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SOLITARY WAVE SOLUTIONS OF SPACE-TIME FDES USING THE GENERALIZED KUDRYASHOV METHOD

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Abstract. In this paper, the generalized Kudryashov method is presented to establish traveling wave solutions for two nonlinear space-time fractional differential equations (FDEs) in the sense of modified Riemann-Liouville derivatives, namely the space-time fractional modified Benjamin-Bona-Mahony (mBBM) equation and the space-time fractional potential Kadomstev-Petviashvili (pKP) equation. The proposed method is effective and convenient for solving nonlinear evolution equations with fractional order.

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1. Introduction

The investigation of traveling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena [1]. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics and so on. Because of the increased concentration in the theory of solitary waves, a large variety of analytic and computational methods have been established in the analysis of the nonlinear models for example, tanh function method [2], extended tanh function method [3, 4], sine-cosine method [5], Jacobi elliptic function method [6, 7], F-expansion method [8, 9], exp-function method [10], (G'/G)-expansion method [11, 12], Kudryashov method [13] and so on.

FPDEs are generalizations of classical PDEs of integer order and have recently proved to be valuable tools to the modeling of many physical phenomena and have been the focus of many studies due to their frequent appearance in various applications in many fields. In order to obtain exact solutions for FPDEs, many powerful and efficient methods have been proposed so far (e.g., see [14-21]).

The objective of this paper is to apply the generalized Kudryashov method [22] for solving two FPDEs in the sense of the modified Riemann-Liouville derivative which has been derived by Jumarie [23]. These equations can be reduced into nonlinear ordinary differential equations (ODE) with integer orders using some fractional complex transformations.

2. The modified Riemann-Liouville derivative

In this section we give some definitions and properties of the modified Riemann-Liouville derivative which are used further in this paper. Assume that $f : R \to R,\ \ t \to f(t)$, denote a continuous function, the Jumarie modified Riemann-Liouville derivative of order α is defined by

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \frac{d}{dt} \int\limits_0^t (t-\eta)^{-\alpha-1} [f(\eta) - f(0)] d\eta, & \alpha < 1, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_0^t (t-\eta)^{-\alpha} [f(\eta) - f(0)] d\eta, & 0 < \alpha \leq 1, \\ \left[f^{(n)}(t)\right]^{(\alpha-n)}, & n \leq \alpha < n+1,\ n \geq 1 \end{cases} \tag{1}$$

We list some important properties for the fractional modified Riemann-Liouville derivative as follows:

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha},\ \ r > 0 \tag{2}$$

$$D_t^{\alpha} \left[f(t) g(t)\right] = f(t) D_t^{\alpha} g(t) + g(t) D_t^{\alpha} f(t) \tag{3}$$

$$D_t^{\alpha} \left[f(g(t))\right] = f'_g(g(t)) D_t^{\alpha} g(t) = D_g^{\alpha} f(g(t)) [g'(t)]^{\alpha}, \tag{4}$$

which are direct consequence of the equality $d^{\alpha} x(t) = \Gamma(1+\alpha) dx(t)$.

3. The generalized Kudryashov method

Suppose that we have a nonlinear evolution equation with fractional order in the form:

$$F\left(u, u_t, D_t^{\alpha} u, D_x^{\alpha} u, ...\right) = 0,\ \ 0 < \alpha \leq 1, \tag{5}$$

where $D_t^{\alpha} u, D_x^{\alpha} u$ are Jumaries modified Riemann-Liouville derivative of $u = u(x,t)$, u is an unknown function, F is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step1. Using the fractional complex transformation

$$u(x,t)=u(\xi),\ \ \xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}+\xi_0, \tag{6}$$

where ξ_0 is an arbitrary constant and $k,c,$ are constants to be determined. Then Eq. (5) reduces to a nonlinear ordinary differential equation of the form

$$P\left(u,u_{\xi},u_{\xi\xi},...\right)=0, \tag{7}$$

Step2. Suppose that the solution of Eq. (7) has the following form:

$$u(\xi)=\frac{\sum\limits_{i=0}^{N}a_iQ^i(\xi)}{\sum\limits_{j=0}^{M}b_jQ^j(\xi)}=\frac{A[Q(\xi)]}{B[Q(\xi)]}, \tag{8}$$

where $a_i(i=0,1,...,N)$and $b_j(j=0,1,...,M)$ are constants to be determined such that $a_N\neq 0$, $b_M\neq 0$ and

$$Q(\xi)=\frac{1}{1\pm e^{\xi}}, \tag{9}$$

Is the solution of the equation

$$Q_{\xi}=Q^2-Q. \tag{10}$$

Step3. Determine the positive integer numbers N and M in Eq. (8) by using the homogeneous balance method between the highest order derivatives and the nonlinear terms in Eq. (7).

Step4. Substitute $u(\xi)$ and its necessary derivatives into Eq.(7)

$$u_{\xi}=\left(Q^2-Q\right)\left(\frac{A'B-AB'}{B^2}\right), \tag{11}$$

$$\begin{aligned}u_{\xi\xi}=&\tfrac{\left(Q^2-Q\right)^2}{B^3}\left\{B(BA''-AB'')-2B'(A'B-AB')\right\}+\\&(2Q-1)\left(Q^2-Q\right)\left(\tfrac{A'B-AB'}{B^2}\right),\end{aligned} \tag{12}$$

$$\begin{aligned}u_{\xi\xi\xi}=&\tfrac{\left(Q^2-Q\right)^3}{B^3}\left\{\begin{array}{l}B(BA'''-AB''')+3B''(AB'-A'B)+3B'(AB''-BA'')+\\ \tfrac{6B'^2}{B}(A'B-AB')\end{array}\right\}+\\&\tfrac{3\left(Q^2-Q\right)^2(2Q-1)}{B^3}\left\{B(BA''-AB'')+2B'(AB'-A'B)\right\}+\\&\left(\tfrac{A'B-AB'}{B^2}\right)\left(Q^2-Q\right)\left(6Q^2-6Q+1\right).\end{aligned} \tag{13}$$

where the prime $'$ denotes the derivative $\frac{d}{dQ}$. As a result of this substitution, we get a polynomial of $\frac{Q^i}{Q^j}$, $(i,j=0,1,2,...)$. In this polynomial we gather all terms of same powers and equating them to zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica to get the unknown parameters $a_i(i=0,1,...,N)$, $b_j(j=0,1,...,M)$,k,c. Consequently, we obtain the exact solutions of Eq. (5).

4. Applications

In this section, we apply the generalized Kudryashov method to find the traveling solutions of the following space-time FDEs:

4.1. The space-time fractional modified Benjamin-Bona-Mahony (mBBM) equation

This equation is well known and has the form

$$D_t^{\alpha}u + D_x^{\alpha}u - vu^2 D_x^{\alpha}u + D_x^{3\alpha}u = 0, \tag{14}$$

where $0<\alpha<1$, $t>0$. Eq. (14) has been solved in [24] using the modified Kudryashov method and [25] using a fractional sub equation method based on fractional Riccati equation . Let us now solve Eq. (14) by using the generalized Kudruashov method. To this end, we use the wave transformation (6) to reduce Eq. (14) to the following ODE :

$$(c+k)u_{\xi} - vku^2u_{\xi} + k^3u_{\xi\xi\xi} = 0, \tag{15}$$

Integrating Eq. (15) with respect to ξ, we get

$$(c+k)u - \frac{vk}{3}u^3 + k^3u_{\xi\xi} + R = 0, \tag{16}$$

where R is the integration constant. Balancing $u_{\xi\xi}$ with u^3 in (16), then we get the formula $N=M+1$ If we choose $M=1$ and $N=2$, then

$$A = a_0 + a_1Q + a_2Q^2, \;\; B = b_0 + b_1Q, \tag{17}$$

$$u = \frac{a_0 + a_1Q + a_2Q^2}{b_0 + b_1Q}. \tag{18}$$

Substituting A,B and their necessary derivatives (with respect to Q) with (8) and (12) into (16) and equating all the coefficients of $\frac{Q^i}{Q^j}$, $(i,j=0,1,2,...)$ to zero, we obtain

$$-\frac{1}{3}kva_2^3 + 2k^3a_2b_1^2 = 0, \tag{19}$$

$$-kva_1a_2^2+6k^3a_2b_0b_1-3k^3a_2b_1^2=0, \tag{20}$$

$$-kva_1^2a_2-kva_0a_2^2+6k^3a_2b_0^2-9k^3a_2b_0b_1+ca_2b_1^2+ka_2b_1^2+k^3a_2b_1^2=0, \tag{21}$$

$$\begin{aligned}&-\tfrac{1}{3}kva_1^3-2kva_0a_1a_2+2k^3a_1b_0^2-10k^3a_2b_0^2-2k^3a_0b_0b_1+k^3a_1b_0b_1+\\&2ca_2b_0b_1+2ka_2b_0b_1+3k^3a_2b_0b_1-k^3a_0b_1^2+ca_1b_1^2+ka_1b_1^2+Rb_1^3=0,\end{aligned} \tag{22}$$

$$\begin{aligned}&-kva_0a_1^2-kva_0^2a_2-3k^3a_1b_0^2+ca_2b_0^2+ka_2b_0^2+4k^3a_2b_0^2+3k^3a_0b_0b_1+\\&2ca_1b_0b_1+2ka_1b_0b_1-k^3a_1b_0b_1+ca_0b_1^2+ka_0b_1^2+k^3a_0b_1^2+3Rb_0b_1^2\end{aligned} \tag{23}$$

$$-kva_0^2a_1+ca_1b_0^2+ka_1b_0^2+k^3a_1b_0^2+2ca_0b_0b_1+2ka_0b_0b_1-k^3a_0b_0b_1+3Rb_0^2b_1=0, \tag{24}$$

$$-\frac{1}{3}kva_0^3+ca_0b_0^2+ka_0b_0^2+Rb_0^3=0. \tag{25}$$

Solving the system of equations (19)-(25) using Mathematica, we obtain

Case1.

$$a_0=0,\ a_1=\mp\sqrt{\frac{3}{2v}}kb_1,\ a_2=\pm\sqrt{\frac{6}{v}}kb_1,\ b_0=0,\ c=\frac{1}{2}k(k^2-2),\ R=0 \tag{26}$$

where b_1, k are arbitrary constants. The solution of Eq. (14) corresponding to (26) is

$$u_{1,2}(x,t)=\mp\sqrt{\frac{3}{2v}}k\,tanh\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)}+\frac{k(k^2-2)t^\alpha}{4\Gamma(1+\alpha)}+\frac{\xi_0}{2}\right], \tag{27}$$

$$u_{3,4}(x,t)=\mp\sqrt{\frac{3}{2v}}k\ \cot h\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)}+\frac{k(k^2-2)t^\alpha}{4\Gamma(1+\alpha)}+\frac{\xi_0}{2}\right], \tag{28}$$

Case2.

$$a_0=0,\ a_1=\mp2\sqrt{\frac{6}{v}}kb_0,\ a_2=\pm2\sqrt{\frac{6}{v}}kb_0,\ b_1=-2b_0,\ c=-k(k^2+1),\ R=0 \tag{29}$$

where b_0, k are arbitrary constants. The solution of Eq. (14) corresponding to (29) is

$$u_{5,6}(x,t)=\pm\sqrt{\frac{6}{v}}k\,csch\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{k(k^2+1)t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right], \tag{30}$$

Case3.

$$a_0=\mp\sqrt{\frac{6}{v}}kb_0,\ a_1=\pm2\sqrt{\frac{6}{v}}kb_0,\ a_2=\mp2\sqrt{\frac{6}{v}}kb_0,\ b_1=-2b_0,\ c=k(2k^2-1), R=0. \tag{31}$$

where b_0, k are arbitrary constants. The solution of Eq. (14) corresponding to (31) is

$$u_{7,8}(x,t)=\pm\sqrt{\frac{6}{v}}k\,coth\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{k(2k^2-1)t^\alpha}{\Gamma(1+\alpha)}+\xi_0\right]. \tag{32}$$

Case4.

$$a_0 = \mp\sqrt{\frac{6}{v}}kb_0, a_1 = 0, a_2 = \mp 2\sqrt{\frac{6}{v}}kb_0, b_1 = 2b_0, c = k(5k^2-1), R = \pm 3\sqrt{\frac{6}{v}}k^4 \quad (33)$$

where b_0, k are arbitrary constants. The solution of Eq. (14) corresponding to (33) is

$$u_{9,10}(x,t) = \pm\sqrt{\frac{3}{2v}}k\left(tanh\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)} + \frac{k(5k^2-1)t^\alpha}{2\Gamma(1+\alpha)} + \frac{\xi_0}{2}\right] + \frac{3}{tanh\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)} + \frac{k(5k^2-1)t^\alpha}{2\Gamma(1+\alpha)} + \frac{\xi_0}{2}\right]-2}\right), \quad (34)$$

$$u_{11,12}(x,t) = \pm\sqrt{\frac{3}{2v}}k\left(\cot h\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)} + \frac{k(5k^2-1)t^\alpha}{2\Gamma(1+\alpha)} + \frac{\xi_0}{2}\right] + \frac{3}{\cot h\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)} + \frac{k(5k^2-1)t^\alpha}{2\Gamma(1+\alpha)} + \frac{\xi_0}{2}\right]-2}\right), \quad (35)$$

4.2. The space-time fractional potential Kadomstev-Petviashvili (pKP) equation

This equation is well known and has the form

$$\frac{1}{4}D_x^{4\alpha}u + \frac{3}{2}D_x^{\alpha}u\,D_x^{2\alpha}u + \frac{3}{4}D_y^{2\alpha}u + D_t^{\alpha}(D_x^{\alpha}u) = 0, \quad (36)$$

where $0 < \alpha < 1$, $t > 0$. Eq. (36) has been solved in [24] using the modified Kudryashov method and [26] using Exp-function and (G'/G)-expansion methods. Let us now solve Eq. (36) by using the generalized Kudruashov method. To this end, we use the wave transformation

$$u(x,y,t) = u(\xi),\ \ \xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ly^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} + \xi_0, \quad (37)$$

to reduce Eq. (36) to the following ODE :

$$\frac{1}{4}k^4u_{\xi\xi\xi\xi} + \frac{3}{2}k^3u_\xi u_{\xi\xi} + \left(\frac{3}{4}l^2 + ck\right)u_{\xi\xi} = 0. \quad (38)$$

Integrating Eq. (38) with respect to ξ, we obtain

$$\frac{1}{4}k^4u_{\xi\xi\xi} + \frac{3}{4}k^3{u_\xi}^2 + \left(\frac{3}{4}l^2 + ck\right)u_\xi + R = 0. \quad (39)$$

where is the integration constant. Balancing $u_{\xi\xi\xi}$ with ${u_\xi}^2$ in (39), then we get the formula $N = M + 1$ If we choose $M = 1$ and $N = 2$, then

$$A = a_0 + a_1Q + a_2Q^2,\ \ B = b_0 + b_1Q, \quad (40)$$

$$u = \frac{a_0 + a_1 Q + a_2 Q^2}{b_0 + b_1 Q}. \tag{41}$$

Substituting A, B and their necessary derivatives (with respect to Q) with (11) and (13) into (39) and equating all the coefficients of $\frac{Q^i}{Q^j}$, $(i = 0, 1, 2, \ j = 0, 1)$ to zero, we obtain

$$\frac{3}{4}k^3 a_2 b_1^2 (a_2 + 2kb_1) = 0, \tag{42}$$

$$\frac{3}{2}k^3 a_2 (2b_0 - b_1) b_1 (a_2 + 2kb_1) = 0, \tag{43}$$

$$\begin{gathered}\tfrac{1}{4}a_2(3k^3a_2(4b_0^2 - 8b_0b_1 + b_1^2)+ \\ b_1(6k^3a_1b_0 + 36k^4b_0^2 - 48k^4b_0b_1 + b_1(-6k^3a_0 + (4ck + 7k^4 + 3l^2)b_1))) = 0,\end{gathered} \tag{44}$$

$$\begin{gathered}-\tfrac{1}{4}a_2(-24k^4b_0^3 - 12k^3a_1b_0(b_0 - b_1) + 12k^3a_2b_0(2b_0 - b_1) + 12k^3a_0b_0b_1 + 72k^4b_0^2b_1 \\ -12k^3a_0b_1^2 - 16ckb_0b_1^2 - 28k^4b_0b_1^2 - 12l^2b_0b_1^2 + 4ckb_1^3 + k^4b_1^3 + 3l^2b_1^3)\end{gathered} \tag{45}$$

$$\begin{gathered}\tfrac{1}{4}(3k^3a_1^2b_0^2 + 12k^3a_2^2b_0^2 + a_1b_0(6k^4b_0^2 + 6k^4b_0b_1 + 6k^3a_2(-4b_0 + b_1)+ \\ b_1(-6k^3a_0 + (4ck + k^4 + 3l^2)b_1)) + a_2(-54k^4b_0^3 + (20ck + 41k^4 + 15l^2)b_0^2b_1 - \\ 6k^3a_0b_1^2 - 4b_0b_1(-6k^3a_0 + (4ck + k^4 + 3l^2)b_1)) + b_1(3k^3a_0^2b_1 + 4Rb_1^3 - \\ a_0(6k^4b_0^2 + 6k^4b_0b_1 + (4ck + k^4 + 3l^2)b_1^2))) = 0,\end{gathered} \tag{46}$$

$$\begin{gathered}\tfrac{1}{4}(-6k^3a_1^2b_0^2 + a_2b_0((8ck + 38k^4 + 6l^2)b_0^2 - 12k^3a_0b_1 - 5(4ck + k^4 + 3l^2)b_0b_1) - \\ a_1b_0(-12k^3a_2b_0 + 12k^4b_0^2 - 2(4ck - 5k^4 + 3l^2)b_0b_1 + b_1(-12k^3a_0 + (4ck + k^4 + 3l^2)b_1)) + \\ b_1(-6k^3a_0^2b_1 + 16Rb_0b_1^2 + a_0(12k^4b_0^2 - 2(4ck - 5k^4 + 3l^2)b_0b_1 + (4ck + k^4 + 3l^2)b_1^2))) = 0,\end{gathered} \tag{47}$$

$$\begin{gathered}\tfrac{1}{4}(3k^3a_1^2b_0^2 - 2(4ck + 4k^4 + 3l^2)a_2b_0^3 + a_1b_0((4ck + 7k^4 + 3l^2)b_0^2 - 6k^3a_0b_1 - \\ 2(4ck - 2k^4 + 3l^2)b_0b_1) + b_1(3k^3a_0^2b_1 + 24Rb_0^2b_1 - a_0b_0((4ck + 7k^4 + 3l^2)b_0 - \\ 2(4ck - 2k^4 + 3l^2)b_1))) = 0,\end{gathered} \tag{48}$$

$$\frac{1}{4}b_0^2(-(4ck + k^4 + 3l^2)a_1b_0 + ((4ck + k^4 + 3l^2)a_0 + 16Rb_0)b_1) = 0, \tag{49}$$

$$Rb_0^4 = 0. \tag{50}$$

Solving the system of equations (42)-(50) using Mathematica, we obtain

Case1.

$$a_1 = \frac{-2kb_0^2 + a_0 b_1}{b_0},\ a_2 = -2kb_1,\ c = -\frac{k^4 + 3l^2}{4k},\ R = 0. \tag{51}$$

where a_0, b_0, k, l are arbitrary constants.

The solution of Eq. (36) corresponding to (51) is

$$u_1(x,y,t) = \frac{a_0}{b_0} + k\left(tanh\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)} + \frac{ly^\alpha}{2\Gamma(1+\alpha)} - \frac{\left(k^4 + 3l^2\right)t^\alpha}{8k\Gamma(1+\alpha)} + \frac{\xi_0}{2}\right] - 1\right), \tag{52}$$

$$u_2(x,y,t) = \frac{a_0}{b_0} + k\left(\cot h\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)} + \frac{ly^\alpha}{2\Gamma(1+\alpha)} - \frac{\left(k^4 + 3l^2\right)t^\alpha}{8k\Gamma(1+\alpha)} + \frac{\xi_0}{2}\right] - 1\right), \tag{53}$$

Case2.

$$a_0 = 0,\ a_2 = -2kb_1,\ b_0 = 0,\ c = -\frac{k^4 + 3l^2}{4k},\ R = 0. \tag{54}$$

where a_1, b_1, k, l are arbitrary constants.

The solution of Eq. (36) corresponding to (54) is

$$u_3(x,y,t) = \frac{a_1}{b_1} + k\left(tanh\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)} + \frac{ly^\alpha}{2\Gamma(1+\alpha)} - \frac{\left(k^4 + 3l^2\right)t^\alpha}{8k\Gamma(1+\alpha)} + \frac{\xi_0}{2}\right] - 1\right), \tag{55}$$

$$u_4(x,y,t) = \frac{a_1}{b_1} + k\left(\cot h\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)} + \frac{ly^\alpha}{2\Gamma(1+\alpha)} - \frac{\left(k^4 + 3l^2\right)t^\alpha}{8k\Gamma(1+\alpha)} + \frac{\xi_0}{2}\right] - 1\right), \tag{56}$$

Case3.

$$a_1 = -2a_0,\ a_2 = 4kb_0,\ b_1 = -2b_0,\ c = -\frac{4k^4 + 3l^2}{4k},\ R = 0. \tag{57}$$

where a_0, b_0, k, l are arbitrary constants.

The solution of Eq. (36) corresponding to (57) is

$$u_5(x,y,t) = \frac{a_0}{b_0} + 2k\left(coth\left[\frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ly^\alpha}{\Gamma(1+\alpha)} - \frac{\left(4k^4 + 3l^2\right)t^\alpha}{4k\Gamma(1+\alpha)} + \xi_0\right] - 1\right), \tag{58}$$

5. Conclusions

In this paper, we have proposed the generalized Kudryashov method for solving two nonlinear space-time FDEs in the sense of modified Riemann-Liouville derivatives, namely the space-time fractional modified Benjamin-Bona-Mahony (mBBM) equation and the space-time fractional potential Kadomstev-Petviashvili (pKP) equation. A new results have been obtained in this work using this method. This method is effective and can be extended for solving many systems of nonlinear FPDEs.

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