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# ON SOME DIFFERENTIAL SANDWICH THEOREMS USING A MULTIPLIER TRANSFORMATION AND RUSCHEWEYH DERIVATIVE

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Abstract. In this work we study a new operator $IR_{\lambda,l}^{m,n}$ defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative $R^n$, given by $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$, $IR_{\lambda,l}^{m,n} f(z) = (I(m,\lambda,l) * R^n) f(z)$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + ..., z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. The purpose of this paper is to derive certain subordination and superordination results involving the operator $IR_{\lambda.l}^{m,n}$ and we establish differential sandwich-type theorems.

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## 1. Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots$.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ and $\mathcal{A} = \mathcal{A}_1$.

Let the functions $f$ and $g$ be analytic in $U$. We say that the function $f$ is subordinate to $g$, written $f \prec g$, if there exists a Schwarz function $w$, analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and $h$ be an univalent function in $U$. If $p$ is analytic in $U$ and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1}$$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all $p$ satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants $q$ of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of $U$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h$ analytic in $U$. If $p$ and $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ are univalent and if $p$ satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \qquad z \in U, \tag{2}$$

then $p$ is a solution of the differential superordination (2) (if $f$ is subordinate to $F$, then $F$ is called to be superordinate to $f$). An analytic function $q$ is called a subordinant if $q \prec p$ for all $p$ satisfying (2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants $q$ of (2) is said to be the best subordinant.

Miller and Mocanu [9] obtained conditions $h$, $q$ and $\psi$ for which the following implication holds

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc $U$, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j.$$

**Definition 1.** *[7] For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the multiplier transformation $I(m, \lambda, l) f(z)$ is defined by the following infinite series*

$$I(m, \lambda, l) f(z) := z + \sum_{j=2}^{\infty} \left( \frac{1 + \lambda(j-1) + l}{1 + l} \right)^m a_j z^j.$$

**Remark 1.** *We have*

$$(l+1) I(m+1, \lambda, l) f(z) = (l + 1 - \lambda) I(m, \lambda, l) f(z) + \lambda z \left(I(m, \lambda, l) f(z)\right)', \quad z \in U.$$

**Remark 2.** *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ for $\lambda = 1$.*

**Definition 2.** *(Ruscheweyh [11]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the Ruscheweyh derivative $R^n$ is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\ldots \\ (n+1) R^{n+1} f(z) &= z \left( R^n f(z) \right)' + n R^n f(z), \quad z \in U. \end{aligned}$$

**Remark 3.** *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ for $z \in U$.*

**Definition 3.** *Let $\lambda, l \geq 0$ and $n, m \in \mathbb{N}$. Denote by $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the multiplier transformation $I(m, \lambda, l)$ and the Ruscheweyh derivative $R^n$,*

$$IR_{\lambda,l}^{m,n} f(z) = \left( I(m, \lambda, l) * R^n \right) f(z),$$

*for any $z \in U$ and each nonnegative integers $m, n$.*

**Remark 4.** *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

**Remark 5.** *For $l = 0$, $\lambda \geq 0$, we obtain the Hadamard product $IR_{\lambda,0}^{m,n} f(z) = DR_{\lambda}^{m,n} f(z)$, which was introduced in [5].*

*For $l = 0$ and $\lambda = 1$ we obtain the operator $IR_{1,0}^{m,n} f(z) = SR^{m,n} f(z)$, which was introduced in [8].*

*For $m = n$, we obtain the Hadamard product $IR_{\lambda,l}^{m}$ which was studied in [1], [2].*

Using simple computation one obtains the next result.

**Proposition 1.** *[3]For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$IR_{\lambda,l}^{m+1,n} f(z) = \frac{1+l-\lambda}{l+1} IR_{\lambda,l}^{m,n} f(z) + \frac{\lambda}{l+1} z \left( IR_{\lambda,l}^{m,n} f(z) \right)' \tag{3}$$

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of Selvaraj and Karthikeyan [13], Shanmugam, Ramachandran, Darus and Sivasubramanian [14] and Srivastava and Lashin [15].

In order to prove our subordination and superordination results, we make use of the following known results.

**Definition 4.** *[10] Denote by $Q$ the set of all functions $f$ that are analytic and injective on $\overline{U}\backslash E(f)$, where $E(f)=\{\zeta\in\partial U:\lim\limits_{z\to\zeta}f(z)=\infty\}$, and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(f)$.*

**Lemma 1.** *[10] Let the function $q$ be univalent in the unit disc $U$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\phi(q(z))$ and $h(z)=\theta(q(z))+Q(z)$. Suppose that*

*1. $Q$ is starlike univalent in $U$ and*

*2. $Re\left(\frac{zh'(z)}{Q(z)}\right)>0$ for $z\in U$.*

*If $p$ is analytic with $p(0)=q(0)$, $p(U)\subseteq D$ and*

$$\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z)),$$

*then $p(z)\prec q(z)$ and $q$ is the best dominant.*

**Lemma 2.** *[6] Let the function $q$ be convex univalent in the open unit disc $U$ and $\nu$ and $\phi$ be analytic in a domain $D$ containing $q(U)$. Suppose that*

*1. $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)>0$ for $z\in U$ and*

*2. $\psi(z)=zq'(z)\phi(q(z))$ is starlike univalent in $U$.*

*If $p(z)\in\mathcal{H}[q(0),1]\cap Q$, with $p(U)\subseteq D$ and $\nu(p(z))+zp'(z)\phi(p(z))$ is univalent in $U$ and*

$$\nu(q(z))+zq'(z)\phi(q(z))\prec\nu(p(z))+zp'(z)\phi(p(z)),$$

*then $q(z)\prec p(z)$ and $q$ is the best subordinant.*

## 2. Main results

We begin with the following

**Theorem 3.** *Let $\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}(U)$ and let the function $q(z)$ be analytic and univalent in $U$ such that $q(z)\neq 0$, for all $z\in U$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. Let*

$$Re\left(\frac{\alpha+\mu}{\mu}+\frac{2\beta}{\mu}q(z)+\frac{zq''(z)}{q'(z)}\right)>0,\qquad(4)$$

*for $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\neq 0$, $z\in U$ and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z):=\frac{\mu(l+1)}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}+\qquad(5)$$

$$\alpha\frac{IR_{\lambda,l}^{m+1,n}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}+\left(\beta-\frac{\mu\left(l+1\right)}{\lambda}\right)\left(\frac{IR_{\lambda,l}^{m+1,n}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\right)^{2}.$$

*If $q$ satisfies the following subordination*

$$\psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right)\prec\alpha q\left(z\right)+\beta\left(q\left(z\right)\right)^{2}+\mu zq'\left(z\right), \tag{6}$$

*for $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\neq0$, then*

$$\frac{IR_{\lambda,l}^{m+1,n}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\prec q\left(z\right), \tag{7}$$

*and $q$ is the best dominant.*

*Proof.* Let the function $p$ be defined by $p\left(z\right):=\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}$, $z\in U$, $z\neq0$, $f\in\mathcal{A}$. We have $p'\left(z\right)=\frac{\left(IR_{\lambda,l}^{m+1,n}f(z)\right)'IR_{\lambda,l}^{m,n}f(z)-IR_{\lambda,l}^{m+1,n}f(z)\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2}}=\frac{\left(IR_{\lambda,l}^{m+1,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}-\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\cdot\frac{\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$. Then $zp'\left(z\right)=\frac{z\left(IR_{\lambda,l}^{m+1,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}-\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\cdot\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$.

By using the identity (3), we obtain

$$zp'\left(z\right)=\frac{l+1}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}-\frac{l+1}{\lambda}\left(\frac{IR_{\lambda,l}^{m+1,n}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\right)^{2}. \tag{8}$$

By setting $\theta\left(w\right):=\alpha w+\beta w^{2}$ and $\phi\left(w\right):=\mu$, it can be easily verified that $\theta$ is analytic in $\mathbb{C}$, $\phi$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi\left(w\right)\neq0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q\left(z\right)=zq'\left(z\right)\phi\left(q\left(z\right)\right)=\mu zq'\left(z\right)$ and $h\left(z\right)=\theta\left(q\left(z\right)\right)+Q\left(z\right)=\alpha q\left(z\right)+\beta\left(q\left(z\right)\right)^{2}+\mu zq'\left(z\right)$, we find that $Q\left(z\right)$ is starlike univalent in $U$.

We have $h'\left(z\right)=\left(\alpha+\mu\right)q'\left(z\right)+2\beta q\left(z\right)q'\left(z\right)+\mu zq''\left(z\right)$ and $\frac{zh'(z)}{Q(z)}=\frac{zh'(z)}{\mu zq'(z)}=\frac{\alpha+\mu}{\mu}+\frac{2\beta}{\mu}q\left(z\right)+\frac{zq''(z)}{q'(z)}$.

We deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\alpha+\mu}{\mu}+\frac{2\beta}{\mu}q\left(z\right)+\frac{zq''(z)}{q'(z)}\right)>0$.

By using (8), we obtain

$\alpha p\left(z\right)+\beta\left(p\left(z\right)\right)^{2}+\mu zp'\left(z\right)=\frac{\mu(l+1)}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}+\alpha\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}+\left(\beta-\frac{\mu(l+1)}{\lambda}\right)\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{2}$.

By using (6), we have $\alpha p\left(z\right)+\beta\left(p\left(z\right)\right)^{2}+\mu zp'\left(z\right)\prec\alpha q\left(z\right)+\beta\left(q\left(z\right)\right)^{2}+\mu zq'\left(z\right)$.

By an application of Lemma 1, we have $p\left(z\right)\prec q\left(z\right)$, $z\in U$, i.e. $\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\prec q\left(z\right)$, $z\in U$ and $q$ is the best dominant.

**Corollary 4.** *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (4) holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha \frac{1+Az}{1+Bz} + \beta \left(\frac{1+Az}{1+Bz}\right)^2 + \mu \frac{(A-B)z}{(1+Bz)^2},$$

*for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then*

$$\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \prec \frac{1+Az}{1+Bz},$$

*and $\frac{1+Az}{1+Bz}$ is the best dominant.*

*Proof.* For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 3 we get the corollary.

**Corollary 5.** *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (4) holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha \left(\frac{1+z}{1-z}\right)^{\gamma} + \beta \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\mu\gamma z}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^{\gamma-1},$$

*for $\alpha, \beta, \mu \in \mathbb{C}$, $0 < \gamma \leq 1$, $\mu \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then*

$$\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma},$$

*and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.*

*Proof.* Corollary follows by using Theorem 3 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$.

**Theorem 6.** *Let $q$ be analytic and univalent in $U$ such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in $U$. Assume that*

$$Re\left(\frac{\alpha}{\mu} q'(z) + \frac{2\beta}{\mu} q(z) q'(z)\right) > 0, \text{ for } \alpha, \beta, \mu \in \mathbb{C}, \ \mu \neq 0. \tag{9}$$

*If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$ is univalent in $U$, where $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$ is as defined in (5), then*

$$\alpha q(z) + \beta (q(z))^2 + \mu z q'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \tag{10}$$

*implies*

$$q(z) \prec \frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}, \quad z \in U, \tag{11}$$

*and $q$ is the best subordinant.*

*Proof.* Let the function $p$ be defined by $p(z) := \frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$ it can be easily verified that $\nu$ is analytic in $\mathbb{C}$, $\phi$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)[\alpha+2\beta q(z)]}{\mu}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\alpha}{\mu} q'(z) + \frac{2\beta}{\mu} q(z) q'(z)\right) > 0$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$.

By using (8) and (10) we obtain

$$\alpha q(z) + \mu (q(z))^2 + \mu z q'(z) \prec \alpha p(z) + \beta (p(z))^2 + \mu z p'(z).$$

Using Lemma 2, we have

$$q(z) \prec p(z) = \frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}, \quad z \in U,$$

and $q$ is the best subordinant.

**Corollary 7.** *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (9) holds. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and*

$$\alpha \frac{1+Az}{1+Bz} + \beta \left(\frac{1+Az}{1+Bz}\right)^2 + \mu \frac{(A-B)z}{(1+Bz)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z),$$

*for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then*

$$\frac{1+Az}{1+Bz} \prec \frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)},$$

*and $\frac{1+Az}{1+Bz}$ is the best subordinant.*

*Proof.* For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 6 we get the corollary.

**Corollary 8.** *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (9) holds. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and*

$$\alpha \left(\frac{1+z}{1-z}\right)^{\gamma} + \beta \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\mu\gamma z}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^{\gamma-1} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z),$$

*for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $0 < \gamma \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)},$$

*and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.*

*Proof.* For $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$ in Theorem 6 we get the corollary.

Combining Theorem 3 and Theorem 6, we state the following sandwich theorem.

**Theorem 9.** *Let $q_1$ and $q_2$ be analytic and univalent in $U$ such that $q_1(z)\neq 0$ and $q_2(z)\neq 0$, for all $z\in U$, with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that $q_1$ satisfies (4) and $q_2$ satisfies (9). If $f\in\mathcal{A}$, $\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ is as defined in (5) univalent in $U$, then*

$$\alpha q_1(z)+\beta(q_1(z))^2+\mu zq_1'(z)\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha q_2(z)+\beta(q_2(z))^2+\mu zq_2'(z),$$

*for $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\neq 0$, implies*

$$q_1(z)\prec\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\prec q_2(z),$$

*and $q_1$ and $q_2$ are respectively the best subordinant and the best dominant.*

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$, where $-1\leq B_2<B_1<A_1<A_2\leq 1$, we have the following corollary.

**Corollary 10.** *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (4) and (9) hold. If $f\in\mathcal{A}$, $\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and*

$$\alpha\frac{1+A_1z}{1+B_1z}+\beta\left(\frac{1+A_1z}{1+B_1z}\right)^2+\mu\frac{(A_1-B_1)z}{(1+B_1z)^2}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$$

$$\prec\alpha\frac{1+A_2z}{1+B_2z}+\beta\left(\frac{1+A_2z}{1+B_2z}\right)^2+\mu\frac{(A_2-B_2)z}{(1+B_2z)^2},$$

*for $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\neq 0$, $-1\leq B_2\leq B_1<A_1\leq A_2\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then*

$$\frac{1+A_1z}{1+B_1z}\prec\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\prec\frac{1+A_2z}{1+B_2z},$$

*hence $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0<\gamma_1<\gamma_2\leq 1$, we have the following corollary.

**Corollary 11.** *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (4) and (9) hold. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and*

$$\alpha\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\beta\left(\frac{1+z}{1-z}\right)^{2\gamma_1}+\frac{2\mu\gamma_1 z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\gamma_1-1} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$$

$$\prec \alpha\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\beta\left(\frac{1+z}{1-z}\right)^{2\gamma_2}+\frac{2\mu\gamma_2 z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\gamma_2-1},$$

*for $\alpha,\beta,\mu \in \mathbb{C}$, $\mu \neq 0$, $0<\gamma_1<\gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2},$$

*hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

We have also

**Theorem 12.** *Let $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta} \in \mathcal{H}(U)$, $f \in \mathcal{A}$, $z \in U$, $\delta \in \mathbb{C}, \delta \neq 0$, $m,n \in \mathbb{N}$, $\lambda, l \geq 0$ and let the function $q(z)$ be convex and univalent in $U$ such that $q(0)=1$, $z \in U$. Assume that*

$$Re\left(1+\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^2(z)-z\frac{q'(z)}{q(z)}+z\frac{q''(z)}{q'(z)}\right)>0, \tag{12}$$

*for $\alpha,\xi,\mu,\beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, and*

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z) := \alpha+\xi\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}+ \tag{13}$$

$$\mu\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{2\delta}+\frac{\beta\delta(l+1)}{\lambda}\left[\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}-\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right].$$

*If $q$ satisfies the following subordination*

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z) \prec \alpha+\xi q(z)+\mu q^2(z)+\frac{\beta z q'(z)}{q(z)}, \tag{14}$$

*for $\alpha,\xi,\mu,\beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, then*

$$\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta} \prec q(z), \quad z \in U,\ \delta \in \mathbb{C},\ \delta \neq 0, \tag{15}$$

*and $q$ is the best dominant.*

*Proof.* Let the function $p$ be defined by $p(z) := \left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function $p$ is analytic in $U$ and $p(0) = 1$

We have $zp'(z) = \delta\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\left[\frac{z\left(IR_{\lambda,l}^{m+1,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} - \frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \cdot \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\right]$.

By using the identity (3), we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\delta(l+1)}{\lambda}\left[\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)} - \frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right]. \tag{16}$$

By setting $\theta(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$, it can be easily verified that $\theta$ is analytic in $\mathbb{C}$, $\phi$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \frac{\beta z q'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in $U$.

Let $h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \mu q^2(z) + \frac{\beta z q'(z)}{q(z)}$.

We have $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(1 + \frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}q^2(z) - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right) > 0$.

By using (16), we obtain $\alpha + \xi p(z) + \mu(p(z))^2 + \beta\frac{zp'(z)}{p(z)} = \alpha + \xi\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta} + \mu\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{2\delta} + \frac{\beta\delta(l+1)}{\lambda}\left[\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)} - \frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right]$.

By using (14), we have $\alpha + \xi p(z) + \mu(p(z))^2 + \beta\frac{zp'(z)}{p(z)} \prec \alpha + \xi q(z) + \mu q^2(z) + \frac{\beta z q'(z)}{q(z)}$.

From Lemma 1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta} \prec q(z)$, $z \in U$, $\delta \in \mathbb{C}, \delta \neq 0$ and $q$ is the best dominant.

**Corollary 13.** *Let $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (12) holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi\frac{1+Az}{1+Bz} + \mu\left(\frac{1+Az}{1+Bz}\right)^2 + \beta\frac{(A-B)z}{(1+Az)(1+Bz)},$$

*for $\alpha, \xi, \mu, \beta, \delta \in \mathbb{C}$, $\beta, \delta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (13), then*

$$\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta} \prec \frac{1+Az}{1+Bz},$$

*and $\frac{1+Az}{1+Bz}$ is the best dominant.*

*Proof.* For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 12 we get the corollary.

**Corollary 14.** *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (12) holds. If* $f \in \mathcal{A}$ *and*

$$\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2},$$

*for* $\alpha, \xi, \mu, \beta, \delta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta, \delta \neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (13), then*

$$\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\gamma},$$

*and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

*Proof.* Corollary follows by using Theorem 12 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$.

**Theorem 15.** *Let* $q$ *be convex and univalent in* $U$ *such that* $q(0) = 1$. *Assume that*

$$Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0, \text{ for } \alpha, \xi, \mu, \beta \in \mathbb{C}, \ \beta \neq 0. \tag{17}$$

*If* $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$ *is univalent in* $U$, *where* $\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$ *is as defined in (13), then*

$$\alpha + \xi q(z) + \mu q^2(z) + \frac{\beta z q'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \tag{18}$$

*implies*

$$q(z) \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta}, \quad \delta \in \mathbb{C}, \ \delta \neq 0, \ z \in U, \tag{19}$$

*and* $q$ *is the best subordinant.*

*Proof.* Let the function $p$ be defined by $p(z) := \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta}$, $z \in U$, $z \neq 0$, $\delta \in \mathbb{C}, \delta \neq 0$, $f \in \mathcal{A}$. The function $p$ is analytic in $U$ and $p(0) = 1$.

By setting $\nu(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that $\nu$ is analytic in $\mathbb{C}$, $\phi$ is analytic in $\mathbb{C} \backslash \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \backslash \{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0$, for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (18) we obtain

$$\alpha+\xi q(z)+\mu q^2(z)+\frac{\beta z q'(z)}{q(z)} \prec \alpha+\xi p(z)+\mu p^2(z)+\frac{\beta z p'(z)}{p(z)}, \quad z\in U.$$

From Lemma 2, we have

$$q(z) \prec p(z)=\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}, \quad z\in U,\ \delta\in\mathbb{C},\ \delta\neq 0,$$

and $q$ is the best subordinant.

**Corollary 16.** *Let* $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. *Assume that (17) holds. If* $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$, $\delta\in\mathbb{C}$, $\delta\neq 0$ *and*

$$\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^2+\beta\frac{(A-B)z}{(1+Az)(1+Bz)} \prec \psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z),$$

*for* $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\le B<A\le 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (13), then*

$$\frac{1+Az}{1+Bz} \prec \left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}, \delta\in\mathbb{C},\ \delta\neq 0,$$

*and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

*Proof.* For $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, in Theorem 15 we get the corollary.

**Corollary 17.** *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. *Assume that (17) holds. If* $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z),$$

*for* $\alpha,\xi,\mu,\beta,\delta\in\mathbb{C}$, $0<\gamma\le 1$, $\beta\ ,\delta\neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (13), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^{\delta},$$

*and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

*Proof.* Corollary follows by using Theorem 15 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$.

Combining Theorem 12 and Theorem 15, we state the following sandwich theorem.

**Theorem 18.** *Let $q_1$ and $q_2$ be convex and univalent in $U$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$. Suppose that $q_1$ satisfies (12) and $q_2$ satisfies (17). If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$ , $\delta \in \mathbb{C}, \delta \neq 0$ and $\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$ is as defined in (13) univalent in $U$, then*

$$\alpha + \xi q_1(z) + \mu q_1^2(z) + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi q_2(z) + \mu q_2^2(z) + \frac{\beta z q_2'(z)}{q_2(z)},$$

*for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, implies*

$$q_1(z) \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta} \prec q_2(z), \quad z \in U, \ \delta \in \mathbb{C}, \ \delta \neq 0,$$

*and $q_1$ and $q_2$ are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

**Corollary 19.** *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (12) and (17) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$ and*

$$\alpha + \xi \frac{1+A_1 z}{1+B_1 z} + \mu \left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \beta \frac{(A_1 - B_1) z}{(1+A_1 z)(1+B_1 z)} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$$

$$\prec \alpha + \xi \frac{1+A_2 z}{1+B_2 z} + \mu \left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \beta \frac{(A_2 - B_2) z}{(1+A_2 z)(1+B_2 z)}, \quad z \in U,$$

*for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then*

$$\frac{1+A_1 z}{1+B_1 z} \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta} \prec \frac{1+A_2 z}{1+B_2 z}, \quad z \in U, \ \delta \in \mathbb{C}, \ \delta \neq 0,$$

*hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \le 1$, we have the following corollary.

**Corollary 20.** *Let $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (12) and (17) hold for $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$ and*

$$\alpha + \xi\left(\frac{1+z}{1-z}\right)^{\gamma_1} + \mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1} + \frac{2\beta\gamma_1 z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$$

$$\prec \alpha + \xi\left(\frac{1+z}{1-z}\right)^{\gamma_2} + \mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2} + \frac{2\beta\gamma_2 z}{1-z^2}, \quad z \in U,$$

*for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \ne 0$, $0 < \gamma_1 < \gamma_2 \le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}, \quad z \in U, \ \delta \in \mathbb{C}, \ \delta \ne 0,$$

*hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

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