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SOME ALGEBRAIC PROPERTIES OF GENERALIZED CLONE AUTOMORPHISMS

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Abstract. Generalized hypersubstitutions are mappings from the set of all fundamental operations into the set of all terms of the same language do not necessarily preserve the arities. In this paper we prove that the group of all generalized clone automorphisms of an algebra $\mathcal{A}$ is isomorphic to a certain group of generalized hypersubstitutions supposed the variety $V(\mathcal{A})$ generated by $\mathcal{A}$ is strongly solid.

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1. Introduction

Let $\tau = (n_i)_{i\in I}$ be a type, indexed by a set I, with n_i-ary operation f_i. Let $X = \{x_1, x_2, ...\}$ be a countably infinite set of variables, and for each $n \geq 1$ let $X_n = \{x_1, x_2, ..., x_n\}$.

An *n-ary term* of type τ is defined inductively as follows:

(i) Every variable $x_i \in X_n$ is an *n-ary term* of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an *n-ary term* of type τ.

We denote by $W_\tau(X)$ and $W_\tau(X_n)$ the sets of all terms, and of all n-ary terms of type τ, respectively. These two sets are the universes of two absolutely free algebras,

$$\mathcal{F}_\tau(X) := (W_\tau(X); \overline{(f_i)}_{i\in I})$$

and

$$\mathcal{F}_\tau(X_n) := (W_\tau(X_n); \overline{(f_i)}_{i\in I})$$

respectively. The operations $\overline{f_i}$ are defined by

$$\overline{f_i}(t_1, t_2, ..., t_{n_i}) := f_i(t_1, t_2, ..., t_{n_i})$$

The algebras $\mathcal{F}_\tau(X)$ and $\mathcal{F}_\tau(X_n)$ are examples of algebras of type τ. Let $Alg(\tau)$ be the class of all algebras of type τ. Another operation on sets of terms is the composition or superposition of terms which plays an important role in universal algebra, clone theory and theoretical computer science. For each pair of natural numbers m and n greater than zero, the superposition operation

$$S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$$

is defined inductively, by the following steps :

(i) If $x_j \in X_n$ is a variable and $t_1, .., t_n \in W_\tau(X_m)$, then $S^n_m(x_j, t_1, ..., t_n) := t_j$, for $1 \le j \le n$.

(ii) If $f_i(s_1, ..., s_{n_i})$ is a composite term where $s_1, .., s_{n_i} \in W_\tau(X_n)$,$t_1, .., t_n \in W_\tau(X_m)$, then $S^n_m(f_i(s_1, ..., s_{n_i}), t_1, ..., t_n) := f_i(S^n_m(s_1, t_1, ..., t_n), ..., S^n_m(s_{n_i}, t_1, ..., t_n))$.

Using these operations, we form the heterogeneous or multi-based algebra

$$clone\tau := ((W_\tau(X_n))_{n>0}; (S^n_m)_{n,m>0}, (x_i)_{i\le n,n>0}).$$

It is well-known and easy to check that this algebra satisfies the clone axioms

(C1) $\widetilde{S}^p_m(X_0, \widetilde{S}^n_m(Y_1, X_1, ..., X_n), ..., \widetilde{S}^n_m(Y_p, X_1, ..., X_n)) \approx$
$\widetilde{S}^n_m(\widetilde{S}^p_m(X_0, Y_1, ..., Y_p), X_1, ..., X_n)$,

(C2) $\widetilde{S}^n_m(\lambda_i, X_1, ..., X_n) \approx X_i, 1 \le i \le n$,

(C3) $\widetilde{S}^n_m(X_1, \lambda_1, ..., \lambda_n) \approx X_1$,

where $\widetilde{S}^p_m$ and $\widetilde{S}^n_m$ are operation symbols corresponding to the operations S^p_m and S^n_m of $clone(\tau)$ where $\lambda_1, ..., \lambda_n$ are nullary operation symbols and $X_1, ..., X_n, Y_1, ..., Y_p$ are variables.

Since later on we have to consider subalgebras and congruences of heterogeneous algebras, we recall these concepts. A subalgebra of $clone(\tau)$ consists of a sequence $(T^{(n)})_{n>0}$, where $T^{(n)} \subseteq W_\tau(X_n)$ for all $n > 0$ which is closed under all operations of $clone(\tau)$. A congruence on $clone(\tau)$ is a sequence $(\theta_n)_{n>0}$ of binary relations, where $\theta_n \subseteq W_\tau(X_n) \times W_\tau(X_n)$, which is preserved by all operations from $clone(\tau)$.

Since the set $W_\tau(X_n)$ of all n-ary terms of type τ is closed under the superposition operation $S^n := S^n_n$ there is a homogeneous analogue of this structure. The algebra $(W_\tau(X_n); S^n, x_1, x_2, ..., x_n)$ is an algebra of type $(n+1, 0, 0, ..., 0)$, which still satisfies the clone axioms above for the case that $p = m = n$. Such an algebra is called *a unitary Menger algebra of rank n*.

Let $n-clone(\tau) := (W_\tau(X_n); S^n)$ be the reduct of the unitary Menger algebra $(W_\tau(X_n); S^n, x_1, x_2, ..., x_n)$ of rank n. The algebra $n - clone(\tau)$ is called a *Menger algebra of rank n*.

An algebra with similar properties can be obtained if we define a superposition operation for n-ary operations on a set A. We consider the set $O(A)$ of all finitary operations on A.

Definition 1. *Let $O^n(A), n \geq 1$, be the set of all n-ary operations defined on the set A. Then the $(n+1)$-ary superposition operation (for operations) $S^{n,A}$: $O^n(A)^{n+1} \longrightarrow O^n(A)$ is defined by*

$$S^{n,A}(f^A, g_1^A, ..., g_n^A)(a_1, ..., a_n) := f^A(g_1^A(a_1, ..., a_n), ..., g_n^A(a_1, ..., a_n))$$

for every $(a_1, ..., a_n) \in A^n$. Here $g_1^A, ..., g_n^A$ as well as are n-ary. This can be generalized to an operation $S^{n,A}_m : O^n(A) \times (O^m(A))^n \longrightarrow O^m(A)$, $m \geq 1$ defined by

$$S^{n,A}_m(f^A, g_1^A, ..., g_n^A)(a_1, ..., a_m) := f^A(g_1^A(a_1, ..., a_m), ..., g_n^A(a_1, ..., a_m)).$$

The $O(A)$ is closed under the operations. Particular operations on $O^n(A)$ are projections $e_i^{n,A}$, mapping each n-tuple of elements from A to the i-th component, that is $e_i^{n,A}(a_1, ..., a_n) := a_i$.

Definition 2. *Let $\mathcal{A}$ be an algebra of type $\tau = (n_i)_{i\in I}$ and $t \in W_\tau(X)$. Then t induces an n-ary term operation $t^A : A^n \longrightarrow A$ called the n-ary term operation induced by the term t on the algebra $\mathcal{A}$, via the following steps :*

(i) If $t = x_j \in X_n$, then $t^A = x_j^A := e_j^{n,A}$ where $e_j^{n,A} : (a_1, \ldots, a_n) \mapsto a_j$ is n-ary projection onto the j-th coordinate.

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ and $t_1^A, ..., t_{n_i}^A$ are the n-ary term operations which are induced by $t_1, ..., t_{n_i}$, then $t^A = S^{n_i,A}(f_i^A, t_1^A, ..., t_{n_i}^A)$.

We will denote by $W_\tau(X_n)^{\mathcal{A}}$ the set of all n-ary term operations of the algebra $\mathcal{A}$, and by $W_\tau(X)^{\mathcal{A}}$ the set of all (finitary) term operations on $\mathcal{A}$. Make a remark that the elements of $W_\tau(X_n)^{\mathcal{A}}$ are also called n-ary term operations induced by terms from $W_\tau(X_n)$.

Sets of operations defined on A containing all projections and being closed under the application of the superposition operation are called *clones of operations*. We denote by $T^{(n)}(\mathcal{A})$ the clone of all n-ary operation generated by fundamental operations $\{f_i^{\mathcal{A}}|i \in I\}$ of algebra $\mathcal{A}$. Further we have $T(\mathcal{A}) := \bigcup_{n\geq 1} T^{(n)}(\mathcal{A}) = W_\tau(X)^{\mathcal{A}}$.

For an algebra $\mathcal{A}$ of type τ, we denote by $V(\mathcal{A})$ the variety generated by $\mathcal{A}$ and by $Id\mathcal{A}$ the set of all equations of type τ which are satisfied as identities in $\mathcal{A}$, i.e.

$$Id\mathcal{A} := \{s \approx t|\ \ s, t \in W_\tau(X) \ \text{ and } \ s^{\mathcal{A}} = t^{\mathcal{A}}\}.$$

For the variety $V(\mathcal{A})$, we denote by $IdV(\mathcal{A})$ the set of all identities which are satisfied in every algebra of $V(\mathcal{A})$, i.e.

$$IdV(\mathcal{A}) := \{s \approx t|\ \ s, t \in W_\tau(X) \ \text{ and } \ \forall \mathcal{B} \in V(\mathcal{A}) \ \ (s^{\mathcal{B}} = t^{\mathcal{B}})\}.$$

2. Generalized Hypersustitution

An arbitrary mapping $\sigma : \{f_i|i \in I\} \longrightarrow W_\tau(X)$ which maps every n_i-ary operation symbol of type τ to an n_i-ary term of the same type that is preserve the arity is called a *hypersubstitution* of type τ. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. Any hypersubstitution σ induces a mapping

$$\hat{\sigma} : W_\tau(X) \to W_\tau(X)$$

in the following inductive way:

(i) $\hat{\sigma}[t] := t$ if $t \in X$.

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, if t is a compound term $f_i(t_1, \ldots, t_{n_i})$.

Using the induced maps $\hat{\sigma}$, a binary operation $\circ_h$ can be defined on the set $Hyp(\tau)$. For any hypersubstitutions $\sigma_1, \sigma_2 \in Hyp(\tau)$, $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ i.e.

$$\forall i \in I, (\sigma_1 \circ_h \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)].$$

Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $\underline{Hyp(\tau)} = (Hyp(\tau); \circ_h, \sigma_{id})$ is a monoid where σ_{id} is the identity element.

In [4] S. Leeratanavalee and K. Denecke generalized the concept of a hypersubstitution to a generalized hypersubstitution. Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$.

Definition 3. *A generalized hypersubstitution of type τ, for simply, a generalized hypersubstitution is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which maps each n_i-ary operation symbol of type τ to a term of this type which does not necessarily preserve the arity. We denoted the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$.*

Firstly, we define inductively the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

To define a binary operation on $Hyp_G(\tau)$, we extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$.

(ii) $\hat{\sigma}[t] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if t is a compound term, $f_i(t_1, \ldots, t_{n_i})$.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then we have $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid where σ_{id} is the identity element.

Let $\underline{M}$ be a submonoid of $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ and V be a variety of type τ. The variety V is called *M-strongly solid variety* if

$$\forall s \approx t \in IdV, \forall \sigma \in M(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV).$$

An identity $s \approx t \in IdV$ is called *M-strong hyperidentity* if

$$\forall \sigma \in M(V \models \hat{\sigma}[s] \approx \hat{\sigma}[t]).$$

If $\underline{M} = \underline{Hyp(\tau)}$, then we speak of *strongly solid variety* and *strong hyperidentity*, respectively.

Definition 4. *Let $\mathcal{A}$ be an algebra of type $\tau = (n_i)_{i \in I}$. Let $t \in W_\tau(X)$. Then t induces an n-ary term operation $t^A : A^n \longrightarrow A$ called the generalized n-ary term operation induced by the term t on the algebra $\mathcal{A}$, via the following steps :*

(i) If $t = x_j \in X_n$, then $t^A = x_j^A := e_j^{n,A}$ where $e_j^{n,A} : (a_1, \ldots, a_n) \mapsto a_j$ is an n-ary projection onto the j-th coordinate.

(ii) *If* $t = x_j \in X \setminus X_n$, *then* $t^A = x_j^A := c_a^n$ *is the* n*-ary constant operation on* A *with value* a *and each element from* A *is uniquely induced by an element from* $X \setminus X_n$.

(iii) *If* $t = f_i(t_1, \ldots, t_{n_i})$ *and* $t_1^A, ..., t_{n_i}^A$ *are the* n*-ary term operations which are induced by* $t_1, ..., t_{n_i}$, *then* $t^A = S^{n_i,A}(f_i^A, t_1^A, ..., t_{n_i}^A)$.

Definition 5. *Let* V *be a variety of type* τ. *A generalized hypersubstitution* σ *of type* τ *is called a* V-proper generalized hypersubstitution *if for every identity* $s \approx t$ *of* V, *the identity* $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ *also holds in* V.

Definition 6. *Let* V *be a variety of type* τ. *Two generalized hypersubstitutions* σ_1 *and* σ_2 *of type* τ *are called a* V-generalized equivalent *if* $\sigma_1(f_i) \approx \sigma_2(f_i)$ *are identities in* V *for all* $i \in I$. *In this case, we write* $\sigma_1 \sim_{VG} \sigma_2$.

Theorem 1. *Let* V *be a variety of algebras of type* τ*,and let* $\sigma_1, \sigma_2 \in Hyp_G(\tau)$.

(i) *If* $\sigma_1 \sim_{VG} \sigma_2$, *then for any term* $t \in W_\tau(X)$ *the equations* $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ *are identities in* V.

(ii) *If* $\sigma_1 \sim_{VG} \sigma_2$ *and* $\hat{\sigma}_1[t] \approx \hat{\sigma}_1[s] \in IdV$, *then* $\hat{\sigma}_2[t] \approx \hat{\sigma}_2[s] \in IdV$.

(iii) *If* V *is solid, then* $\sim_{VG}$ *is a congruence relation on* $Hyp_G(\tau)$.

Proposition 1. *The endomorphism monoid* $End(n-clone(\tau))$ *is isomorphic to the monoid* $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

Proposition 2. *Let* $\mathcal{A}$ *be an algebra of type* τ. *Then* $V(\mathcal{A})$ *is strongly solid iff the clone of all terms operations of* $\mathcal{A}$, *i.e. the heterogeneous algebra* $\mathcal{T}(\mathcal{A})$ *is free with respect to itself, freely generated by* $\{f_i^{\mathcal{A}} | i \in I\}$ *that means, every heterogeneous mapping from* $\{f_i^{\mathcal{A}} | i \in I\}$ *to* $\mathcal{T}(\mathcal{A})$ *can be extended to an endomorphism of* $\mathcal{T}(\mathcal{A})$.

3. The Kernel Monoid of Generalized Hypersustitution

In this section, we will introduce a new monoid of generalized hypersustitution which is defined by using the *kernel of a generalized hypersustitution.*

Definition 7. *Let* σ *be a generalized hypersustitution of type* τ *and let* V *be a variety of type* τ. *The set*

$$ker_V^G(\sigma) := \{(s,t) | s, t \in W_\tau(X) \text{ and } \hat{\sigma}[s] = \hat{\sigma}[t] \in IdV\}$$

is called the semanticl kernel of the generalized hypersustitution σ *with respect to the variety* V. *If* $V = Alg(\tau)$ *is the variety of all algebras of the type* τ, *we will write* $ker^G(\sigma)$ *and speak of the syntactical kernel of* σ.

Proposition 3. *Let σ be a generalized hypersubstitution of type $\tau = (n_i)_{i\in I}$ with $n_i \geq 1$ for all $i \in I$. Then $ker^G_V(\sigma)$ is a fully invariant congruence relation on the absolutely free algebra $\mathcal{F}_\tau(X)$.*

If $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ is an algebra of type τ, then we consider the following set $M^{\mathcal{A}}_{ker^G}$ of generalized hypersubstitutions:

$$M^{\mathcal{A}}_{ker^G} := \{\sigma | \sigma \in Hyp_G(\tau) \text{ and } ker^G_{V(\mathcal{A})} = IdV(\mathcal{A}) \text{ and } \hat{\sigma}[W_\tau(X)]^{\mathcal{A}} = T(\mathcal{A})\}.$$

Then we have:

Lemma 2. *For every algebra $\mathcal{A}$ of type τ, the set $M^{\mathcal{A}}_{ker^G}$ forms a submonoid of the monoid $Hyp_G(\tau)$ of all generalized hypersubstitutions of type τ.*

Proof. First, we prove that $M^{\mathcal{A}}_{ker^G}$ is closed under the multiplication of generalized hypersustitutions. Let $\sigma_1, \sigma_2 \in M^{\mathcal{A}}_{ker^G}$. Then we have

$$\begin{aligned}(s,t) \in ker^G_{V(\mathcal{A})}(\sigma_1 \circ_G \sigma_2) &\iff (\hat{\sigma}_1 \circ \hat{\sigma}_2)[s] \approx (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t] \in IdV(\mathcal{A}) \\ &\iff (\hat{\sigma}_1[\hat{\sigma}_2[s]] \approx (\hat{\sigma}_1[\hat{\sigma}_2[t]] \in IdV(\mathcal{A}) \\ &\iff (\hat{\sigma}_2[s], \hat{\sigma}_2[t]) \in ker^G_{V(\mathcal{A})}(\sigma_1)\end{aligned}$$

by definition of the semantical kernel. Since $ker^G_{V(\mathcal{A})}(\sigma_1) = IdV(\mathcal{A})$, we obtain $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV(\mathcal{A})$ and again, by definition of the kernel, we have $(s,t) \in ker^G_{V(\mathcal{A})}(\sigma_2)$. Since $ker^G_{V(\mathcal{A})} = IdV(\mathcal{A})$, we obtain $s \approx t \in IdV(\mathcal{A})$ and then $ker^G_{V(\mathcal{A})}(\sigma_1 \circ_G \sigma_2) = IdV(\mathcal{A})$.

If $(\hat{\sigma}_i[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A}), i = 1, 2$, then from $(\hat{\sigma}_1[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$ we obtain that for every $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s \in W_\tau(X)$ such that $(\hat{\sigma}_1[s])^{\mathcal{A}} = t^{\mathcal{A}}$ and then $(\hat{\sigma}_1[s]) \approx t \in IdV(\mathcal{A})$. For $s^{\mathcal{A}} \in T(\mathcal{A})$, there is a term $s' \in W_\tau(X)$ such that $(\hat{\sigma}_2[s'])^{\mathcal{A}} = s^{\mathcal{A}}$ and then $(\hat{\sigma}_2[s']) \approx s \in IdV(\mathcal{A})$. Applying $\hat{\sigma}_1$ on both side and using that $IdV(\mathcal{A}) = ker^G_{V(\mathcal{A})}(\sigma_1)$, we have $\hat{\sigma}_1[\hat{\sigma}_2[s']] \approx \hat{\sigma}_1[s] \approx t \in IdV(\mathcal{A})$, thus $(\hat{\sigma}_1[\hat{\sigma}_2[s']])^{\mathcal{A}} = t^{\mathcal{A}}$ and this means that for $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s' \in W_\tau(X)$ such that $((\hat{\sigma}_1 \circ \hat{\sigma}_2)[s'])^{\mathcal{A}} = t^{\mathcal{A}}$ and thus $((\sigma_1 \circ_G \sigma_2)[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$.

Finally, we show that $M^{\mathcal{A}}_{ker^G}$ contains the identity generalized hypersubstitution since

$$\begin{aligned}(s,t) \in ker_{VG(\mathcal{A})}(\sigma_{id}) &\iff \hat{\sigma}_{id}[s] \approx \hat{\sigma}_{id}[t] \in IdV(\mathcal{A}) \\ &\iff s \approx t \in IdV(\mathcal{A})\end{aligned}$$

and $(\hat{\sigma}_{id}[W_\tau(X)])^{\mathcal{A}} = W_\tau(X)\mathcal{A} = T(\mathcal{A})$.

We call $M^{\mathcal{A}}_{ker^G}$ the *kernel monoid of generalized hypersustitutions with respect to $\mathcal{A}$*. An interesting property of the kernel monoid $M^{\mathcal{A}}_{ker^G}$ is that it consists of full blocks of the equivalence relation $\sim_{VG(\mathcal{A})}$, i.e. this relation saturates the kernel monoid $M^{\mathcal{A}}_{ker^G}$.

Proposition 4. *The kernel monoid with respect to an algebra $\mathcal{A}$ of type τ is a union of equivalence classes of the relation $\sim_{VG(\mathcal{A})}$.*

Proof. Let $\sigma_1 \in M^{\mathcal{A}}_{ker^G}$ and $\sigma_1 \sim_{VG(\mathcal{A})} \sigma_2$. By Theorem 1(ii), we get that if $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV(\mathcal{A})$ then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV(\mathcal{A})$ and similar if $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV(\mathcal{A})$ then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV(\mathcal{A})$. Thus $ker^G_{V(\mathcal{A})}(\sigma_1) = ker^G_{V(\mathcal{A})}(\sigma_2)$.

Since $\sigma_1 \in M^{\mathcal{A}}_{ker^G}$, $(\hat{\sigma}_1[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$. Then for every $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s \in W_\tau(X)$ such that $(\hat{\sigma}_1[s])^{\mathcal{A}} = t^{\mathcal{A}}$. Since $\sigma_1 \sim_{VG(\mathcal{A})} \sigma_2$ and by Theorem 1(i), $\hat{\sigma_2}[s] \approx \hat{\sigma_1}[s] \in IdV(\mathcal{A})$ and thus $(\hat{\sigma_2}[s])^{\mathcal{A}} = (\hat{\sigma_1}[s])^{\mathcal{A}} = t^{\mathcal{A}}$. Thus for every $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s \in W_\tau(X)$ such that $(\hat{\sigma}_2[s])^{\mathcal{A}} = t^{\mathcal{A}}$ and this means $(\hat{\sigma}_2[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$.

Therefore, we conclude that if $\sigma_1 \in M^{\mathcal{A}}_{ker^G}$ and $\sigma_1 \sim_{VG(\mathcal{A})} \sigma_2$ then $ker^G_{V(\mathcal{A})}(\sigma_1) = IdV(\mathcal{A}) = ker^G_{V(\mathcal{A})}(\sigma_2)$ and $(\hat{\sigma}_2[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$ and thus $\sigma_2 \in M^{\mathcal{A}}_{ker^G}$.

Corollary 3. *If $V(\mathcal{A})$ is a strongly solid variety, then $M^{\mathcal{A}}_{ker^G}/\sim_{VG(\mathcal{A})}$ is a monoid.*

Proof. This is clear, since for strongly solid variety the relation $\sim_{VG}$ is congruence.

4. Generalized Clone Automorphism

We mentioned already that extended generalized hypersubstitution correspond to endomorphism of $n - clone(\tau)$. Since $n - clone(\tau)$ is free in the variety of all unitary Menger algebra of rank n, the Menger algebra $(T^{(n)}(\mathcal{A}); S^{n,A}, e^{n,A}_1, ..., e^{n,A}_n)$ is a homomorphic image of $n - clone(\tau)$.

Here we ask whether the group of all automorphisms of the multi-based algebra $T(\mathcal{A})$ can be described by generalized hypersubstitutions. Let $Aut(T(\mathcal{A}))$ be the set of all automorphisms on $T(\mathcal{A})$. Indeed, we make the following lemma:

Lemma 4. *Every clone automorphism $\varphi \in Aut(T(\mathcal{A}))$ corresponds to a class of $M^{\mathcal{A}}_{ker^G}/\sim_{VG(\mathcal{A})}$.*

Proof. Let $\varphi \in Aut(T(\mathcal{A}))$ be an automorphism on $T(\mathcal{A})$. Then φ maps the fundamental operation $f_i^{\mathcal{A}}$ to a term operation $t_i^{\mathcal{A}}$ and so we assign the class $A_\varphi = \{\sigma | \sigma \in Hyp_G(\tau)$ and $\sigma(f_i)^{\mathcal{A}} = t_i^{\mathcal{A}}\}$. Therefore A_φ is an equivalence class with respect to $\sim_{VG(\mathcal{A})}$.

Let $\sigma, \sigma' \in A_\varphi$. Then $\sigma(f_i)^{\mathcal{A}} = t_i^{\mathcal{A}} = \sigma'(f_i)^{\mathcal{A}}$ and so $\sigma(f_i) \approx \sigma'(f_i) \in IdV(\mathcal{A})$, which means $\sigma \sim_{VG(\mathcal{A})} \sigma'$. Since from $\sigma \in A_\varphi$ and $\sigma \sim_{VG(\mathcal{A})} \sigma'$ there follows

$\sigma^{'}(f_i)^{\mathcal{A}} = \sigma(f_i)^{\mathcal{A}} = t_i^{\mathcal{A}}$, $\sigma^{'} \in A_{\varphi}$. Thus A_{φ} is a full equivalence class. Therefore, φ maps $f_i^{\mathcal{A}}$ to $[\sigma]_{\sim_{VG(\mathcal{A})}}$ with $\sigma(f_i)^{\mathcal{A}} = \varphi(f_i^{\mathcal{A}})$

Next, we show that $\sigma \in M^{\mathcal{A}}_{ker^G}/\sim_{VG(\mathcal{A})}$. By Proposition 4,we get $[\sigma]_{\sim_{VG(\mathcal{A})}} \subseteq M^{\mathcal{A}}_{ker^G}/\sim_{VG(\mathcal{A})}$. Then we show that $\sigma(f_i)^{\mathcal{A}} = \varphi(f_i^{\mathcal{A}})$ for every term t there follows $\varphi(t^{\mathcal{A}}) = \hat{\sigma}[t]^{\mathcal{A}}$. We prove by induction on the complexity of a term t.
If $t = x_i \in X_n$, then $\varphi(x_i^{\mathcal{A}}) = \varphi(e_i^{n,A}) = e_i^{n,A} = x_i^{\mathcal{A}} = \hat{\sigma}[x_i]^{\mathcal{A}}$.
If $t = x_i \in X \setminus X_n$,then $\varphi(x_i^{\mathcal{A}}) = \varphi(c_a^n) = c_a^n = x_i^{\mathcal{A}} = \hat{\sigma}[x_i]^{\mathcal{A}}$ where c_a^n is the n-ary constant operation on A with value a and each element from A is uniquely induced by an element from $X \setminus X_n$.
If $t = f_i(t_1, ..., t_{n_i})$ is a composite term and assume that $\varphi(t_i^{\mathcal{A}}) = \hat{\sigma}[t_i]^{\mathcal{A}}$ for $i = 1, 2, ..., n_i$ then from $\sigma(f_i)^{\mathcal{A}} = \varphi(f_i^{\mathcal{A}})$ we get by superposition $\varphi(f_i^{\mathcal{A}})(\varphi(t_1^{\mathcal{A}}), ..., \varphi(t_{n_i}^{\mathcal{A}})) = \sigma(f_i)^{\mathcal{A}}(\hat{\sigma}[t_1]^{\mathcal{A}}, ..., \hat{\sigma}[t_{n_i}]^{\mathcal{A}}) = (\hat{\sigma}[f(t_1, ..., t_{n_i})])^{\mathcal{A}}$.
Now, using the property of $\varphi \in Aut(T(\mathcal{A}))$ is an automorphism of $T(\mathcal{A})$, we have

$$\begin{aligned} s \approx t \in IdV(\mathcal{A}) &\iff s^{\mathcal{A}} = t^{\mathcal{A}} \\ &\iff \varphi(s^{\mathcal{A}}) = \varphi(t^{\mathcal{A}}) \\ &\iff (\hat{\sigma}[s])^{\mathcal{A}} = (\hat{\sigma}[t])^{\mathcal{A}} \\ &\iff \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV(\mathcal{A}) \\ &\iff (s,t) \in ker_{VG(\mathcal{A})}(\sigma). \end{aligned}$$

This implies $IdV(\mathcal{A}) = ker_{VG(\mathcal{A})}(\sigma)$. Since φ is surjective for every $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term operation $s^{\mathcal{A}} \in T(\mathcal{A})$ such that $\varphi(s^{\mathcal{A}}) = t^{\mathcal{A}}$. This means that for every $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s \in W_{\tau}(X)$ such that $\hat{\sigma}[t]^{\mathcal{A}} = t^{\mathcal{A}}$ and then $\hat{\sigma}[W_{\tau}(X)]^{\mathcal{A}} = T(\mathcal{A})$. Altogether, we have $\sigma \in M^{\mathcal{A}}_{ker^G}$.

Lemma 5. *If $V(\mathcal{A})$ is strongly solid, then a clone automorphism corresponds to every class of $M^{\mathcal{A}}_{ker^G}/\sim_{VG(\mathcal{A})}$.*

Proof. Let $[\sigma]_{\sim_{VG(\mathcal{A})}}$ be a class from $M^{\mathcal{A}}_{ker^G}/\sim_{VG(\mathcal{A})}$. For this class we define a mapping φ by $\varphi(f_i^{\mathcal{A}}) := (\hat{\sigma}[f_i(x_1, ..., x_n)])^{\mathcal{A}}$. Clearly, if $\sigma \sim_{VG(\mathcal{A})} \sigma^{'}$, we have $(\hat{\sigma}[f_i(x_1, ..., x_n)])^{\mathcal{A}} = (\hat{\sigma^{'}}[f_i(x_1, ..., x_n)])^{\mathcal{A}}$. Then we get that φ is well-defined since $f_i^{\mathcal{A}} = f_j^{\mathcal{A}} \Longrightarrow i = j \Longrightarrow f_i = f_j \Longrightarrow \sigma(f_i) = \sigma(f_j) \Longrightarrow (\sigma(f_i))^{\mathcal{A}} = (\sigma(f_j))^{\mathcal{A}}$. By Proposition 2 we have $T(\mathcal{A})$ is free with respect to itself and that $\{f_i | i \in I\}$ is an independent set of generators. Since $\varphi(f_i^{\mathcal{A}}) = \varphi(f_j^{\mathcal{A}}) \Longrightarrow (\sigma(f_i))^{\mathcal{A}} = (\sigma(f_j))^{\mathcal{A}} \Longrightarrow \sigma(f_i) \approx \sigma(f_j) \in IdV(\mathcal{A}) \Longrightarrow f_i(x_1, ..., x_{n_i}) \approx f_j(x_1, ..., x_{n_i}) \in IdV(\mathcal{A})$, the mapping φ is one-to-one.

For the last step, we used $ker_{VG(\mathcal{A})} = IdV(\mathcal{A})$. The surjectivity of φ follows from $(\hat{\sigma}[W_{\tau}(X)])^{\mathcal{A}} = T(\mathcal{A})$. We show that $\varphi\mid_{T(\mathcal{A})}$ is an automorphism of

$$T(\mathcal{A}) = (T^{(n)}(\mathcal{A}); S^{n,A}, e_1^{n,A}, ..., e_n^{n,A})$$

for every n.
Indeed,

$$\begin{aligned}\varphi(S^{n,A}(t^{\mathcal{A}}, t_1^{\mathcal{A}}, ..., t_n^{\mathcal{A}})) &= (\hat{\sigma}[S^n(t, t_1, t_2, ..., t_n)])^{\mathcal{A}} \\ &= S^{n,A}((\hat{\sigma}[t])^{\mathcal{A}}, (\hat{\sigma}[t_1])^{\mathcal{A}}, ..., (\hat{\sigma}[t_n])^{\mathcal{A}}) \\ &= \varphi(S^n(t, t_1, ..., t_n^{\mathcal{A}})) \\ &= (S^n(\hat{\sigma}[t]), (\hat{\sigma}[t_1]), ..., (\hat{\sigma}[t_n]))^{\mathcal{A}} \\ &= S^{n,A}(\varphi(t^{\mathcal{A}}), \varphi(t_1^{\mathcal{A}}), ..., \varphi(t_n^{\mathcal{A}})).\end{aligned}$$

This work can be done in the same way if we apply the more general operators $S_m^{n,A}$ to sets of term operations of different arities. Here we used that $\hat{\sigma}$ is an endomorphism of $n-clone(\tau)$. Finally, we have $\varphi(e_i^{n,A}) = \varphi(x_i^{\mathcal{A}}) = \hat{\sigma}[x_i]^{\mathcal{A}} = x_i^{\mathcal{A}} = e_i^{n,A}$ for all $i \in I$. Therefore, φ is an automorphism of $\mathcal{T}(\mathcal{A})$.

Theorem 6. *If $V(\mathcal{A})$ is a strongly solid variety, then the group $Aut(\mathcal{T}(\mathcal{A}))$ of all clone automorphisms of $\mathcal{T}(\mathcal{A})$ is isomorphic to $M_{ker^G}^{\mathcal{A}}/\sim_{VG(\mathcal{A})}$.*

Proof. By Lemma 4 we define $\psi : Aut(\mathcal{T}(\mathcal{A})) \longrightarrow M_{ker^G}^{\mathcal{A}}/\sim_{VG(\mathcal{A})}$ by $\psi(\varphi) = [\hat{\sigma}]_{\sim VG(\mathcal{A})}$ with $(\hat{\sigma}[t])^{\mathcal{A}} = \varphi(t^{\mathcal{A}})$. We will show that ψ is a bijection. In fact, we have

$$\begin{aligned}\varphi_1 = \varphi_2 &\iff \forall t^A \in T(\mathcal{A})(\varphi_1(t^{\mathcal{A}}) = \varphi_2(t^{\mathcal{A}})) \\ &\iff \exists[\sigma_1]_{\sim_{VG(\mathcal{A})}}, [\sigma_2]_{\sim_{VG(\mathcal{A})}} \in M_{ker^G}^{\mathcal{A}}/\sim_{VG(\mathcal{A})} (\varphi_1(t^{\mathcal{A}}) = \varphi_2(t^{\mathcal{A}})) \\ &\iff \forall t \in W_\tau(X)(\sigma_1[t] \approx \sigma_2[t] \in IdV(\mathcal{V})) \\ &\iff \sigma_1 \sim_{VG(\mathcal{A})} \sigma_2 \\ &\iff [\sigma_1]_{\sim_{VG(\mathcal{A})}} = [\sigma_2]_{\sim_{VG(\mathcal{A})}}.\end{aligned}$$

The surjective of ψ follows from Lemma 5. We will show the compatibility of ψ with the operations. Let us note that $\psi(\varphi_1 \circ \varphi_2) = [\hat{\sigma_1} \circ \hat{\sigma_2}]_{\sim_{VG(\mathcal{A})}} = [\hat{\sigma_1}]_{\sim_{VG(\mathcal{A})}} \circ [\hat{\sigma_2}]_{\sim_{VG(\mathcal{A})}} = \psi(\varphi_1) \circ \psi(\varphi_2)$, since $(\varphi_1 \circ \varphi_2)(t^{\mathcal{A}}) = \varphi_1(\varphi_2(t^{\mathcal{A}})) = \varphi_1((\hat{\sigma_2}[t])^{\mathcal{A}}) = (\hat{\sigma_1}[\hat{\sigma_2}[t]])^{\mathcal{A}} = ((\hat{\sigma_1} \circ \hat{\sigma_2})[t])^{\mathcal{A}}$. For the identical automorphism, we have $\psi(\varphi_{id}) = [\sigma_{id}]_{\sim_{VG(\mathcal{A})}}$ since $\varphi_{id}(t^{\mathcal{A}}) = t^{\mathcal{A}} = (\sigma_{id}[t])^{\mathcal{A}}$.

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