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HIGHER $*$-DERIVATIONS IN NON-ARCHIMEDEAN RANDOM C^*-ALGEBRAS AND LIE HIGHER $*$-DERIVATIONS IN NON-ARCHIMEDEAN RANDOM LIE C^*-ALGEBRAS

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Abstract. Using fixed point method, we establish the generalized Hyers-Ulam stability of higher $*$-derivations in non-Archimedean random C^*-algebras and Lie higher $*$-derivations in non-Archimedean random Lie C^*-algebras associated to the following Cauchy-Jensen additive functional equation:

$$f(\frac{x+y+z}{2}) + f(\frac{x-y+z}{2}) = f(x) + f(z).$$

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1. Introduction and preliminaries

Let K denote a field and function (valuation absolute) $|\ .\ |$ from K into $[0, \infty)$. A non-Archimedean valuation is a function $|\ .\ |$ that satisfies the strong triangle inequality; namely, $|x+y| \leq \max\{|x|, |y|\} \leq |x|+|y|$ for all $x, y \in K$. The associated field K is referred to as a non-Archimedean field. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\ .\ |$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|\ .\ |$ is nontrivial, i.e., there is a $z \in K$ such that $|z| \neq 0, 1$.

Let X be a linear space over a field K with a non-Archimedean nontrivial valuation $|\ .\ |$. A function $\|\ .\ \| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it is a norm over K with the strong triangle inequality (ultrametric); namely, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. Then $(X, \|\cdot\|)$ is called a non-Archimedean

normed space. In any such a space a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy if and only if $\{x_{n+1}-x_n\}_{n\in\mathbb{N}}$ converges to zero. By a complete non–Archimedean space, we mean one in which every Cauchy sequence is convergent.
For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$,where a and b are integers non divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.
A non-Archimedean Banach algebra is a complete non-Archimedean algebra A which satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [6, 13].
If $\mathcal{I}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{I}$ is a mapping $t \mapsto t^*$ from $\mathcal{I}$ into $\mathcal{I}$ which satisfies

(i) $t^{**} = t$ for $t \in \mathcal{I}$;

(ii) $(\alpha s + \beta t)^* = \bar{\alpha}s^* + \bar{\beta}t^*$;

(iii) $(st)^* = t^*s^*$ for $s, t \in \mathcal{I}$.

If, in addition, $\|t^*t\| = \|t\|^2$ for $t \in \mathcal{I}$, then $\mathcal{I}$ is a non-Archimedean C^*-algebra.

The stability problem of functional equations was originated from a question of Ulam [14] concerning the stability of group homomorphisms. Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group (a metric which is defined on a set with a group property) with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x*y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of a homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable (see also [9, 10]).
Let X be a nonempty set and $d : X \times X \to [0,\infty]$ satisfying: $d(x,y) = 0$ if and only if $x = y$, $d(x,y) = d(y,x)$ and $d(x,z) \leq d(x,y) + d(y,z)$ (strong triangle inequality), for all $x, y, z \in X$. Then (X, d) is called a generalized metric space. (X, d) is called complete if every d-Cauchy sequence in X is d-convergent.
Using the strong triangle inequality in the proof of the main result of [5], we get to the following result:

Theorem 1. *[5] Let (Ω, d) be a complete generalized metric space and let $\mathcal{F} : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $L \in (0,1)$. Then, for a given element $x \in \Omega$, exactly one of the following assertions is true:*
either
(1) *$d(\mathcal{F}^n x, \mathcal{F}^{n+1} x) = \infty$ for all $n \geq 0$ or*

(2) *there exists* n_0 *such that* $d(\mathcal{F}^n x, \mathcal{F}^{n+1} x) < \infty$ *for all* $n \geq n_0$.
Actually, if (2) *holds, then the sequence* $\{\mathcal{F}^n x\}$ *is convergent to a fixed point* x^* *of* $\mathcal{F}$ *and*
(3) x^* *is the unique fixed point of* $\mathcal{F}$ *in* $\Lambda := \{y \in \Omega, d(\mathcal{F}^{n_0} x, y) < \infty\}$;
(4) $d(y, x^*) \leq \frac{d(y, \mathcal{F}y)}{1-L}$ *for all* $y \in \Lambda$.

In this paper we consider a mapping $f : X \to Y$ satisfying the following Cauchy-Jensen functional equation

$$f(\frac{x+y+z}{2}) + f(\frac{x-y+z}{2}) = f(x) + f(z) \tag{1}$$

for all $x, y, z \in X$ and establish the higher $*$-derivations in non-Archimedean random C^*-algebras and Lie higher $*$-derivations in non-Archimedean random Lie C^*-algebras for the functional equation (1).

2. Random spaces

In the section, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces as in [1, 2, 3, 4, 7, 11, 12]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. $\mathcal{D}^+$ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$.

Definition 1. *[11] A mapping* $T : [0,1] \times [0,1] \to [0,1]$ *is a continuous triangular norm (briefly, a continuous t-norm) if* T *satisfies the following conditions:*

(1) T *is commutative and associative;*

(2) T *is continuous;*

(3) $T(a, 1) = a$ *for all* $a \in [0,1]$*;*

(4) $T(a, b) \leq T(c, d)$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Definition 2. *[12] A non-Archimedean random normed space (briefly, NA-RN-space) is a triple* (X, μ, T)*, where* X *is a vector space,* T *is a continuous t-norm, and* μ *is a mapping from* X *into* $\mathcal{D}^+$ *such that the following conditions hold:*

$(RN1)$ $\mu_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$;

$(RN2)$ $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ *for all* $x \in X$, $\alpha \neq 0$;

$(RN3)$ $\mu_{x+y}(\max\{t, s\}) \geq T(\mu_x(t), \mu_y(s))$.

It is easy to see that if $(RN3)$ *holds, then we have*

$(RN4)$ $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$.

Definition 3. *[8] A non-Archimedean random normed algebra* (X, μ, T, T') *is a non-Archimedean random normed space* (X, μ, T) *with an algebraic structure such that*

$(RN5)$ $\mu_{xy}(t) \geq T'(\mu_x(t), \mu_y(t))$

for all $x, y \in X$ *and all* $t > 0$, *in which* T' *is a continuous* t*-norm.*

Definition 4. *Let* (X, μ, T) *and* (Y, μ, T) *be non-Archimedean random normed algebras.*

(1) *An* $\mathbb{R}$*-linear mapping* $f : X \to Y$ *is called a homomorphism if* $f(xy) = f(x)f(y)$ *for all* $x, y \in X$.

(2) *An* $\mathbb{R}$*-linear mapping* $f : X \to X$ *is called a derivation if* $f(xy) = f(x)y + xf(y)$ *for all* $x, y \in X$.

Definition 5. *Let* $(\mathcal{I}, \mu, T, T')$ *be a non-Archimedean random Banach algebra, then an involution on* $\mathcal{I}$ *is a mapping* $u \mapsto u^*$ *from* $\mathcal{I}$ *into* $\mathcal{I}$ *which satisfies*

(i) $u^{**} = u$ *for* $u \in \mathcal{I}$;

(ii) $(\alpha u + \beta v)^* = \bar{\alpha} u^* + \bar{\beta} v^*$;

(iii) $(uv)^* = v^* u^*$ *for* $u, v \in \mathcal{I}$.

If, in addition, $\mu_{u^*u}(t) = T'(\mu_u(t), \mu_u(t))$ *for* $u \in \mathcal{I}$ *and* $t > 0$, *then* $\mathcal{I}$ *is a non-Archimedean random* C^**-algebra.*

Definition 6. *Let* (X, μ, T) *be an NA-RN-space.*

(1) *A sequence* $\{x_n\}$ *in* X *is said to be convergent to* x *in* X *if, for every* $\epsilon > 0$ *and* $\lambda > 0$, *there exists a positive integer* N *such that* $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ *whenever* $n \geq N$.

(2) *A sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence if, for every* $\epsilon > 0$ *and* $\lambda > 0$, *there exists a positive integer* N *such that* $\mu_{x_n - x_{n+1}}(\epsilon) > 1 - \lambda$ *whenever* $n \geq N$.

(3) *An RN-space* (X, μ, T) *is said to be complete if and only if every Cauchy sequence in* X *is convergent to a point in* X.

3. Higher $*$-derivations in non-Archimedean random C^*-algebras

In this section, we will assume that $\mathcal{A}$ and $\mathcal{B}$ are two non-Archimedean random Banach C^*-algebras with the norm $\mu^{\mathcal{A}}$ and $\mu^{\mathcal{B}}$, respectively. For convenience, for each $n \in \mathbb{N}_0$, we use the following abbreviations for each given mapping $f_n : \mathcal{A} \to \mathcal{B}$:

$$D_\nu f_n(x,y,z) := \nu f_n(\frac{x+y+z}{2}) + \nu f_n(\frac{x-y+z}{2}) - f_n(\nu x) - f_n(\nu z)$$

for all $\nu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in \mathcal{A}$.

Definition 7. *Let $\mathbb{N}$ be the set of natural numbers. Form $m \in \mathbb{N} \cup \{0\}$, a sequence $H = \{h_0, h_1, \ldots, h_m\}$ (resp. $H = \{h_0, h_1, \ldots, \ldots\}$) of mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a higher $*$-derivation of rank m (resp. infinite rank) from $\mathcal{A}$ into $\mathcal{B}$ if*
(i) $f_n(x^*) = (f_n(x))^*$, for all $x \in \mathcal{A}$ and for each $n \in \{0, 1, \ldots, m\}$ (resp. $n \in \mathbb{N}_0$.)
(ii) $f_n(xy) = \sum_{i=0}^n f_i(x) f_{n-i}(y)$ holds for each $n \in \{0, 1, \ldots, m\}$ (resp. $n \in \mathbb{N}_0$) and all $x, y \in \mathcal{A}$.

We are going to investigate the generalized Hyers-Ulam stability of higher $*$-derivations in non-Archimedean random C^*-algebras for the functional equation $D_\nu f_n(x,y,z) = 0$.

Theorem 2. *Let $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to$ and $\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{D}^+$ be functions. Suppose that $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$, $f_n(0) = 0$,*

$$\mu^{\mathcal{B}}_{D_\nu f_n(x,y,z)}(t) \geq \varphi_{x,y,z}(t), \tag{2}$$

$$\mu^{\mathcal{B}}_{f_n(x^*) - f_n(x)^*}(t) \geq \varphi_{x,0,0}(t), \tag{3}$$

$$\mu^{\mathcal{B}}_{f_n(xy) - \sum_{i=0}^n f_i(x) f_{n-i}(y)}(t) \geq \psi_{x,y}(t) \tag{4}$$

for all $\nu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}; |\lambda| = 1\}$, all $x, y, z \in \mathcal{A}$ and all $t > 0$. Assume that $|2| < 1$ is far from zero and there exists an $0 \leq L < 1$ such that

$$\varphi_{2x,2y,2z}(|2|Lt) \geq \varphi_{x,y,z}(t), \tag{5}$$

$$\psi_{2x,2y}(|2|^2 Lt) \geq \psi_{x,y}(t) \tag{6}$$

for all $x, y, z \in \mathcal{A}$ and $t > 0$. Then there exists a unique higher $*$-derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$,

$$\mu^{\mathcal{B}}_{f_n(x) - h_n(x)}(t) \geq \varphi_{x,2x,x}\Big(|2|(1-L)t\Big) \tag{7}$$

holds for all $x \in \mathcal{A}$ and $t > 0$.

Proof. Fix $n \in \mathbb{N}_0$. Setting $\nu = 1$ and replacing (x, y, z) by $(x, 2x, x)$ in (2) implies

$$\mu^{\mathcal{B}}_{f_n(2x)-2f_n(x)}(t) \geq \varphi_{x,2x,x}(t) \tag{8}$$

for all $x \in \mathcal{A}$ and $t > 0$.
Let $\mathcal{Z}$ be the set of all functions $g : \mathcal{A} \to \mathcal{B}$. We define the metric d on $\mathcal{Z}$ as follows:

$$d(g, h) = \inf\Big\{k \in (0, \infty) : \mu^{\mathcal{B}}_{g(x)-h(x)}(kt) > \varphi_{x,2x,x}(t), \forall x \in \mathcal{A}, t > 0\Big\}.$$

One has the operator $J : \mathcal{Z} \to \mathcal{Z}$ by $(Jh)(x) := \frac{1}{2}h(2x)$. Then J is a contraction with Lipschitz constant L; in fact, for arbitrarily elements $f, g \in \mathcal{Z}$, we have

$$\begin{aligned} d(f, g) < k &\Rightarrow \mu^{\mathcal{B}}_{f(x)-g(x)}(kt) > \varphi_{x,2x,x}(t) \\ &\Rightarrow \mu^{\mathcal{B}}_{f(2x)-g(2x)}(kt) > \varphi_{2x,2(2x),2x}(t) \\ &\Rightarrow \mu^{\mathcal{B}}_{\frac{1}{2}f(2x)-\frac{1}{2}g(2x)}(kt) > \varphi_{2x,2(2x),2x}(|2|t) \\ &\Rightarrow \mu^{\mathcal{B}}_{\frac{1}{2}f(2x)-\frac{1}{2}g(2x)}(kLt) > \varphi_{x,2x,x}(t) \Rightarrow d(Jf, Jg) < kL \end{aligned}$$

for all $x \in \mathcal{A}$ and $t > 0$. Hence we see that

$$d(Jf, Jg) \leq Ld(f, g).$$

On the other hand, by (8) we have

$$\mu^{\mathcal{B}}_{Jf_n(x)-f_n(x)}(\frac{1}{|2|}t) \geq \varphi_{x,2x,x}(t) \Rightarrow d(Jf_n, f_n) \leq \frac{1}{|2|} < \infty.$$

Therefore, it follows from Theorem (1) that there exists a mapping $h_n : \mathcal{A} \to \mathcal{B}$ such that h_n is a fixed point of J that is $h_n(2x) = 2h_n(x)$ for all $x \in \mathcal{A}$. By Theorem (1) $\lim_{m\to\infty} d(J^m f_n, f_n) = 0$ we conclude that

$$\lim_{m\to\infty} \frac{f_n(2^m x)}{2^m} = h_n(x) \tag{9}$$

for all $x \in \mathcal{A}$. The mapping h_n is a unique fixed point of J in the set $\mathcal{U}_n = \{g \in \mathcal{Z} : d(f_n, g) < \infty\}$. Thus, h_n is a unique mapping such that there exists $k \in (0, \infty)$ satisfying $\mu^{\mathcal{B}}_{f_n(x)-h_n(x)}(kt) > \varphi_{x,2x,x}(t)$ for all $x \in \mathcal{A}$ and $t > 0$.
Again, by Theorem (1), we have

$$d(f_n, h_n) \leq \frac{1}{1-L} d(f_n, Jf_n) \leq \frac{1}{|2|(1-L)},$$

so

$$\mu^{\mathcal{B}}_{f_n(x)-h_n(x)}(t) \geq \varphi_{x,2x,x}\Big(|2|(1-L)t\Big).$$

This implies that the inequality (7) holds. Furthermore, it follows from (2), (5) and (9) that

$$\mu^{\mathcal{B}}_{D_\nu h_n(x,y,z)}(t) = \lim_{m\to\infty} \mu^{\mathcal{B}}_{\frac{1}{2^m} D_\nu f_n(2^m x, 2^m y, 2^m z)}(t) \geq \lim_{m\to\infty} \varphi_{2^m x, 2^m y, 2^m z}(|2|^m t) \to 1$$

for all $x, y, z \in \mathcal{A}$ and $t > 0$. So the mapping h_n is additive. By a similar method to the above, we have $\nu h_n(x) = h_n(\nu x)$ for all $\nu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus, one can show that the mapping $h_n : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear for each $n \in \mathbb{N}_0$. Using (4), (6) and (9), we get

$$\begin{aligned}\mu^{\mathcal{B}}_{h_n(xy)-\sum_{i=0}^n h_i(x)h_{n-i}(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{f_n(2^{2m}(xy))-\sum_{i=0}^n f_i(2^m x)f_{n-i}(2^m y)}(|2|^{2m} t) \\ &\geq \lim_{m\to\infty} \psi_{2^m x, 2^m y}(|2|^{2m} t) \to 1\end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$. So, $h_n(xy) = \sum_{i=0}^n h_i(x)h_{n-i}(y)$ for all $x, y \in \mathcal{A}$. By (3),

$$\mu^{\mathcal{B}}_{\frac{1}{2^m} f_n(2^m x^*) - \frac{1}{2^m} f_n(2^m x)^*}(t) \geq \varphi_{2^m x, 0, 0}(|2|^m t)$$

for all $x \in \mathcal{A}$ and $t > 0$. Passing to the limit as $m \to \infty$, we get $h_n(x^*) = h_n(x)^*$ for all $x \in \mathcal{A}$. This completes the proof.

Corollary 3. *Let $p > 1$, ξ be nonnegative real number and let $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$, $f_n(0) = 0$,*

$$\mu^{\mathcal{B}}_{D_\nu f_n(x,y,z)}(t) \geq \frac{t}{t + \xi\Big[\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}}\Big]},$$

$$\mu^{\mathcal{B}}_{f_n(x^*)-f_n(x)^*}(t) \geq \frac{t}{t + \xi\|x\|^p_{\mathcal{A}}},$$

$$\mu^{\mathcal{B}}_{f_n(xy)-\sum_{i=0}^n f_i(x)f_{n-i}(y)}(t) \geq \frac{t}{t + \xi\Big[\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}}\Big]}$$

for all $\nu \in \mathbb{T}^1$, all $x, y, z \in \mathcal{A}$ and all $t > 0$. Then there exists a unique higher $*$-derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$,

$$\mu^{\mathcal{B}}_{f_n(x)-h_n(x)}(t) \geq \frac{\Big(|2| - |2|^p\Big)t}{\Big(|2| - |2|^p\Big)t + \xi\Big(|2| + |2|^p\Big)\|x\|^p_{\mathcal{A}}}$$

holds for all $x \in \mathcal{A}$ and $t > 0$.

Proof. Put $\varphi_{x,y,z}(t) = \frac{t}{t+\xi\left[\|x\|^p_{\mathcal{A}}+\|y\|^p_{\mathcal{A}}+\|z\|^p_{\mathcal{A}}\right]}$, $\psi_{x,y}(t) = \frac{t}{t+\xi\left[\|x\|^p_{\mathcal{A}}+\|y\|^p_{\mathcal{A}}\right]}$ and let $L = |2|^{p-1}$ in the Theorem (2).
Then there exists a sequence $H = \{h_0, h_1, \dots, h_n, \dots\}$ with the required properties.

Similar to Theorem (2), we can prove the following theorem:

Theorem 4. *Let $\varphi : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \to \mathcal{D}^+$ and $\psi : \mathcal{A}\times\mathcal{A} \to \mathcal{D}^+$ be functions. Assume that $F = \{f_0, f_1, \dots, f_n, \dots\}$ be a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$, $f_n(0) = 0$,*

$$\begin{aligned} &\mu^{\mathcal{B}}_{D_\nu f_n(x,y,z)}(t) \geq \varphi_{x,y,z}(t), \\ &\mu^{\mathcal{B}}_{f_n(x^*)-f_n(x)^*}(t) \geq \varphi_{x,0,0}(t), \\ &\mu^{\mathcal{B}}_{f_n(xy)-\sum_{i=0}^n f_i(x)f_{n-i}(y)}(t) \geq \psi_{x,y}(t) \end{aligned} \tag{10}$$

for all $\nu \in \mathbb{T}^1$, all $x, y, z \in \mathcal{A}$ and all $t > 0$. Suppose that $|2| < 1$ is far from zero and there exists an $0 \leq L < 1$ such that

$$\varphi_{\frac{x}{2},\frac{y}{2},\frac{z}{2}}(\frac{L}{|2|}t) \geq \varphi_{x,y,z}(t), \tag{11}$$

$$\psi_{\frac{x}{2},\frac{y}{2}}(\frac{L}{|2|^2}t) \geq \psi_{x,y}(t) \tag{12}$$

for all $x, y, z \in \mathcal{A}$ and $t > 0$. Then there exists a unique higher $*$-derivation $H = \{h_0, h_1, \dots, h_n, \dots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$,

$$\mu^{\mathcal{B}}_{f_n(x)-h_n(x)}(t) \geq \varphi_{x,2x,x}\Big(\frac{|2|(1-L)}{L}t\Big) \tag{13}$$

holds for all $x \in \mathcal{A}$ and $t > 0$.

Proof. Fix $n \in \mathbb{N}_0$. Putting $\nu = 1$ in (10). Let $\mathcal{Z}$ be the set of all functions $g : \mathcal{A} \to \mathcal{B}$. We define the metric d on $\mathcal{Z}$ as in the proof of Theorem (2). One has the operator $J : \mathcal{Z} \to \mathcal{Z}$ by $(Jh)(x) = 2h(\frac{x}{2})$ for all $h \in \mathcal{Z}$. For arbitrarily elements $f, g \in \mathcal{Z}$, we have

$$\begin{aligned} d(f,g) < k &\Rightarrow \mu^{\mathcal{B}}_{f(x)-g(x)}(kt) > \varphi_{x,2x,x}(t) \\ &\Rightarrow \mu^{\mathcal{B}}_{f(\frac{x}{2})-g(\frac{x}{2})}(kt) > \varphi_{\frac{x}{2},x,\frac{x}{2}}(t) \\ &\Rightarrow \mu^{\mathcal{B}}_{2f(\frac{x}{2})-2g(\frac{x}{2})}(kt) > \varphi_{\frac{x}{2},x,\frac{x}{2}}(\frac{1}{|2|}t) \\ &\Rightarrow \mu^{\mathcal{B}}_{2f(\frac{x}{2})-2g(\frac{x}{2})}(kLt) > \varphi_{x,2x,x}(t) \Rightarrow d(Jf,Jg) < kL. \end{aligned}$$

Thus, J is a contraction with the Lipschitz constant L. Now, by Theorem (1) there exists a unique mapping $h_n : \mathcal{A} \to \mathcal{B}$ such that h_n is a fixed point of J that is $2h_n(\frac{x}{2}) = h_n(x)$ for all $x \in \mathcal{A}$. By Theorem (1),

$$\lim_{m\to\infty} 2^m f_n(\frac{x}{2^m}) = h_n(x)$$

for all $x \in \mathcal{A}$. By Theorem (1), (8) and (11), we have

$$\mu^{\mathcal{B}}_{f_n(x)-2f_n(\frac{x}{2})}(\frac{L}{|2|}t) \geq \varphi_{x,2x,x}(t) \Rightarrow d(f_n, Jf_n) \leq \frac{L}{|2|} < \infty$$

for all $x \in \mathcal{A}$ and all $t > 0$. This implies that

$$d(f_n, h_n) \leq \frac{1}{1-L} d(f_n, Jf_n) \leq \frac{L}{|2|(1-L)},$$

that is

$$\mu^{\mathcal{B}}_{f_n(x)-h_n(x)}(t) \geq \varphi_{x,2x,x}\Big(\frac{|2|(1-L)}{L}t\Big)$$

for all $x \in \mathcal{A}$ and all $t > 0$. The rest of the proof is similar to that of the proof of Theorem (2).

The following corollary is similar to Corollary (3) for the case where $0 \leq p < 1$.

Corollary 5. *Let $0 \leq p < 1$, ξ be nonnegative real number and let $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mapping from $\mathcal{A}$ into $\mathcal{B}$ such that $f_n(0) = 0$ and*

$$\mu^{\mathcal{B}}_{D_\nu f_n(x,y,z)}(t) \geq \frac{t}{t + \xi\Big[\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}}\Big]},$$

$$\mu^{\mathcal{B}}_{f_n(x^*)-f_n(x)^*}(t) \geq \frac{t}{t + \xi\|x\|^p_{\mathcal{A}}},$$

$$\mu^{\mathcal{B}}_{f_n(xy)-\sum_{i=0}^{n} f_i(x)f_{n-i}(y)}(t) \geq \frac{t}{t + \xi\Big[\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}}\Big]}$$

for all $\nu \in \mathbb{T}^1$, all $x, y, z \in \mathcal{A}$ and all $t > 0$. Then there exists a unique higher $*$-derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$,

$$\mu^{\mathcal{B}}_{f_n(x)-h_n(x)}(t) \geq \frac{\Big(|2|^p - |2|\Big)t}{\Big(|2|^p - |2|\Big)t + \xi\Big(|2| + |2|^p\Big)\|x\|^p_{\mathcal{A}}}$$

holds for all $x \in \mathcal{A}$ and $t > 0$.

Proof. Let $\varphi_{x,y,z}(t) = \frac{t}{t+\xi\left[\|x\|^p_{\mathcal{A}}+\|y\|^p_{\mathcal{A}}+\|z\|^p_{\mathcal{A}}\right]}$, $\psi_{x,y}(t) = \frac{t}{t+\xi\left[\|x\|^p_{\mathcal{A}}+\|y\|^p_{\mathcal{A}}\right]}$ and let $L = |2|^{1-p}$ in the Theorem (4).
Then there exists a sequence $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ with the required properties.

4. Lie higher $*$-derivations in non-Archimedean random Lie C^*-algebras

A non-Archimedean random C^*-algebra $\mathcal{N}$, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$ on $\mathcal{N}$, is called a non-Archimedean random Lie C^*-algebra.

Definition 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean random Lie C^*-algebras. Let $\mathbb{N}$ be the set of natural numbers. Form $m \in \mathbb{N} \cup \{0\}$, a sequence $H = \{h_0, h_1, \ldots, h_m\}$ (resp. $H = \{h_0, h_1, \ldots, \ldots\}$) of mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a Lie higher $*$-derivation of rank m (resp. infinite rank) from $\mathcal{A}$ into $\mathcal{B}$ if*
(i) $f_n(x^*) = (f_n(x))^*$, for all $x \in \mathcal{A}$ and for each $n \in \{0, 1, \ldots, m\}$ (resp. $n \in \mathbb{N}_0$.)
(ii) $f_n[x, y] = \sum_{i=0}^{n}[f_i(x), f_{n-i}(y)]$ holds for each $n \in \{0, 1, \ldots, m\}$ (resp. $n \in \mathbb{N}_0$) and all $x, y \in \mathcal{A}$.

In this section, assume that $\mathcal{A}$ is a non-Archimedean random Lie C^*-algebra with norm $\mu^{\mathcal{A}}$ and that $\mathcal{B}$ is a non-Archimedean random Lie C^*-algebra with norm $\mu^{\mathcal{B}}$. We are going to investigate the generalized Hyers-Ulam stability of Lie higher $*$-derivations in non-Archimedean random Lie C^*-algebras for the functional equation $D_\nu f_n(x, y, z) = 0$.

Theorem 6. *Let $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{D}^+$ and $\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{D}^+$ be functions such that (2) and (3) hold. Suppose that $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$, $f_n(0) = 0$,*

$$\mu^{\mathcal{B}}_{f_n([x,y])-\sum_{i=0}^{n}[f_i(x),f_{n-i}(y)]}(t) \geq \psi_{x,y}(t) \tag{14}$$

for all $\nu \in \mathbb{T}^1$, all $x, y, z \in \mathcal{A}$ and all $t > 0$. Assume that $|2| < 1$ is far from zero and there exists an $0 \leq L < 1$ and (5), (6) hold. Then there exists a unique Lie higher $*$-derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$, (7) holds.

Proof. By the same reasoning as in the proof of Theorem (2), there is a mapping $h_n : \mathcal{A} \to \mathcal{B}$ which is $*$-preserving for each $n \in \mathbb{N}_0$ and satisfy (7). The mapping $h_n : \mathcal{A} \to \mathcal{B}$ is given by

$$h_n(x) = \lim_{m\to\infty} \frac{f_n(2^m x)}{2^m}$$

for all $x \in \mathcal{A}$. By (6) and (14),

$$\mu^{\mathcal{B}}_{f_n(2^{2m}[x,y])-\sum_{i=0}^{n}[f_i(2^m x), f_{n-i}(2^m y)]}(|2|^{2m}t) \\ \geq \psi_{2^m x, 2^m y}(|2|^{2m}t) \to 1 \qquad when \ \ m \to \infty$$

for all $x, y \in \mathcal{A}$ and all $t > 0$. Therefore,

$$h_n[x,y] = \sum_{i=0}^{n}[h_i(x), h_{n-i}(y)]$$

for all $x, y \in \mathcal{A}$. Thus $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ is Lie higher $*$-derivation.

Corollary 7. *Let $p > 1$, ξ be nonnegative real numbers and let $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$, $f_n(0) = 0$,*

$$\begin{aligned} &\mu^{\mathcal{B}}_{D_\nu f_n(x,y,z)}(t) \geq \frac{t}{t + \xi\Big[\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}}\Big]}, \\ &\mu^{\mathcal{B}}_{f_n(x^*)-f_n(x)^*}(t) \geq \frac{t}{t + \xi\|x\|^p_{\mathcal{A}}}, \\ &\mu^{\mathcal{B}}_{f_n([x,y])-\sum_{i=0}^{n}[f_i(x), f_{n-i}(y)]}(t) \geq \frac{t}{t + \xi\Big[\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}}\Big]} \end{aligned} \tag{15}$$

for all $\nu \in \mathbb{T}^1$, all $x, y, z \in \mathcal{A}$ and all $t > 0$. Then there exists a unique Lie higher $*$-derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$,

$$\mu^{\mathcal{B}}_{f_n(x)-h_n(x)}(t) \geq \frac{\Big(|2| - |2|^p\Big)t}{\Big(|2| - |2|^p\Big)t + \xi\Big(|2| + |2|^p\Big)\|x\|^p_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. The proof follows from Theorem (6) by taking $\varphi_{x,y,z}(t) = \frac{t}{t+\xi\Big[\|x\|^p_{\mathcal{A}}+\|y\|^p_{\mathcal{A}}+\|z\|^p_{\mathcal{A}}\Big]}$, $\psi_{x,y}(t) = \frac{t}{t+\xi\Big[\|x\|^p_{\mathcal{A}}+\|y\|^p_{\mathcal{A}}\Big]}$ for all $x, y, z \in \mathcal{A}$ and $t > 0$. Then $L = |2|^{p-1}$ and we get the desired result.

Theorem 8. *Let $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{D}^+$ and $\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{D}^+$ be functions such that (2) and (3) hold. Suppose that $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ satisfying $f_n(0) = 0$, for each $n \in \mathbb{N}_0$, and (14). Assume that $|2| < 1$ is far from zero and there exists an $0 \leq L < 1$ and (11), (12) hold. Then there exists a unique Lie higher $*$-derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$, (13) holds.*

Corollary 9. *Let $0 \leq p < 1$, ξ be nonnegative real numbers and let $F = \{f_0, f_1, \dots, f_n, \dots\}$ be a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that satisfying $f_n(0) = 0$, for each $n \in \mathbb{N}_0$, and (15). Then there exists a unique Lie higher $*$-derivation $H = \{h_0, h_1, \dots, h_n, \dots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$,*

$$\mu^{\mathcal{B}}_{f_n(x)-h_n(x)}(t) \geq \frac{\Big(|2|^p - |2|\Big)t}{\Big(|2|^p - |2|\Big)t + \xi\Big(|2| + |2|^p\Big)\|x\|^p_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. The proof follows from Theorem (8) by taking $\varphi_{x,y,z}(t) = \frac{t}{t+\xi\left[\|x\|^p_{\mathcal{A}}+\|y\|^p_{\mathcal{A}}+\|z\|^p_{\mathcal{A}}\right]}$, $\psi_{x,y}(t) = \frac{t}{t+\xi\left[\|x\|^p_{\mathcal{A}}+\|y\|^p_{\mathcal{A}}\right]}$ for all $x, y \in \mathcal{A}$ and $t > 0$. Then $L = |2|^{1-p}$ and we get the desired result.

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