^{m}\frac{k_j}{\psi_j}\frac{s_m}{k_m}||A_m^jA_j^nP_n^2x||\leq$$

$$\leq Bh_ns_n||P_n^1x||+Br_m\frac{s_m}{\psi_m}||A_m^nP_n^2x||\leq$$

$$\leq\varphi_nd_n||P_n^1x||+\frac{d_m}{\psi_m}||A_m^nP_n^2x||,\quad \text{for all}\quad (m,n,x)\in\Delta\times X;$$

(D_2)

$$\sum_{j=n}^{+\infty}\frac{1}{\nu_j}||A_j^nP_n^3x||\leq\sum_{j=n}^{+\infty}\frac{\mu_j}{\nu_j}\frac{s_n}{\mu_n}||P_n^3x||=$$

$$= \frac{s_n}{\mu_n} \sum_{j=n}^{+\infty} \frac{\psi_j}{k_j} ||P_n^3 x|| \leq B \frac{s_n}{\mu_n} ||P_n^3 x|| \leq \frac{d_n}{\nu_n} ||P_n^3 x||,$$

for all $(n, x) \in \mathbb{N} \times X$;

(D_3)

$$\sum_{j=n}^{m} \nu_j ||A_j^n P_n^3 x|| = \sum_{j=n}^{m} \nu_j ||P_j^3 A_j^n x|| \leq$$

$$\leq \sum_{j=n}^{m} \frac{\nu_j}{\mu_j} \mu_m s_m ||A_m^j A_j^n P_n^3 x|| = \mu_m s_m \sum_{j=n}^{m} \frac{k_j}{\psi_j} ||A_m^n P_n^3 x|| \leq$$

$$\leq B \mu_m s_m r_m \frac{k_m}{\psi_m} ||A_m^n P_n^3 x|| \leq d_m \nu_m ||A_m^n P_n^3 x||, \quad \text{for all} \quad (m, n, x) \in \Delta \times X.$$

Corollary 2. *If the pair $(\mathcal{A}, \mathcal{P})$ is exponentially trichotomic then there exists the constants $a, b, c > 0$ and a nondecresing sequence $(d_n)_n$, $d_n \geq 1$ with:*

$$(eD_1) \quad \sum_{j=n}^{+\infty} e^{ja} ||A_j^n P_n^1 x|| + \sum_{j=n}^{m} e^{-jb} ||A_j^n P_n^2 x|| \leq$$
$$\leq e^{na} d_n ||P_n^1 x|| + e^{-mb} d_m ||A_m^n P_n^2 x||, \quad \textit{for all} \quad (m, n, x) \in \Delta \times X;$$

$$(eD_2) \quad \sum_{j=n}^{+\infty} e^{-cj} ||A_j^n P_n^3 x|| \leq e^{-nc} d_n ||P_n^3 x||, \quad \textit{for all} \quad (n, x) \in \mathbb{N} \times X;$$

$$(eD_3) \quad \sum_{j=n}^{m} e^{cj} ||A_j^n P_n^3 x|| \leq e^{cm} d_m ||A_m^n P_n^3 x||, \quad \textit{for all} \quad (m, n, x) \in \Delta \times X.$$

Proof. It is a particular case of Theorem 1 and Remark 3 for

$$h_n = e^{n\alpha}, \ \varphi_n = e^{na}, \ a \in (0, \alpha),$$

$$k_n = e^{n\beta}, \ \psi_n = e^{nb}, \ b \in (0, \beta),$$

$$\mu_n = e^{n\gamma}, \ \gamma > 0, \ \nu_n = e^{nc}, \ c = \gamma + \beta - b.$$

Corollary 3. *If the pair $(\mathcal{A},\mathcal{P})$ is polynomially trichotomic then there exists the constants $a,b,c>0$ and a nondecresing sequence $(d_n)_n$, $d_n \geq 1$ with:*

$$(pD_1)\quad \sum_{j=n}^{+\infty}(j+1)^a||A_j^nP_n^1x||+\sum_{j=n}^{m}(j+1)^{-b}||A_j^nP_n^2x|| \leq$$
$$\leq (n+1)^a d_n||P_n^1x||+(m+1)^{-b}d_m||A_m^nP_n^2x||, \quad for\ all \quad (m,n,x)\in\Delta\times X;$$

$$(pD_2)\quad \sum_{j=n}^{+\infty}(j+1)^{-c}||A_j^nP_n^3x|| \leq (n+1)^{-c}d_n||P_n^3x||, \quad for\ all \quad (n,x)\in\mathbb{N}\times X;$$

$$(pD_3)\quad \sum_{j=n}^{m}(j+1)^{c}||A_j^nP_n^3x|| \leq (m+1)^{c}d_m||A_m^nP_n^3x||, \quad for\ all \quad (m,n,x)\in\Delta\times X.$$

Proof. It is similar with Corollary 2, using Theorem 1 and Remark 4.

Definition 6. *A mapping $L=(L_1,L_2):\Delta\times X\to\mathbb{R}_+^2$ is called (h,k,μ)-Lyapunov function for the pair $(\mathcal{A},\mathcal{P})$ if:*

$$(Lh_1)\quad L_1(m,n,P_n^1x)-L_1(m+1,n,P_n^1x)\geq h_m||A_m^nP_n^1x||;$$
$$(Lk_2)\quad k_{m+1}L_2(m+1,n,P_n^2x)-k_{m+1}L_2(m,n,P_n^2x)\geq ||A_{m+1}^nP_n^2x||;$$
$$(L\mu_3)\quad \mu_mL_1(m,n,P_n^3x)-\mu_mL_1(m+1,n,P_n^3x)\geq ||A_m^nP_n^3x||;$$
$$(L\mu_4)\quad L_2(m+1,n,P_n^3x)-L_2(m,n,P_n^3x)\geq \mu_{m+1}||A_{m+1}^nP_n^3x||,$$

for all $(m,n,x)\in\Delta\times X$.

In particular, if

(i) $h_n=e^{n\alpha}$, $k_n=e^{n\beta}$, $\mu_n=e^{n\gamma}$, with $\alpha,\beta,\gamma>0$ then the (h,k,μ)-Lyapunov function is called *exponential Lyapunov function;*

(ii) $h_n=(n+1)^\alpha$, $k_n=(n+1)^\beta$, $\mu_n=(n+1)^\gamma$ with $\alpha,\beta,\gamma>1$ then the (h,k,μ)-Lyapunov function is called *polynomial Lyapunov function.*

Another necessary condition for (h,k,μ)-trichotomy is given by

Theorem 4. *If $(\mathcal{A},\mathcal{P})$ is (h,k,μ)-trichotomic with $h\in\mathcal{H}$ and $k\in\mathcal{K}$, then there exists $L:\Delta\times X\to\mathbb{R}_+^2$ a (φ,ψ,ν)-Lyapunov function for $(\mathcal{A},\mathcal{P})$, where φ,ψ are given by $\mathcal{H}$, respectively $\mathcal{K}$ and ν by Theorem 1.*

Proof. Let $L=(L_1,L_2):\Delta\times X\to\mathbb{R}_+^2$, where

$$L_1(m,n,x)=\sum_{j=m}^{+\infty}\varphi_j||A_j^nP_n^1x||+\sum_{j=m}^{+\infty}\frac{1}{\nu_j}||A_j^nP_n^3x||,$$

respectively

$$L_2(m,n,x) = \sum_{j=n}^{m} \frac{1}{\psi_j} ||A_j^n P_n^2 x|| + \sum_{j=n}^{m} \nu_j ||A_j^n P_n^3 x||.$$

A simple computation shows that the conditions from Definition 6 are verified.

As consequences of the above result, for $h(t) = e^{\alpha t},\ k(t) = e^{\beta t},\ \alpha, \beta > 0$ we obtain

Corollary 5. *If the pair $(\mathcal{A}, \mathcal{P})$ is exponentially trichotomic, then there exists an exponential Lyapunov function for $(\mathcal{A}, \mathcal{P})$.*

Proof. It results from Theorem 4.

In a similar manner, for $h(t) = (t+1)^\alpha,\ k(t) = (t+1)^\beta,\ \alpha, \beta > 1$ we give

Corollary 6. *If the pair $(\mathcal{A}, \mathcal{P})$ is polynomially trichotomic then there exists an polynomial Lyapunov function for $(\mathcal{A}, \mathcal{P})$.*

Proof. It is immediate, using Theorem 4.

3. Sufficient conditions for (h, k, μ)-trichotomy

In this section we obtain sufficient criteria for the (h, k, μ)-trichotomy of a pair $(\mathcal{A}, \mathcal{P})$, with $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ invariant projections sequence for the linear discrete-time system $(\mathcal{A})$.

A first result in this context is emphasized by

Theorem 7. *If there exists a nondecreasing sequence $(d_n)_n$, with $d_n \geq 1$ such that*

$$(i) \quad \sum_{j=n}^{+\infty} h_j ||A_j^n P_n^1 x|| + \sum_{j=n}^{m} \frac{||A_j^n P_n^2 x||}{k_j} \leq$$

$$\leq h_n d_n ||P_n^1 x|| + \frac{d_m}{k_m} ||A_m^n P_n^2 x||, \quad \text{for all} \quad (m,n,x) \in \Delta \times X;$$

$$(ii) \quad \sum_{j=n}^{+\infty} \frac{1}{\mu_j} ||A_j^n P_n^3 x|| \leq \frac{d_n}{\mu_n} ||P_n^3 x||, \quad \text{for all} \quad (n,x) \in \mathbb{N} \times X;$$

$$(iii) \quad \sum_{j=n}^{m} \mu_j ||A_j^n P_n^3 x|| \leq \mu_m d_m ||A_m^n P_n^3 x||, \quad \text{for all} \quad (m,n,x) \in \Delta \times X,$$

then $(\mathcal{A}, \mathcal{P})$ is (h, k, μ)-trichotomic.

Proof. We show that the conditions from Definition 5 are satisfied.

$$(t_1) \qquad h_m||A_m^n P_n^1 x|| + k_m||P_n^2 x|| \le \sum_{j=n}^{+\infty} h_j||A_j^n P_n^1 x|| + k_m k_n \sum_{j=n}^{m} \frac{||A_j^n P_n^2 x||}{k_j} \le$$

$$\le h_n d_n ||P_n^1 x|| + k_n d_m ||A_m^n P_n^2 x||, \quad for\ all \quad (m,n,x) \in \Delta \times X;$$

(t_2) It results for $j = m$ in (ii) and $j = m$ in (iii).
So, $(\mathcal{A}, \mathcal{P})$ is (h, k, μ)-trichotomic.

Corollary 8. *The pair $(\mathcal{A}, \mathcal{P})$ is exponentially trichotomic if and only if there exists the constants $a, b, c > 0$ and a nondecreasing sequence $(d_n)_n,\ d_n \ge 1$ such that the relations $(eD_1) - (eD_3)$ from Corollary 2 are satisfied.*

Proof. Necessity. It follows from Corollary 2.
Sufficiency. It results from Theorem 7, for $h_n = e^{na},\ k_n = e^{nb},\ \mu_n = e^{nc}$.

Corollary 9. *If there exists a nondecreasing sequence $(d_n)_n,\ d_n \ge 1$ and the constants $a, b, c > 1$ such that the conditions from Corollary 3 are verified, then $(\mathcal{A}, \mathcal{P})$ is polynomially trichotomic.*

Proof. It is immediate by Theorem 7.

Another sufficient condition for the (h, k, μ)-trichotomy is given by

Theorem 10. *If there exists $L : \Delta \times X \to \mathbb{R}_+^2$, a (h, k, μ)-Lyapunov function for $(\mathcal{A}, \mathcal{P})$ and there exists a nondecreasing sequence $(u_n)_n$, with $u_n \ge 1$ such that*

(i) $L_1(m, n, P_n^1 x) + \mu_n L_1(n, n, P_n^3 x) \le u_n(||P_n^1 x|| + ||P_n^3 x||)$, $for\ all\ (m, n, x) \in \Delta \times X$;

(ii) $k_m L_2(m, n, P_n^2 x) + L_2(m, n, P_n^3 x) \le u_m(||A_m^n P_n^2 x|| + ||A_m^n P_n^3 x||)$ $for\ all\ (m, n, x) \in \Delta \times X$;

(iii) $k_n L_2(n, n, P_n^2 x) \ge ||P_n^2 x||, \quad L_2(n, n, P_n^3 x) \ge \mu_n ||P_n^3 x||$, $for\ all\ (n, x) \in \mathbb{N} \times X$,

then $(\mathcal{A}, \mathcal{P})$ is (h, k, μ)-trichotomic.

Proof. We use the inequalities from Definition 6 and $(i) - (iii)$ from the hypothesis.
(t_1) From (Lh_1) we have

$$L_1(n, n, P_n^1 x) - L_1(n+1, n, P_n^1 x) \ge h_n ||A_n^n P_n^1 x||$$

$$L_1(n+1,n,P_n^1x) - L_1(n+2,n,P_n^1x) \geq h_{n+1}||A_{n+1}^n P_n^1 x||$$
$$L_1(n+2,n,P_n^1x) - L_1(n+3,n,P_n^1x) \geq h_{n+2}||A_{n+2}^n P_n^1 x||$$
$$\dots$$

and then

$$L_1(n,n,P_n^1x) \geq \sum_{j=n}^{+\infty} h_j ||A_j^n P_n^1 x|| \geq h_m ||A_m^n P_n^1 x||.$$

From (i) we obtain

$$h_m ||A_m^n P_n^1 x|| \leq h_n u_n ||P_n^1 x||, \tag{1}$$

for all $(m,n,x) \in \Delta \times X$.
Now, by (Lk_2) and (iii) we deduce

$$k_m L_2(m,n,P_n^2x) - k_m L_2(m-1,n,P_n^2x) \geq ||A_m^n P_n^2 x||$$
$$k_{m-1} L_2(m-1,n,P_n^2x) - k_{m-1} L_2(m-2,n,P_n^2x) \geq ||A_{m-1}^n P_n^2 x||$$
$$\dots$$
$$k_n L_2(n,n,P_n^2x) \geq ||P_n^2 x||,$$

which implies

$$L_2(m,n,P_n^2x) \geq \sum_{j=n}^{m} \frac{1}{k_j} ||A_j^n P_n^2 x|| \geq \frac{1}{k_n} ||P_n^2 x||$$

and by (ii) it follows that

$$\frac{k_m}{k_n} ||P_n^2 x|| \leq u_m ||A_m^n P_n^2 x||, \tag{2}$$

for all $(m,n,x) \in \Delta \times X$.

By the relations (1) and (2) it results the condition (t_1).
(t_2) According to $(L\mu_3)$ and (i) we obtain

$$\mu_n L_1(n,n,P_n^3x) - \mu_n L_1(n+1,n,P_n^3x) \geq ||A_n^n P_n^3 x||$$
$$\mu_{n+1} L_1(n+1,n,P_n^3x) - \mu_{n+1} L_1(n+2,n,P_n^3x) \geq ||A_{n+1}^n P_n^3 x||$$
$$\dots$$

which imply

$$\sum_{j=n}^{+\infty} \frac{1}{\mu_j} ||A_j^n P_n^3 x|| \leq L_1(n, n, P_n^3 x).$$

Then,

$$\frac{\mu_n}{\mu_m} ||A_m^n P_n^3 x|| \leq u_n ||P_n^3 x||, \tag{3}$$

for all $(m, n, x) \in \mathbb{N} \times X$.
In an analog manner, as above, using $(L\mu_4)$ and (ii) we deduce

$$\frac{\mu_n}{\mu_m} \frac{1}{u_m} ||P_n^3 x|| \leq ||A_m^n P_n^3 x||, \tag{4}$$

for all $(m, n, x) \in \Delta \times X$.
By (3) and (4) it follows (t_2).
In conclusion, $(\mathcal{A}, \mathcal{P})$ is (h, k, μ)-trichotomic.

Corollary 11. *The pair $(\mathcal{A}, \mathcal{P})$ is exponentially trichotomic if and only if there exists an exponential Lyapunov function for $(\mathcal{A}, \mathcal{P})$.*

Proof. Necessity. It results by Corollary 5.
Sufficiency. It is immediate from Theorem 10.

Corollary 12. *If there exists a polynomial Lyapunov function for the pair $(\mathcal{A}, \mathcal{P})$, then it is polynomially trichotomic.*

Proof. It is a particular case of Theorem 10.

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