Acta Universitatis Apulensis No. 42/2015
ISSN: 1582-5329 pp. 201-214
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.42.13

MINIMIZING POLYNOMIALS ON COMPACT SETS

V.T. Phan

Abstract. In this paper, the problem of minimizing a polynomial $g_* = \inf_{x\in S(F)} g(x)$ in the compact case is investigated. It is known that such problem is severely ill-posed. We use results of positive performed theorems of Putinar ([8]) and Schmüdgen ([9]) to solve it. A numerical example is given to illustrate the efficiency of the proposed method works.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: global optimization, polynomials, sum of squares, semidefinite programming.

1. Introduction

Given a polynomial function $g \in \mathbb{R}[x] = \mathbb{R}[x_1, x_2, \ldots, x_n]-$ the polynomial ring. Fix a finite subset $F = \{f_1, f_2, \ldots, f_m\} \subset \mathbb{R}[x]$. Denote

$$S(F) := \{x \in \mathbb{R}^n |\, f_i(x) \geq 0, i = 1, \ldots, m\}$$

is the basic closed semialgebraic set generated by F. We consider the problem of minimizing a polynomial g on $S(F)$: $g_* = \inf_{x\in S(F)} g(x)$. (*)

Finding the optimal solution of the problem (*) is NP-hard problem (see [2], [4]). Based on the results of performing non-negative polynomials on the semi algebraic sets, some authors (eg, [1], [3], [7], ...) have developed a series of positive semidefinited programming ((SDP for short) (see [2], [4]) which their optimal values converge monotonically increasing to the optimum value of the problem (*). The idea traces back to work of Shor 1987 ([12]) and is further developed by Parrilo 2000 ([6]), by Lasserre 2001 ([1])and by Parrilo and Sturmfels 2003 ([7]).

In [1] Lasserre describes an extension of the method to minimizing a polynomial on an arbitrary basic closed semialgebraic set and uses a result due to Putinar ([8]) to prove that the method produces the exact minimum in the compact case. In the

general case it produces a lower bound for the minimum. However, the assumption that S is compact set is strict and not to be missed in the methods of Lasserre.
The purpose of this paper is to introdue the problem of minimizing a polymonial $g_* = \inf_{x \in S(F)} g(x)$ in the compact case. Uses results of positive performed theorems of Putinar ([8]) and Schmüdgen ([9]), we will build a series of positive semidefinited programming which their optimal values converge monotonically increasing to the optimum value g_*.

2. Preliminaries

Given a finite subset $F = \{f_1, f_2, \dots, f_m\} \subset \mathbb{R}[x]$. Denote

$$S(F) := \{x \in \mathbb{R}^n | f_i(x) \geq 0, i = 1, \dots, m\}$$

is the basic closed semialgebraic set generated by F;

$$M(F) := \left\{\sigma_0 + \sigma_1 f_1 + \cdots + \sigma_m f_m \,|\, \sigma_i \in \sum \mathbb{R}[x]^2\right\}$$

is the quadratic module in $\mathbb{R}[x]$;

$$P(F) := \left\{ \sum_{e \in \{0,1\}^m} \sigma_e f^e \,|\, \sigma_e \in \sum \mathbb{R}[x]^2, \forall e \in \{0,1\}^m \right\}$$

is the preordering generated by F.

Property 1. $M(F)$ is the quadratic module, that is

$$M(F) + M(F) \subset M(F),\ a^2 M(F) \subset M(F), \forall a \in \mathbb{R}[x] \text{ and } 1 \in M(F).$$

Property 2. $P(F)$ is the preordering, that is

$$P(F) + P(F) \subset P(F),\ P(F).P(F) \subset P(F) \text{ and } a^2 \in P(F), \forall a \in \mathbb{R}[x].$$

Definition 1. $M(F)$ is archimedean if $\exists\, k \geq 1 \,|\, k - \sum_{i=1}^{n} x_i^2 \in M(F)$.

Example 1. Take $n = 1, F = \{-x^2\} \subset \mathbb{R}[x]$. We have

$$M(F) = \{\sigma_0 - \sigma_1 x^2 \,|\, \sigma_i \in \sum \mathbb{R}[x]^2\}.$$

Take $k = 1$. Then $k - x^2 = 1 - x^2 \in M(F)$. Thus $M(F)$ is archimedean.

Example 2. Take $n = 2, F = \{x - \frac{1}{2}, y - \frac{1}{2}, 1 - xy\} \subset \mathbb{R}[x, y]$. Then

$$M(F) = \left\{ \sigma_0 + \sigma_1(x - \frac{1}{2}) + \sigma_2(y - \frac{1}{2}) + \sigma_3(1 - xy) \mid \sigma_i \in \sum \mathbb{R}[x, y]^2 \right\}.$$

We be alble to build quadratic module $Q \subset \mathbb{R}[x, y]$ ([4, Example 7.3.1]) which satisfies

$$\begin{cases} Q \cup -Q = \mathbb{R}[x, y], Q \cap -Q = \{0\}, \\ x - \frac{1}{2}, y - \frac{1}{2}, 1 - xy \in Q, (\texttt{ to } M(G) \subset Q), \\ k - (x^2 + y^2) \notin Q, \forall k \in \mathbb{Z}, k \geq 1. \end{cases}$$

Then $M(F) \subset Q$, $k - (x^2 + y^2) \notin Q, \forall k \in \mathbb{Z}, k \geq 1$, and

$$k - (x^2 + y^2) \notin M(F), \forall k \in \mathbb{Z}, k \geq 1.$$

Thus $M(F)$ is not archimedean.

Theorem 1. *([9]) Suppose $S(F)$ is compact and $g \in \mathbb{R}[x]$. If $g > 0$ on $S(F)$, then $g \in P(F)$.*

Theorem 2. *([8]) Suppose $M(F)$ is archimedean and $g \in \mathbb{R}[x]$. If $g > 0$ on $S(F)$, then $g \in M(F)$.*

Remark 1. If $M(F)$ is archimedean, then $S(F)$ is compact.

The opposite of Remark 1 is not true. For example, we consider Example 2, we have

$$S(F) = \{(x, y) \in \mathbb{R}^2 \mid x - \frac{1}{2} \geq 0, y - \frac{1}{2} \geq 0, 1 - xy \geq 0\}$$

is compact, and

$$M(F) = \{\sigma_0 + \sigma_1(x - \frac{1}{2}) + \sigma_2(y - \frac{1}{2}) + \sigma_3(1 - xy) \mid \sigma_i \in \sum \mathbb{R}[x, y]^2\}$$

is not archimedean.

3. Semidefinited programming (SDP)

The problem SDP:

$$\begin{cases} \inf \sum_{i=1}^{n} c_i x_i, \\ G(x) := G_0 + x_1 G_1 + \cdots + x_n G_n \succeq 0, \end{cases} \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and $G_i \in Sym(\mathbb{R}^{d \times d})$ is the symmetric matrix $(i = 0, \ldots, n)$.

Remark 2. Problem (1) can not achieve min. This can be seen in the following example.

Example 3. Consider the problem SDP

$$\begin{cases} \inf\ x_1, \\ \begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix} \succeq 0. \end{cases}$$

We have $n = d = 2, c^T x = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and

$$F(x) = \begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{F_0} + x_1 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{F_1} + x_2 \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{F_2}.$$

Consider the equation $\det \begin{pmatrix} x_1 - \lambda & 1 \\ 1 & x_2 - \lambda \end{pmatrix} = 0, \lambda \in \mathbb{R}$. Reduced, we obtain

$$\lambda^2 - (x_1 + x_2)\lambda + x_1 x_2 - 1 = 0. \tag{2}$$

The condition $G(x) \succeq 0$ is equivalent to eigenvalues of matric $G(x)$ is non negative. This is equivalent to Equation (2) has two non negative solutions, that is $S = \frac{-b}{a} = x_1 + x_2 \geq 0$ and $P = \frac{c}{a} = x_1 x_2 - 1 \geq 0$. Then $x_1 > 0, x_2 > 0$ and the objective function $c^T x = x_1$ can not achieve min on $\{x = (x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 + x_2 \geq 0, x_1 x_2 - 1 \geq 0\}$, and $p_* = 0$.

The dual problem (DP for short) of (1) is

$$\begin{cases} \sup\ -\langle G_0, Z \rangle, \\ \langle G_i, Z \rangle = c_i, i = 1, \ldots, n, \\ Z \succeq 0. \end{cases} \tag{3}$$

Remark 3.

- *SDP and DP are convex optimization problems. Using the polynomial algorithm to solve them.*
- *Opt - value (SDP)* $\geq$ *Opt - value (DP).*

4. The case $M(F)$ is archimedean

For $g \in \mathbb{R}[x]$ and $S(F)$ is the basic closed semialgebraic set generated by F, we consider the problem

$$g_* := \inf \{g(x) \mid x \in S(F)\}.$$

Remark 4. This is NP-hard problem. There is no efficient algorithm to solve it, unless the case g is linear, $S(F)$ is convex polyhedron, then using the simplex algorithm to solve it.

Remark 5. For $\gamma \in \mathbb{R}$, test $g - \gamma \geq 0$ on $S(F)$ is generally difficult. However, test $g - \gamma \in M(F)$ can do (using SDP).

Property 3. $\sup\limits_{g-\gamma \in M(F)} \gamma \leq g_*$.

Fix a positive integer $N \geq \deg g$. Denote

$$M_N(F) := \left\{ \sum_{i=0}^{m} \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[x]^2, \deg(\sigma_i g_i) \leq N, i = 0, \ldots, m \right\},$$

$$\chi_N := \{L : \mathbb{R}[x]_N \to \mathbb{R} \texttt{ linear} \mid L(1) = 1, \texttt{ and } L \geq 0 \texttt{ on } M_N(F)\},$$

$$g_{+,N} := \inf\{L(g) \mid L \in \chi_N\}, \tag{4}$$

$$g_N^* := \sup\{\gamma \in \mathbb{R} \mid g - \gamma \in M_N(F)\}. \tag{5}$$

Proposition 1. *([3])*

(a) $g_N^* \leq g_{+,N} \leq g_*$.

(b) $g_{+,N} \leq g_{+,N+1}; g_N^* \leq g_{N+1}^*$.

(c) If $M(F)$ is archimedean, then $\lim\limits_{N\to\infty} g_N^* = g_*$. *Hence* $\lim\limits_{N\to\infty} g_{+,N} = g_*$.

Proposition 2. *Problem (4) is SDP.*

Proof. Without loss generality, we assume $f_i \not\equiv 0$ and $\deg f_i \leq N, i = 1, \ldots, m$. Because if $\deg(\sigma_i f_i) \leq N$ and $\deg f_i > N$, then $\sigma_i = 0$, so $\sigma_i f_i = 0$: not have any contribution to $M_N(F)$. We see $\mathbb{R}[x]_N$ generated by the basic set $\{x^\alpha \mid |\alpha| \leq N\}$, number of elements of that basic is C_{n+N}^N. We consider linear mapping

$$L : \mathbb{R}[x]_N \longrightarrow \mathbb{R}, L(p) = L\left(\sum_{|\alpha|\leq N} p_\alpha x^\alpha\right) = \sum_{|\alpha|\leq N} p_\alpha L(x^\alpha).$$

Putting $y_\alpha = L(x^\alpha)$, $|\alpha| \leq N$ then L corresponds to a vector $(y_\alpha), |\alpha| \leq N, y_\alpha \in \mathbb{R}$. We have $y_0 = 1$. $L \geq 0$ on $M_N(F)$ is equivalent to

$$L\left(\sum_{i=0}^{m} \sigma_i f_i\right) \geq 0, \sigma_i \in \sum \mathbb{R}[x]^2, \deg(\sigma_i f_i) \leq N,$$

or

$$\sum_{i=0}^{m} L(\sigma_i f_i) \geq 0, \sigma_i \in \sum \mathbb{R}[x]^2, \deg(\sigma_i f_i) \leq N,$$

or

$$L(\sigma_i f_i) \geq 0, \forall i, \sigma_i \in \sum \mathbb{R}[x]^2, \deg(\sigma_i f_i) \leq N,$$

or

$$L(p^2 f_i) \geq 0, p \in \mathbb{R}[x], \deg p \leq \frac{N - \deg(f_i)}{2}.$$

Test

$$\deg p \leq \frac{N - \deg f_i}{2}.$$

Indeed, since $p^2 f_i \in M_N(F)$ we have

$$\deg(p^2 f_i) \leq N,$$

or

$$\deg p^2 + \deg f_i \leq N,$$

or

$$2 \deg p + \deg f_i \leq N,$$

or

$$\deg p \leq \frac{N - \deg f_i}{2}.$$

We write $g = \sum\limits_{|\alpha| \leq N} g_\alpha x^\alpha$, thus

$$L(g) = \sum_{|\alpha| \leq N} g_\alpha L(x^\alpha) = \sum_{|\alpha| \leq N} g_\alpha y_\alpha = g_0 + \sum_{|\alpha| \leq N, \alpha \neq 0} g_\alpha y_\alpha.$$

If $p = \sum\limits_{\alpha} p_\alpha x^\alpha$, then $p^2 = \sum\limits_{\alpha, \beta} p_\alpha p_\beta x^{\alpha+\beta}$, therefore

$$L(p^2) = \sum_{\alpha, \beta} p_\alpha p_\beta L(x^{\alpha+\beta}) = \sum_{\alpha, \beta} p_\alpha p_\beta y_{\alpha+\beta}.$$

We write $f_i = \sum_\gamma f_{i\gamma} x^\gamma$. Similar to the above, we have

$$p^2 f_i = \sum_{\alpha,\,\beta} p_\alpha p_\beta x^{\alpha+\beta} f_i = \sum_{\alpha,\,\beta,\,\gamma} p_\alpha p_\beta f_{i\gamma} x^{\alpha+\beta+\gamma}$$

v$L(p^2 f_i) = \sum_{\alpha,\,\beta,\,\gamma} p_\alpha p_\beta f_{i\gamma} y_{\alpha+\beta+\gamma} = \sum_{\alpha,\,\beta} \left(\sum_\gamma f_{i\gamma} y_{\alpha+\beta+\gamma} \right) p_\alpha p_\beta$. Putting

$$M(f_i * y) = \left(\sum_\gamma f_{i\gamma} y_{\alpha+\beta+\gamma} \right)_{\alpha,\,\beta}.$$

Then, $M(f_i * y)$ is the matrix which size is $D_i \times D_i$, where

$$D_i = \#\{\alpha \,|\, |\alpha| \le \frac{N - \deg f_i}{2}\}.$$

Note that $M(1 * y) = M(y)$. Then

$$L(p^2 f_i) = \sum_{\alpha,\,\beta} \left(\sum_\gamma f_{i\gamma} y_{\alpha+\beta+\gamma} \right) p_\alpha p_\beta = p^T M(f_i * y) p.$$

Therefore, condition $L(p^2 f_i) \ge 0$ is equivalent to $p^T M(f_i * y) p \ge 0$. This is equivalent to $M(f_i * y) \succeq 0$. Thus

$$L \in \chi_N \Leftrightarrow \begin{cases} L(1) = 1, \\ L \ge 0 \ \ \texttt{on} \ \ M_N(G) \end{cases} \Leftrightarrow \begin{cases} y_0 = 1, \\ M(f_i * y) \succeq 0, i = 0, \dots, m. \end{cases}$$

Putting $G(y) := diag(M(f_i * y), \dots, M(f_i * y))$. The size of the matrix $G(y)$ is $\sum_{1=0}^m D_i \times \sum_{1=0}^m D_i$. Then,

$$\begin{cases} y_0 = 1, \\ M(f_i * y) \succeq 0, i = 0, \dots, m \end{cases} \Leftrightarrow \begin{cases} y_0 = 1, \\ G(y) \succeq 0. \end{cases}$$

For $|\alpha| \le N$, we define $e^{(\alpha)} := (e_\beta^{(\alpha)})$, where

$$e_\beta^{(\alpha)} := \begin{cases} 0, & \texttt{if} \ \ \beta \ne \alpha \\ 1, & \texttt{if} \ \ \beta = \alpha. \end{cases}$$

So $\{e^{(\alpha)}, \alpha \neq 0\}$ is basic vector of freedom variables space $y = (y_\alpha), |\alpha| \leq N, \alpha \neq 0$, that is $y = \sum y_\alpha e^{(\alpha)}, \forall\, y = (y_\alpha), |\alpha| \leq N, \alpha \neq 0$. Then $G(y) = G_0 + \sum\limits_{|\alpha|\leq N, \alpha\neq 0} y_\alpha G^\alpha$, $G_\alpha := G(e^{(\alpha)})$, and

$$\begin{cases} y_0 = 1, \\ G(y) \succeq 0 \end{cases} \Leftrightarrow \begin{cases} y_0 = 1, \\ G_0 + \sum\limits_{|\alpha|\leq N, \alpha\neq 0} y_\alpha G^\alpha \succeq 0. \end{cases}$$

So $g_{+,N} := \inf\{L(g)|L \in \chi_N\} = \inf\{g_0 + \sum\limits_{\alpha\neq 0} g_\alpha y_\alpha\} = g_0 + \inf \sum\limits_{\alpha\neq 0} g_\alpha y_\alpha$. We see that problem calculate $g_{+,N}$ with constrain $L \in \chi_N$ same as problem calculate $g_0 + \inf \sum\limits_{\alpha\neq 0} g_\alpha y_\alpha$ with constrain

$$\begin{cases} y_0 = 1, \\ G_0 + \sum\limits_{|\alpha|\leq N, \alpha\neq 0} y_\alpha G^\alpha \succeq 0, \end{cases}$$

or with constrain $G(y) \succeq 0$. Therefore Problem (4) is SDP.

Proposition 3. *Problem (5) is duality of Problem (4).*

Proof. Take $\gamma \in \mathbb{R}$ so that $g - \gamma = \sigma_0 + \sigma_1 f_1 + \cdots + \sigma_m f_m$, where

$$\sigma_i \in \sum \mathbb{R}[x]^2, \deg \sigma_i \leq \frac{N - \deg f_i}{2}, i = 0, \ldots, m.$$

For $\sigma_i \in \sum \mathbb{R}[x]^2$, there exists a positive semidefinite (PSD for short) matrix which size is $D_i \times D_i : A^{(i)} = (A^{(i)}_{\delta\beta})_{\delta,\beta}$ so that $\sigma_i = \sum\limits_{\delta,\beta} A^{(i)}_{\delta\beta} x^{\delta+\beta}$. Then

$$g - \gamma = \sum_{i=0}^{m} \sigma_i f_i = \sum_{i=0}^{m} \sum_{\delta,\beta} A^{(i)}_{\delta\beta} x^{\delta+\beta} f_i.$$

We write $f_i = \sum\limits_{\gamma} f_{i\gamma} x^\gamma$. Then

$$g - \gamma = \sum_{i=0}^{m} \sum_{\delta,\beta} \sum_{\gamma} A^{(i)}_{\delta\beta} f_{i\gamma} x^{\delta+\beta+\gamma}.$$

For

$$g = \sum_{\alpha} g_\alpha x^\alpha = g_0 + \sum_{\alpha\neq 0} g_\alpha x^\alpha,$$

we have

$$g_0 + \sum_{\alpha \neq 0} g_\alpha x^\alpha - \gamma = \sum_{i=0}^{m} \sum_{\delta,\beta} \sum_{\gamma} A^{(i)}_{\delta\beta} f_{i\gamma} x^{\delta+\beta+\gamma},$$

or

$$g_0 - \gamma + \sum_{\alpha \neq 0} f_\alpha x^\alpha = \sum_{i=0}^{m} \sum_{\delta,\beta} \sum_{\gamma} A^{(i)}_{\delta\beta} f_{i\gamma} x^{\delta+\beta+\gamma}.$$

Identify coefficients two sides the above equation, we get

$$\begin{cases} g_0 - \gamma = \sum\limits_{i=0}^{m} A^{(i)}_{00} f_{i0} = \langle G_0, A\rangle, \\ g_\alpha = \sum\limits_{i=0}^{m} \sum\limits_{\delta+\beta+\gamma=\alpha} A^{(i)}_{\delta\beta} f_{i\gamma} = \langle G_\alpha, A\rangle, \texttt{for } \alpha \neq 0, \end{cases}$$

where $A := diag(A^{(0)}, \dots, A^{(m)}), G_\alpha := G(e^{(\alpha)})$. We have A is PSD and

$$\begin{aligned} g_N^* &= \sup\{\gamma \,|\, g - \gamma \in M_N(F)\} \\ &= \sup\{g_0 - \langle G_0, A\rangle | A \succeq 0, g_\alpha = \langle G_\alpha, A\rangle, \alpha \neq 0\} \\ &= g_0 + \sup\{-\langle G_0, A\rangle | A \succeq 0, g_\alpha = \langle G_\alpha, A\rangle, \alpha \neq 0\}. \end{aligned}$$

Thus, Problem (5) is duality of Problem (4).

Remark 6. Exist $g \in \mathbb{R}[x]$ such that $g^{sos} < g_*$. For instance, we consider some the following examples.

Example 4. [5, 6.2].
(1) Take $g(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \in \mathbb{R}[x,y]$. Then

$$g_* = 0, g^{sos} = -\infty.$$

(2) Take $g(x,y) = x^4 + x^2 + y^6 - 3x^2y^2 \in \mathbb{R}[x,y]$. Then

$$g_* = 0, g^{sos} = -729/4096.$$

Remark 7. Can happen case $g_N^* \neq g_{+,N}$. However, if $M(F) \cap -M(F) = \{0\}$, then $g_N^* = g_{+,N}$. (See [4, Proporition 10.5.1]).

Example 5. [2, Problem 4.6, 4.7] We consider the optimization problem

$$\begin{cases} \inf\limits_x g(x) := -x_1 - x_2, \\ x_2 \leq 2x_1^4 - 8x_1^3 + 8x_1^2 + 2, \\ x_2 \leq 4x_1^4 - 32x_1^3 + 88x_1^2 - 96x_1 + 36, \\ 0 \leq x_1 \leq 3, 0 \leq x_2 \leq 4. \end{cases}$$

Then $g_4^* = g_* = -5.5079$.

Example 6. [2, Problem 4.6, 4.7] We consider the optimization problem

$$\begin{cases} \inf_x g(x) := -12x_1 - 7x_2 + x_2^2, \\ -2x_1^4 + 2 - x_2 = 0, \\ 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 3. \end{cases}$$

Then $g_5^* = g_* = -16.73889$.

5. The case $M(F)$ is not archimedean

We have the same results as above if we replace the quadratic module $M_N(F)$ by the preordering

$$P_N(F) := \left\{ \sum_{e\in\{0,1\}^m} \sigma_e f^e | \sigma_e \in \sum \mathbb{R}[x]^2, \deg \sigma_e f^e \leq N, e \in \{0,1\}^m \right\}.$$

We denote

$$\chi_N := \{L : \mathbb{R}[x]_N \to \mathbb{R} \ \texttt{linear} \ | L(1) = 1 \, \texttt{and} \, L \geq 0 \ \texttt{on} \ P_N(F)\},$$

$$g_{+,N} := \inf\{L(g) \,|\, L \in \chi_N\}, \quad (6)$$

$$g_N^* := \sup\{\gamma \in \mathbb{R} \,|\, g - \gamma \in P_N(F)\}. \quad (7)$$

Proposition 4.

(a) $g_N^* \leq g_{+,N} \leq g_*$.

(b) $g_{+,N} \leq g_{+,N+1}; g_N^* \leq g_{N+1}^*$.

(c) If $S(F)$ is compact, then $\lim_{N\to\infty} g_N^* = g_*$. *Hence* $\lim_{N\to\infty} g_{+,N} = g_*$.

Proof. (a)We prove $g_{+,N} \leq g_*$. Taking arbitrary $a \in S(F)$, define

$$L_a \colon \mathbb{R}[x]_N \to \mathbb{R}, L_a(q) = q(a).$$

We have $L_a(1) = 1$, $L_a\left(\sum_{e\in\{0,1\}^m} \sigma_e f^e\right) = \sum_{e\in\{0,1\}^m} L_a(\sigma_e f^e) = \sum_{e\in\{0,1\}^m} \sigma_e f_e(a) \geq 0$. Then $L_a \in \chi_N$. Because

$$g_{+,N} := \inf\{L(g) | L \in \chi_N\},$$

we get

$$g_{+,N} \leq L_a(g) = g(a).$$

By $a \in S(F)$ is arbitrary, we have

$$g_{+,N} \leq \inf_{a \in S(F)} g(a) = g_*.$$

Next, we prove $g_N^* \leq g_{+,N}$. Take $\gamma \in \mathbb{R}$ such that $g - \gamma \in P_N(F)$ and $L \in \chi_N$ is arbitrary. We have

$$0 \leq L(g - \gamma) = L(g) - L(\gamma) = L(g) - \gamma.$$

Then $L(g \geq \gamma$. Therefore

$$\inf\{L(g) \,|\, L \in \chi_N\} \geq \sup\{\gamma \in \mathbb{R} \,|\, g - \gamma \in P_N(F)\},$$

that is $g_{+,N} \geq g_N^*$.

(b) We have $P_N(F) \subseteq P_{N+1}(F)$ and $\chi_{N+1} \subseteq \chi_N$. Take $\gamma \in \mathbb{R}$ such that

$$g - \gamma \in P_N(F),$$

we get $g - \gamma \in P_{N+1}(F)$. Thus $g_N^* \leq g_{N+1}^*$.

Next, we prove $g_{+,N} \leq g_{+,N+1}$. Take $L \in \chi_{N+1}$ is abitrary. Put

$$L' := L_{|\mathbb{R}[x]_N},$$

then $L' \in \chi_N$ and $L'(g) = L(g)$. Therefore

$$\inf\{L(g) \,|\, L \in \chi_N\} \leq \inf\{L(g) \,|\, L \in \chi_{N+1}\},$$

that is $g_{+,N} \leq g_{+,N+1}$.

(c) Take $\gamma \in \mathbb{R}, \gamma < g_*$. We have $g - \gamma > 0$ on $S(G)$. From Theorem 1, we get

$$g - \gamma \in P(F), \ \texttt{that is} \ g - \gamma = \sum_{e \in \{0,1\}^m} \sigma_e f^e,$$

where $\sigma_e \in \sum \mathbb{R}[x]^2$. Choose $N = \max \deg(\sigma_e f^e)$, then $g - \gamma \in P_N(F)$, so $\gamma \leq g_N^*$. Thus

$$\gamma \leq g_N^* \leq g_*.$$

For $\gamma \uparrow g_*$, then $g_N^* \uparrow g_*$. From $g_N^* \xrightarrow{N \to \infty} g_*$ and $g_N^* \leq g_{+,N} \leq g_*$, we obtain $g_{+,N} \xrightarrow{N \to \infty} g_*$.

Proposition 5. *Problem (6) is SDP.*

Proof. Similar to the proof of Proposition 2.

Proposition 6. *Problem (7) is duality of Problem (6).*

Proof. Similar to the proof of Proposition 3.

Example 7. *We consider problem*

$$\begin{cases} \inf\limits_{(x,y)\in S}(x,y) = x + y, \\ S = \{(x,y) \in \mathbb{R}^2 \mid x \geq \frac{1}{2}, y \geq \frac{1}{2}, xy \leq 1\}. \end{cases}$$

Then

$$g_2^* = g_* = 1.$$

Example 8. *Problem*

$$\begin{cases} \inf\limits_{(x,y)\in S} g(x,y) = -x - y, \\ S = \{(x,y) \in \mathbb{R}^2 \mid x \geq \frac{1}{2}, y \geq \frac{1}{2}, xy \leq 1\} \end{cases}$$

has

$$g_2^* = g_* = -2,5.$$

6. Conclusion

The paper found out the problem of minimizing a polymonial $g_* = \inf\limits_{x\in S(F)} g(x)$ in case $S(F)$ is compact, where $g \in \mathbb{R}[x]$ and $S(F)$ is the basic closed semialgebraic set generated by F.

The paper presented positive performed theorems:

- Putinar,
- Schmüdgen.

Using results of positive performed theorems of Putinar ([8]) and Schmüdgen ([9]), we can build a series of positive semidefinited programming which their optimal values converge monotonically increasing to the optimum value g_*. Finally, the numerical results show that the proposed method works effectively.

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V.T. Phan
Faculty of Mathematics and Statistics,
Ton Duc Thang University,

Ho Chi Minh City, Viet Nam
email: *tripv@tdt.edu.vn*