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ON UPPER AND LOWER SLIGHTLY δ-β-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper, we introduce and study upper and lower slightly δ-β-continuous multifunctions in topological spaces and obtain some characterizations of these new continuous multifunctions.

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1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. Recently, Hatir and Noiri [2] have introduced a weak form of open sets called δ-β-open sets. In this paper, we introduce and study upper and lower slightly δ-β-continuous multifunctions in topological spaces and obtain some characterizations of these new continuous multifunctions and present several of their properties.

2. Preliminaries

Let A be a subset of a topological space (X, τ). We denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be regular open [6] if $A = \mathrm{Int}(\mathrm{Cl}(A))$. A set $A \subset X$ is said to be δ-open [7] if it is the union of regular open sets of X. The complement of a regular open (resp. δ-open) set is said to be regular closed (resp. δ-closed). The intersection of all δ-closed sets of (X, τ) containing A is said to be the δ-closure [7] of A and is denoted by $\mathrm{Cl}_\delta(A)$. A subset S of a topological space (X, τ) is said to be δ-β-open [2] if $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(S)))$. The complement of a δ-β-open set is said to be δ-β-closed [2]. The intersection of all δ-β-closed sets containing S is called the δ-β-closure of S and is denoted by $\beta\,\mathrm{Cl}_\delta(S)$. The δ-β-interior of S is defined by the union of all δ-β-open sets contained in S and is denoted by $\beta\,\mathrm{Int}_\delta(S)$. The family of all δ-β-open sets of (X, τ) is denoted by $\delta\beta O(X)$. The family of all δ-β-open sets of (X, τ) containing a point $x \in X$ is denoted by $\delta\beta O(X, x)$. By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset A of Y by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A)$ $= \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F(y)$ $= \{x \in X : y \in F(x)\}$ for each point $y \in Y$. A multifunction $F : X \to Y$ is said to be surjective if $F(X) = Y$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be lower δ-β-continuous [4] (resp. upper δ-β-continuous) multifunction if $F^-(V) \in \delta\beta O(X)$ (resp. $F^+(V) \in \delta\beta O(X)$) for every $V \in \sigma$.

3. Slightly δ-β-continuous Multifunctions

Definition 1. *A multifunction $F : X \to Y$ is said to be :*

(i) upper slightly δ-β-continuous at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \delta\beta O(X)$ containing x such that $F(U) \subset V$;

(ii) lower slightly δ-β-continuous at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \delta\beta O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;

(iii) upper (lower) slightly δ-β-continuous if it has this property at each point of X.

Remark 1. *It is clear that every upper δ-β-continuous multifunction is upper slightly δ-β-continuous. But the converse is not true in general, as the following example shows.*

Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\varnothing, \{a\}, X\}$. Then the multifunction $F : (X, \tau) \to (Y, \sigma)$ defined by $F(a) = \{b\}$, $F(b) = \{c\}$ and $F(c) = \{a\}$ is upper slightly δ-β-continuous but not upper δ-β-continuous.

Definition 2. *A sequence (x_n) is said to δ-β-converge to a point x if for every δ-β-open set V containing x, there exists an index x_0 such that for $n \geq n_0$, $x_n \in V$. This is denoted by $x_n \underset{\to}{\delta\beta} x$*

Theorem 1. *For a multifunction $F : X \to Y$, the following statements are equivalent :*

(i) *F is upper slightly δ-β-continuous;*

(ii) *For each $x \in X$ and for each clopen set V such that $x \in F^+(V)$, there exists a δ-β-open set U containing x such that $U \subset F^+(V)$;*

(iii) *For each $x \in X$ and for each clopen set V such that $x \in F^+(Y\backslash V)$, there exists a δ-β-closed set H such that $x \in X\backslash H$ and $F^-(V) \subset H$;*

(iv) *$F^+(V)$ is a δ-β-open set for any clopen set V of Y;*

(v) *$F^-(V)$ is a δ-β-closed set for any clopen set V of Y;*

(vi) *$F^-(Y\backslash V)$ is a δ-β-closed set for any clopen set V of Y;*

(vii) *$F^+(Y\backslash V)$ is a δ-β-open set for any clopen set V of Y.*

(viii) *For each $x \in X$ and for each net (x_n) which δ-β-converges to $x \in X$ and for each clopen set V of Y such that $x \in F^+(V)$, the net (x_n) is eventually in $F^+(V)$.*

Proof. (i)$\Leftrightarrow$(ii): Clear.
(ii)$\Leftrightarrow$(iii): Let $x \in X$ and V be a clopen set of Y such that $x \in F^+(Y\backslash V)$. By (ii), there exists a δ-β-open set U containing x such that $U \subset F^+(Y\backslash V)$. Then $F^-(V) \subset X\backslash U$. Take $H = X\backslash U$. We have $x \in X\backslash H$ and H is δ-β-open. The converse is similar.
(i)$\Leftrightarrow$(iv): Let $x \in F^+(V)$ and V be a clopen set of Y. By (i), there exists a δ-β-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \underset{x\in F^+(V)}{\cup} U_x$. Since any union of δ-β-open sets is δ-β-open, $F^+(V)$ is δ-β-open. The converse can be shown similarly.
(iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi)$\Leftrightarrow$(vii) : Clear.
(i)$\Rightarrow$(viii): Let (x_α) be a net which δ-β-converges to x in X and let V be any clopen set of Y such that $x \in F^+(V)$. Since F is an upper slightly δ-β-continuous multifunction, it follows that there exists a δ-β-open set U of X containing x such that $U \subset F^+(V)$. Since (x_α) δ-β-converges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x_\alpha \in U \subset F^+(V)$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^+(V)$.
(viii) $\Rightarrow$ (i): Suppose that (i) is not true. There exists a point x and a clopen set V with $x \in F^+(V)$ such that $U \nsubseteq F^+(V)$ for each δ-β-open set U of X containing x. Let $x_U \in U$ and $x_U \notin F^+(V)$ for each δ-β-open set U of X containing x. Then for

each δ-β-neighbourhood net (x_U), $x_U \underset{\rightarrow}{\delta\beta} x$, but (x_U) is not eventually in $F^+(V)$. This is a contradiction. Thus, F is an upper slightly δ-β-continuous multifunction.

Theorem 2. *For a multifunction $F : X \to Y$, the following statements are equivalent :*

(i) *F is lower slightly δ-β-continuous;*

(ii) *For each $x \in X$ and for each clopen set V such that $X \in F^-(V)$, there exists a δ-β-open set U containing x such that $U \subset F^-(V)$;*

(iii) *For each $x \in X$ and for each clopen set V such that $x \in F^-(Y\backslash V)$, there exists a δ-β-closed set H such that $x \in X\backslash H$ and $F^+(V) \subset H$;*

(iv) *$F^-(V)$ is a δ-β-open set for any clopen set V of Y;*

(v) *$F^+(V)$ is a δ-β-closed set for any clopen set V of Y;*

(vi) *$F^+(Y\backslash V)$ is a δ-β-closed set for any clopen set V of Y;*

(vii) *$F^-(Y\backslash V)$ is a δ-β-open set for any clopen set V of Y.*

(viii) *For each $x \in X$ and for each net (x_α) which δ-β-converges to $x \in X$ and for each clopen set V of Y such that $x \in F^-(V)$ the net (x_γ) is eventually in $F^-(V)$.*

Proof. The proof is similar to that of Theorem 1.

Lemma 3. *[2] If A is δ-open in X and $B \in \delta\beta O(X)$, then $A \cap B \in \delta\beta O(A)$.*

Theorem 4. *Let $F : X \to Y$ be a multifunction and U is delta-open in X. If F is a lower (upper) slightly δ-β-continuous multifunction, then multifunction $F_{|_U} : U \to Y$ is a lower (upper) slightly δ-β-continuous multifunction.*

Proof. Let V be any clopen set of Y, $x \in U$ and $x \in F^-_{|_U}(V)$. Since F is lower slightly δ-β-continuous multifunction, it follows that there exists a δ-β-open set G containing x such that $G \subset F^-(V)$. From here by Lemma 3, we obtain that $x \in G \cap U \in \gamma O(U)$ and $G \cap U \subset F^-_{|_U}(V)$. This shows that the restriction multifunction $F_{|_U}$ is a lower slightly δ-β-continuous. The proof of the upper slightly δ-β-continuity of $F_{|_U}$ can be done by the same token.

Definition 3. *For a multifunction $F : X \to Y$, the graph multifunction $G_F : x \to X \times Y$ is defined as follows $F_F(x) = \{x\} \times F(x)$ for every $x \in X$ and subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and s denoted by $G(F)$.*

Lemma 5. *For a multifunction $F : X \to Y$, the following holds:*

(i) $G_F^+(A \times B) = A \cap F^+(B)$;

(ii) $G_F^-(A \times B) = A \cap F^-(B)$

for any subset A of X and B of Y.

Theorem 6. *Let $F : X \to Y$ be a multifunction. If the graph multifunction of F is an upper slightly δ-β-continuous, then F is an upper slightly δ-β-continuous.*

Proof. Let $x \in X$ and V be any clopen subset of Y such that $x \in F^+(V)$. We obtain that $x \in G_F^+(X \times V)$ and that $X \times V$ is a clopen set. Since the graph multifunction G_F is upper slightly δ-β-continuous, it follows that there exists a δ-β-open set U of X containing x such that $U \subset G_F^+(X \times V)$. Since $U \subset G_F^+(X \times V) = X \cap F^+(V) = F^+(V)$. We obtain that $U \subset F^+(V)$. Thus, F is upper slightly δ-β-continuous.

Theorem 7. *A multifunction $F : X \to Y$ is lower slightly δ-β-continuous if $G_F : (X, \tau) \to (X \times Y, \tau \times \sigma)$ is lower slightly δ-β-continuous.*

Proof. Suppose that G_F is lower slightly δ-β-continuous. Let $x \in X$ and V be any clopen set of Y such that $x \in F^-(V)$. Then $X \times V$ is clopen in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \varnothing$. Since G_F is lower slightly δ-β-continuous, there exists a δ-β-open U containing x such that $U \subset G_F^-(X \times V)$; hence $U \subset F^-(V)$. This shows that F is lower slightly δ-β-continuous.

Theorem 8. *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is an upper (lower) slightly δ-β-continuous multifunction, then $P_\alpha \circ F$ is an upper (lower) slightly δ-β-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be a clopen set in $(X_{\alpha 0}, \tau_{\alpha 0})$. Then $(P_{\alpha 0} \circ F)^+(V_{\alpha 0}) = F^+(P_{\alpha 0}^+(V_{\alpha 0})) = F^+(V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (resp. $(P_{\alpha 0} \circ F)^-(V_{\alpha 0}) = F^-(P_{\alpha 0}^-(V_{\alpha 0})) = F^-(V_{\alpha 0} \times \prod_{\alpha \neq \alpha 0} X_\alpha)$). Since F is an upper (lower) slightly δ-β-continuous multifunction and since $V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a clopen set, it follows that $F^+(V_{\alpha 0} \times \prod_{\alpha \neq 0} X_\alpha)$ (resp. $F^-(V_{\alpha 0} \times \prod_{\alpha \neq \alpha 0} X_\alpha)$) is a δ-β-open set in (X, τ). This shows that $P_{\alpha 0} \circ F$ is an upper (lower) slightly δ-β-continuous multifunction. Hence, we obtain that $P_\alpha \circ F$ is an upper (lower) slightly δ-β-continuous multifunction for each $\alpha \in J$.

Theorem 9. *Suppose that for each $\alpha \in J$, (X_α, τ_α), $(Y_\alpha, \sigma_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha\in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha\in J} X_\alpha$ to the product space $\prod_{\alpha\in J} Y_\alpha$. If F is an upper (lower) slightly δ-β-continuous multifunction, then each F_α is an upper (lower) slightly δ-β-continuous multifunction for each $\alpha \in J$.*

Proof. Let V_α be a clopen set of Y_α. Then $V_\alpha \times \prod_{\alpha\neq\beta} Y_\beta$ is a clopen set. Since F is an upper (lower) slightly δ-β-continuous multifunction, it follows that $F^+(V_\alpha \times \prod_{\alpha\neq\beta} Y_\beta)$ $= F_\alpha^+(V_\alpha) \times \prod_{\alpha\neq\beta} X_\beta$ (resp. $F^-(V_\alpha \times \prod_{\alpha\neq\beta} Y_\beta) = F_\alpha^-(V_\alpha) \times \prod_{\alpha\neq\beta} X_\beta$) is a δ-β-open set. Consequently, we obtain that $F_\alpha^+(V_\alpha)$ (resp. $F_\alpha^-(V_\alpha)$) is an δ-β-open set. Thus, we show that F_α is an upper (lower) slightly δ-β-continuous multifunction.

Recall that for two multifunctions $F_1 : X_1 \to Y_1$ and $F_2 : X_2 \to Y_2$, the product multifunction $F_1 \times F_2$: $X_1 \times X_2 \to Y_1 \times Y_2$ is defined as follows: $(F_1 \times F_2)$ (x_1, x_2) $= F_1(x_1) \times F_2(x_2)$ for every $x_1 \in X_1$ and $x_2 \in X_2$.

Lemma 10. *[3] If $A \in \delta\beta O(X)$ and $B \in \delta\beta O(Y)$, then $A \times B \in \delta\beta O(X \times Y)$.*

Theorem 11. *Suppose that $F_1 : X_1 \to Y_1$, $F_2 : X_2 \to Y_2$ are multifunctions. If $F_1 \times F_2$ is an upper (lower) slightly δ-β-continuous multifunction, then F_1 and F_2 are upper (lower) slightly δ-β-continuous multifunctions.*

Proof. Let $K \subset Y_1$ and $H \subset Y_2$ be clopen sets. Then $(F_1 \times F_2)^+(K \times H) = F_1^+(K)$ $\times$ $F_2^+(H)$. Since $F_1 \times F_2$ is upper slightly δ-β-continuous multifunction, it follows that $F_1^+(K) \times F_2^+(H)$ is a δ-β-open set. Therefore, $F_1^+(K)$ and $F_2^+(H)$ are δ-β-open sets. Hence, F_1 and F_2 are upper slightly δ-β-continuous multifunctions. The proof of the lower slightly δ-β-continuity of F_1 and F_2 is similar to the above argument.

Theorem 12. *Suppose that $(X, \tau), (Y, \sigma), (Z, \eta)$ are topological spaces and F_1: $X \to Y$, F_2: $X \to Z$ are multifunctions. Let $F_1 \times F_2$: $X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) slightly δ-β-continuous multifunction, then F_1 and F_2 are upper (lower) slightly δ-β-continuous multifunctions.*

Proof. Let $x \in X$, $K \subset Y$ and $H \subset Z$ be clopen sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and thus, $F_1(x)\times F_2(x)$ $= (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper slightly δ-β-continuous multifunction, it follows that there exists a δ-β-open set U containing x such that $U \subset (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subset F_1^+(K)$ and $U \subset F_2^+(H)$. Thus, F_1 and F_2 are upper slightly δ-β-continuous multifunction. The proof of the lower slightly δ-β-continuity of F_1 and F_2 is similar to the above.

Definition 4. *[5] Let (X, τ) be a topological space. X is said to be a strongly normal space if for every disjoint closed subsets K and F of X, there exist two clopen sets U and V such that $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$.*

Recall that a multifunction $F : X \to Y$ is said to be punctually closed if for each $x \in X$, $F(x)$ is closed.

Theorem 13. *If Y is a strongly normal space and $F_i : X_i \to Y$ is an upper slightly δ-β-continuous multifunction such that F_i is punctually closed for i = 1, 2, then a set $\{(x_1, x_2) \in X_1 \times X_2\colon F_1(x_1) \cap F_2(x_2) \neq \varnothing\}$ is a δ-β-closed set in $X_1 \times X_2$.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2\colon F_1(x_1) \cap F_2(x_2) \neq \varnothing\}$ and $(x_1, x_2) \in (X_1 \times X_2) \backslash A$. Then $F_1(x_1) \cap F_2(x_2) = \varnothing$. Since Y is strongly normal and F_i is punctually closed for i = 1, 2, there exist disjoint clopen sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for i = 1, 2. Since F_i is upper slightly δ-β-continuous $F_i^+(V_i)$ is a δ-β-open set for i = 1, 2. Put $U = F_1^+(V_1) \times F_2^+(V_2)$, then U is a δ-β-open set and $(x_1, x_2) \in U \subset (X_1 \times X_2) \backslash A$. This shows that $(X_1, \times X_2) \backslash A$ is δ-β-open and hence A is δ-β-closed in $X_1 \times X_2$.

Recall that a topological space (X, τ) is said to be ultra normal [5] if every two disjoint closed sets of X can be separated by clopen sets.

Theorem 14. *Let F and G be upper slightly δ-β-continuous and punctually closed multifunctions from a topological space (X, τ) to a strongly normal space (Y, σ). Then the set $K = \{x : F(x) \cap G(x) \neq \varnothing\}$ is δ-β-closed in X.*

Proof. Let $x \in X \backslash K$. Then $F(x) \cap G(x) = \varnothing$. Since F and G are punctually closed multifunctions and Y is a strongly normal space, it follows that there exist disjoint clopen sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are upper slightly δ-β-continuous multifunctions, then the sets $F^+(U)$ and $G^+(V)$ are δ-β-open sets containing x. Let $H = F^+(U) \cup G^+(V)$. Then H is an δ-β-open set containing x and $H \cap K = \varnothing$; hence K is δ-β-open in X.

Definition 5. *A topological space (X, τ) is said to be δ-β-T_2 [3] if for each pair of distinct points x and y in X, there exist disjoint δ-β-open sets U and V in X such that $x \in U$ and $y \in V$.*

Theorem 15. *Let $F : X \to Y$ be an upper slightly δ-β-continuous multifunction and punctually closed from a topological space X to a strongly normal space Y and let $F(x) \cap F(y) = \varnothing$ for each pair of distinct points x and y of X. Then X is a δ-β-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \varnothing$. Since Y is strongly normal, it follows that there exist disjoint clopen sets U and V containing $F(x)$ and $F(y)$, respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint δ-β-open sets containing x and y, respectively and hence (X, τ) is δ-β-T_2.

Definition 6. *A topological space (X, τ) is said to be mildly compact [5] (resp. δ-β-compact) if every clopen (resp. δ-β-open) cover of X has a finite subcover.*

Theorem 16. *Let $F : X \to Y$ be an upper slightly δ-β-continuous surjective multifunction such that $F(x)$ is mildly compact for each $x \in X$. If X is δ-β-compact space, then Y is mildly compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a clopen cover of Y. Since $F(x)$ is mildly compact for each $x \in X$, there exists a finite subset $\Lambda(x)$ of Λ such that $F(x) \subset \cup\{V_\alpha\colon \alpha \in \Lambda(x)\}$. Put $V(x) = \cup\{V_\alpha\colon \alpha \in \Lambda(x)\}$. Since F is an upper slightly δ-β-continuous, there exists a δ-β-open set $U(x)$ of X containing x such that $F(U(x)) \subset V(x)$. Then the family $\{U(x) : x \in X\}$ is a δ-β-open cover of X and since X is δ-β-compact, there exists a finite number of points, say, $x_1, x_2, x_3,...$ x_n in X such that $X = \cup\{U(x_i)\colon i = 1, 2,..., n\}$. Hence we have $Y = F(X) = F(\overset{n}{\underset{i=1}{\cup}} U(x_i)) = \overset{n}{\underset{i=1}{\cup}} F(U(x_i)) \subset \overset{n}{\underset{i=1}{\cup}} V(x_i) = \overset{n}{\underset{i=1}{\cup}} \underset{\alpha \in \Lambda(x_i)}{\cup} V_\alpha$. This shows that Y is mildly compact.

Definition 7. *Let $F : X \to Y$ be a multifunction. The multigraph $G(F)$ is said to be δ-β-co-closed if for each $(x, y) \notin G(F)$, there exist δ-β-open set U and clopen set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \varnothing$.*

Definition 8. *[5] A topological space (X, τ) is said to be clopen T_2 (clopen Hausdorff) if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x \in U$ and $y \in V$.*

Theorem 17. *If a multifunction $F : X \to Y$ is an upper slightly δ-β-continuous such that $F(x)$ is mildly compact relative to Y for each $x \in X$ and Y is a clopen Hausdorff space, then the multigraph $G(F)$ of F is δ-β-co-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. That is $y \notin F(x)$. Since Y is clopen Hausdorff, for each $z \in F(x)$, there exist disjoint clopen sets $V(z)$ and $U(z)$ of Y such that $z \in U(z)$ and $y \in V(z)$. Then $\{U(z) : z \in F(x)\}$ is a clopen cover of $F(x)$ and since $F(x)$ is mildly compact, there exists a finite number of points, say, z_1, z_2, ... , z_n in $F(x)$ such that $F(x) \subset \cup\{U(z_i) : i = 1, 2, ..., n\}$. Put $U = \cup\{U(z_i)\colon i = 1, 2, ..., n\}$ and $V = \cap \{V(y_i)\colon i = 1, 2, ..., n\}$. Then U and V are clopen sets in Y such that $F(x) \subset U$, $y \in V$ and $U \cap V = \varnothing$. Since F is upper slightly δ-β-continuous multifunction, there exists a δ-β-open set W of X containing x such that $F(W) \subset U$.

We have $(x, y) \in W \times V \subset (X \times Y) \setminus G(F)$. We obtain that $(W \times V) \cap G(F) = \varnothing$ and hence $G(F)$ is δ-β-co-closed in $X \times Y$.

Theorem 18. *Let $F : X \to Y$ be a multifunction having δ-β-co-closed multigraph $G(F)$. If B is a mildly compact subset relative to Y, then $F^-(B)$ is δ-β-closed in X.*

Proof. Let $x \in X \backslash F^-(B)$. For each $y \in B$, $(x, y) \notin G(F)$ and there exist a δ-β-open set $U(y) \subset X$ and a clopen set $V(y) \subset Y$, containing x and y, respectively, such that $F(U(y)) \cap V(y) = \varnothing$. That is, $U(y) \cap F^-(V(y)) = \varnothing$. Then $\{V(y)\colon y \in B\}$ is a clopen cover of B and since B is mildly compact relative to Y, there exists a finite subset B_0 of B such that $B \subset \cup\{V(y) : y \in B_0\}$. Put $U = \cap\{U(y) : y \in B_0\}$. Then U is δ-β-open in X, $x \in U$ and $U \cap F^-(B) = \varnothing$; that is, $x \in U \subset X \backslash F^-(B)$. This shows that $F^-(B)$ is δ-β-closed in X.

References

[1] C. Berge, *Espaces topologiques functions multivoques*, Paris, Dunod (1959).

[2] E. Hatir and T. Noiri, *Decompositions of continuity and complete continuity*, Acta Math. Hungar., 113 (4) (2006), 281-287.

[3] E. Hatir and T. Noiri, *On δ-β-continuous functions*, to appear in Chaos, Solutions and Fractals.

[4] N. Rajesh, *On upper and lower δ-β-continuous multifunctions* (submitted).

[5] R. Staum, *The algebra of bounded continuous fuctions into a nonarchimedean field*, Pacific J. Math., 50(1974), 169-185.

[6] M. Stone, *Applications of the theory of boolean rings to general topology*, Trans. Amer. Math. Soc., 41(1937), 374-381.

[7] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl.(2), 78(1968), 103-118.

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