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SOME INCLUSION RELATIONS ASSOCIATED WITH LIU-OWA INTEGRAL OPERATOR

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Abstract. In this paper we introduce and study some new subclasses of p-valent starlike, convex, close-to-convex and quasi-convex functions defined by the means of Liu-Owa operator. Some inclusion relationships and their inverse inclusion relationships are established. Integral operator of functions in these subclasses is discussed .

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1. Introduction and Definitions

Let $\mathbb{A}_{n,p}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \qquad (p, n \in N = \{1, 2, 3...\}), \tag{1.1}$$

which are analytic and p-valent in the unit disc $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$.
A function $f(z) \in \mathbb{A}_{n,p}$ is said to be in the class $S^*_{n,p}(\lambda)$ of p-valently starlike of order λ, if it satisfies the inequality

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \lambda \qquad (0 \leq \lambda < p,\ z \in \mathbb{U}). \tag{1.2}$$

Also a function $f(z) \in \mathbb{A}_{n,p}$ is said to be in the class $C_{n,p}(\lambda)$ of p-valently convex of order λ if it satisfies the inequality

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \lambda \qquad (0 \leq \lambda < p,\ z \in \mathbb{U}). \tag{1.3}$$

It follows from (1.2) and (1.3) that

$$f(z) \in C_{n,p}(\lambda) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in S^*_{n,p}(\lambda) \qquad (0 \leq \lambda < p,\ z \in \mathbb{U}). \tag{1.4}$$

The class $S^*_{1,p}(\lambda)$ was introduced by Patil and Thakare [12] and the class $C_{1,p}(\lambda)$ was introduced by Owa [11]. Also we note that the class $S^*_{1,p}(0) = S^*_p$ and $C_{1,p}(0) = C_p$ were introduced by Goodman [3]. Furthermore, a function $f(z) \in \mathbb{A}_{n,p}$ is said to be p-valently close-to-convex of order θ and type λ in $\mathbb{U}$, if there exist a function $g(z) \in S^*_{n,p}(\lambda)$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > \theta \qquad (0 \leq \theta, \quad \lambda < p, \quad z \in \mathbb{U}). \tag{1.5}$$

We denote by $K_{n,p}(\theta,\lambda)$, the subclass of consisting of all such functions, the class $K_{1,p}(\theta,\lambda)$ was studied by Aouf [1]. We note that $K_{1,1}(\theta,\lambda)$=$K(\theta,\lambda)$ is the class of close-to-convex of order θ and type λ, was studied by Libera [5]. Also a function $f(z) \in \mathbb{A}_{n,p}$ is called p-valently quasi-convex of order θ and type λ, if there exists a function $g(z) \in C_{n,p}(\lambda)$ such that

$$Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \theta \qquad (0 \leq \theta, \ \ \lambda < p, \ \ z \in \mathbb{U}). \tag{1.6}$$

We denote this class by $C^*_{n,p}(\theta,\lambda)$, also we observe that $C^*_{1,1}(\theta,\lambda) = C^*(\theta,\lambda)$, is the class of quasi-convex of order θ and type λ $(0 \leq \theta, \ \ \lambda < 1)$, was introduced and studied by Noor [9,10].

It follows from (1.5) and (1.6) that

$$f(z) \in C^*_{n,p}(\theta,\lambda) \qquad \text{if and only if} \qquad \frac{zf'(z)}{p} \in K^*_{n,p}(\theta,\lambda) \qquad (0 \leq \lambda < p, \ \ z \in \mathbb{U}). \tag{1.7}$$

Motivated by Jung *et al.*[4], Liu and Owa [6] considered the integral operator $Q^\alpha_{\beta,p} : \mathbb{A}_{n,p} \to \mathbb{A}_{n,p}$, defined by

$$Q^\alpha_{\beta,p}f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1}\frac{\alpha}{z^\beta}\int_0^z\left(1-\frac{t}{z}\right)^{\alpha-1}t^{\beta-1}f(t)dt \qquad (\alpha > 0, \ \ \beta > -1) \tag{1.8}$$

and

$$Q^0_{\beta,p}f(z) = f(z) \qquad (\alpha = 0, \ \ \beta > -1). \tag{1.9}$$

We note that if $f(z) \in \mathbb{A}_{n,p}$, then from (1.8), we have

$$Q^\alpha_{\beta,p}f(z) = z^p + \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)}\sum_{k=p+n}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)}a_kz^k. \tag{1.10}$$

It is easy to verify that

$$z(Q^{\alpha}_{\beta,p}f(z))' = (p+\alpha+\beta-1)Q^{\alpha-1}_{\beta,p}f(z) - (\alpha+\beta-1)Q^{\alpha}_{\beta,p}f(z). \tag{1.11}$$

Using the linear operator $Q^{\alpha}_{\beta,p}$, we now introduce the following subclasses of $\mathbb{A}_{n,p}$:

$$S^*_{n,p}(\alpha,\lambda) = \Big(f \in \mathbb{A}_{n,p} : \quad Q^{\alpha}_{\beta,p}f(z) \in S^*_{n,p}(\lambda), \quad 0 \le \lambda < p\Big),$$

$$C_{n,p}(\alpha,\lambda) = \Big(f \in \mathbb{A}_{n,p} : \quad Q^{\alpha}_{\beta,p}f(z) \in C_{n,p}(\lambda), \quad 0 \le \lambda < p\Big),$$

$$K_{n,p}(\alpha,\theta,\lambda) = \Big(f \in \mathbb{A}_{n,p} : \quad Q^{\alpha}_{\beta,p}f(z) \in K_{n,p}(\theta,\lambda), \quad 0 \le \lambda < p\Big)$$

and

$$C^*_{n,p}(\alpha,\theta,\lambda) = \Big(f \in \mathbb{A}_{p} : \quad Q^{\alpha}_{\beta,p}f(z) \in C^*_{n,p}(\theta,\lambda), \quad 0 \le \lambda < p\Big).$$

In this paper, we shall establish the various inclusion relationships and their inverse inclusion relationships for these subclasses of $\mathbb{A}_{n,p}$.

2. Preliminary Results

In order to prove our main results, we shall require the following lemma:

Lemma 1. *[8]. let $\psi(u,v)$ be a complex-valued function such that $\psi : D \subset \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $\psi(u,v)$ satisfies each of the following conditions:*
(i) $\psi(u,v)$ is continuous in D;
(ii) $(1,0) \in D$ and $Re\{\psi(1,0)\} > 0$;
(iii) $Re\{\psi(iu_2,v_1)\} \le 0$ for all $(iu_2,v_1) \in D$ such that $v_1 \le \frac{-n(1+u_2^2)}{2}$.
Let $h(z) = 1 + c_nz^n + c_{n+1}z^{n+1} + ...$ be analytic in $\mathbb{U}$, such that $(h(z), zh'(z)) \in D$ for all $z \in \mathbb{U}$. If $Re\{\psi(h(z), zh'(z))\} > 0$, $(z \in \mathbb{U})$ then $Re\{h(z)\} > 0$, $(z \in \mathbb{U})$.

3. Main Results

Theorem 2. *(a) Let $\lambda+\alpha+\beta-2 > 0$, $\alpha \ge 2$, $\beta > -1$. For $f \in \mathbb{A}_{n,p}$, if*

$$Re\left(\frac{z(Q^{\alpha-2}_{\beta,p}f(z))'}{(Q^{\alpha-2}_{\beta,p}f(z)} - \frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}f(z)}\right) > 0$$

and $\frac{z(Q^{\alpha-1}_{\beta,p}f(z))'}{(Q^{\alpha-1}_{\beta,p}f(z)}$ is analytic function, then $S^*_{n,p}(\alpha,\lambda) \subset S^*_{n,p}(\alpha-1,\lambda)$.

(b) Let $\lambda+\alpha+\beta-1 > 0$, $\alpha \ge 1$, $\beta > -1$. For $f \in \mathbb{A}_{n,p}$, if $\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}f(z))}$ is analytic function, then $S^*_{n,p}(\alpha-1,\lambda) \subset S^*_{n,p}(\alpha,\lambda)$.

Proof. (a) Let $f(z) \in S^*_{n,p}(\alpha, \lambda)$ and set

$$\frac{z(Q^{\alpha-1}_{\beta,p} f(z))'}{(Q^{\alpha-1}_{\beta,p} f(z)} = \lambda + (p-\lambda)h(z), \tag{3.1}$$

where $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots$. An easy calculation shows that

$$\frac{\frac{z(Q^{\alpha-1}_{\beta,p} f(z))'}{Q^{\alpha-1}_{\beta,p} f(z)} \left(\frac{z(Q^{\alpha-1}_{\beta,p} f(z))''}{(Q^{\alpha-1}_{\beta,p} f(z))'} + \alpha + \beta - 1 \right)}{\frac{z(Q^{\alpha-1}_{\beta,p} f(z))'}{(Q^{\alpha-1}_{\beta,p} f(z))} + \alpha + \beta - 2} = \frac{z(Q^{\alpha-2}_{\beta,p} f(z))'}{Q^{\alpha-2}_{\beta,p} f(z)}. \tag{3.2}$$

By setting $H(z) = \frac{z(Q^{\alpha-1}_{\beta,p} f(z))'}{(Q^{\alpha-1}_{\beta,p} f(z)}$, we have

$$H(z) + \frac{zH'(z)}{H(z)} = 1 + \frac{z(Q^{\alpha-1}_{\beta,p} f(z))''}{(Q^{\alpha-1}_{\beta,p} f(z))'}. \tag{3.3}$$

By making use of (3.3) in (3.2), we obtain

$$(p-\lambda)h(z) + \frac{z(p-\lambda)h'(z)}{\lambda + (p-\lambda)h(z) + \alpha + \beta - 2} = \frac{z(Q^{\alpha-2}_{\beta,p} f(z))'}{(Q^{\alpha-2}_{\beta,p} f(z))} - \lambda. \tag{3.4}$$

If we consider

$$\psi(u,v) = (p-\lambda)u + \frac{(p-\lambda)v}{\lambda + (p-\lambda)u + \alpha + \beta - 2}$$

with $h(z) = u$ and $zh'(z) = v$, then
(i) $\psi(u,v)$ is continuous in $D = \left(\mathbb{C} \backslash \{\frac{(2-\lambda-\alpha-\beta)}{p-\lambda}\}\right) \times \mathbb{C}$
(ii) $(1,0) \in D$ and $Re(\psi(1,0)) > 0$,
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-n(1+u_2^2)}{2}$

$$\begin{aligned} Re(\psi(iu_2, v_1)) &= \frac{(p-\lambda)(\lambda+\alpha+\beta-2)v_1}{(\lambda+\alpha+\beta-2)^2 + (p-\lambda)^2 u_2^2} \\ &\leq \frac{-n}{2} \quad \frac{(p-\lambda)(\lambda+\alpha+\beta-2)(1+u_2^2)}{(\lambda+\alpha+\beta-2)^2 + (p-\lambda)^2 u_2^2} < 0. \end{aligned}$$

Therefore the function $\psi(u,v)$ satisfies the conditions of Lemma 1 and since in view of the assumption, by considering (3.4), we have $Re(\psi(h(z), zh'(z)) > 0$. Thus we have $Re\{h(z)\} > 0 (z \in \mathbb{U})$, that is $f(z) \in S^*_{n,p}(\alpha - 1, \lambda)$. This completes the proof of Theorem 1(a).

Proof. (b) The proof of this part of Theorem is analogous to part (a) of Theorem 1, so we choose to omit it.

Theorem 3. *(a) Let* $\lambda+\alpha+\beta-2>0,\ \alpha\geq 2,\ \beta>-1$. *For* $f\in\mathbb{A}_{n,p}$ *if*

$$Re\left(\frac{z(Q^{\alpha-2}_{\beta,p}f(z))'}{(Q^{\alpha-2}_{\beta,p}f(z)}-\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}f(z)}\right)>0$$

and $\frac{z(Q^{\alpha-1}_{\beta,p}f(z))'}{(Q^{\alpha-1}_{\beta,p}f(z)}$ *is analytic function, then* $C_{n,p}(\alpha,\lambda)\subset C_{n,p}(\alpha-1,\lambda)$.

(b) Let $\lambda+\alpha+\beta-1>0,\quad \alpha\geq 1,\quad \beta>-1$. *For* $f\in\mathbb{A}_{n,p}$, *if* $\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}f(z))}$ *is analytic function, then* $C_{n,p}(\alpha-1,\lambda)\subset C_{n,p}(\alpha,\lambda)$.

Proof. (a) $f(z)\in C_{n,p}(\alpha,\lambda)\Leftrightarrow Q^{\alpha}_{\beta,p}f(z)\in C_{n,p}(\lambda)\ \Leftrightarrow\frac{z}{p}(Q^{\alpha}_{\beta,p}f(z))'\in S^{*}_{n,p}(\lambda)$
$\Leftrightarrow Q^{\alpha}_{\beta,p}(\frac{zf'(z)}{p})\in S^{*}_{n,p}(\lambda)\Leftrightarrow\frac{zf'(z)}{p}\in S^{*}_{n,p}(\alpha,\lambda)$
$\Rightarrow\frac{zf'(z)}{p}\in S^{*}_{n,p}(\alpha-1,\lambda)\Leftrightarrow Q^{\alpha-1}_{\beta,p}(\frac{zf'(z)}{p})\in S^{*}_{n,p}(\lambda)$
$\Leftrightarrow\frac{z}{p}(Q^{\alpha-1}_{\beta,p}f(z))'\in S^{*}_{n,p}(\lambda)\Leftrightarrow Q^{\alpha-1}_{\beta,p}f(z)\in C_{n,p}(\lambda)$
$\Leftrightarrow f(z)\in C_{n,p}(\alpha-1,\lambda)$.

Proof. (b) The proof of this part of Theorem is analogous to part (a) of Theorem 2, so we choose to omit it.

Theorem 4. *(a) Let* $\lambda+\alpha+\beta-2>0$. *For* $f\in\mathbb{A}_{n,p}$, *if*

$$Re\left(\frac{z(Q^{\alpha-2}_{\beta,p}f(z))'}{(Q^{\alpha-2}_{\beta,p}g(z)}-\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}g(z)}\right)>0$$

and $\frac{z(Q^{\alpha-1}_{\beta,p}f(z))'}{(Q^{\alpha-1}_{\beta,p}g(z)}$ *is analytic function, then* $K_{\alpha}(\beta,\theta,\lambda)\subset K_{\alpha-1}(\beta,\theta,\lambda)$.

(b) Let $\lambda+\alpha+\beta-1>0,\quad \alpha\geq 1,\quad \beta>-1$. *For* $f\in\mathbb{A}_{n,p}$, *if* $\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}f(z))}$ *is analytic function, then* $K_{\alpha-1}(\beta,\theta,\lambda)\subset K_{\alpha}(\beta,\theta,\lambda)$.

Proof. (a) Suppose that

$$f(z)\in K_{\alpha}(\beta,\theta,\lambda)$$

So

$$Re\left(\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}g(z)}\right)>\theta, \tag{3.5}$$

where
$$Re\left(\frac{z(Q^{\alpha}_{\beta,p}g(z))'}{(Q^{\alpha}_{\beta,p}g(z)}\right) > \lambda. \tag{3.6}$$

Set
$$\frac{z(Q^{\alpha-1}_{\beta,p}f(z))'}{(Q^{\alpha-1}_{\beta,p}g(z)} = \theta + (p-\theta)p(z), \tag{3.7}$$

where $\qquad p(z) = 1 + d_n z^n + d_{n+1}z^{n+1} + \ldots$.
Now using recurrence relation for $f(z)$ and $g(z)$ and after some easy calculation we get

$$\frac{\frac{z(Q^{\alpha-1}_{\beta,p}f(z))'}{(Q^{\alpha-1}_{\beta,p}g(z))}\left(\frac{z(Q^{\alpha-1}_{\beta,p}f(z))''}{(Q^{\alpha-1}_{\beta,p}f(z))'} + \alpha + \beta - 1\right)}{\frac{z(Q^{\alpha-1}_{\beta,p}g(z))'}{(Q^{\alpha-1}_{\beta,p}g(z)} + \alpha + \beta - 2} = \frac{z(Q^{\alpha-2}_{\beta,p}f(z))'}{(Q^{\alpha-2}_{\beta,p}g(z))} \tag{3.8}$$

By setting
$$H(z) = \frac{z(Q^{\alpha-1}_{\beta,p}f(z))'}{(Q^{\alpha-1}_{\beta,p}g(z)}, \tag{3.9}$$

we get
$$H(z) = \theta + (p-\theta)p(z) \tag{3.10}$$

or
$$H'(z) = (p-\theta)p'(z). \tag{3.11}$$

Taking logarithmic differentiation of (3.9), we get

$$\frac{zH'(z)}{H(z)} = 1 + \frac{z(Q^{\alpha-1}_{\beta,p}f(z))''}{(Q^{\alpha-1}_{\beta,p}f(z))'} - \frac{z(Q^{\alpha-1}_{\beta,p}g(z))'}{(Q^{\alpha-1}_{\beta,p}g(z))}. \tag{3.12}$$

Again set
$$\frac{z(Q^{\alpha-1}_{\beta,p}g(z))'}{(Q^{\alpha-1}_{\beta,p}g(z))} = \lambda + (p-\lambda)q(z), \tag{3.13}$$

where $q(z) = 1 + e_n z^n + e_{n+1}z^{n+1} + \ldots$.
Now by (3.12) and (3.13) we have

$$\frac{z(Q^{\alpha-1}_{\beta,p}f(z))''}{(Q^{\alpha-1}_{\beta,p}f(z))'} = \frac{zH'(z)}{H(z)} + (\lambda-1) + (p-\lambda)q(z). \tag{3.14}$$

Putting (3.10) in (3.4) we get

$$\frac{H(z)\left(\frac{zH'(z)}{H(z)}+(\lambda-1)+(p-\lambda)q(z)+\alpha+\beta-1\right)}{\lambda+\alpha+\beta-2+(p-\lambda)q(z)}=\frac{z(Q^{\alpha-2}_{\beta,p}f(z))'}{(Q^{\alpha-2}_{\beta,p}g(z))}$$

In view of (3.6) and (3.7), it yields

$$(p-\theta)p(z)+\frac{z(p-\theta)p'(z)}{\lambda+\alpha+\beta-2+(p-\lambda)q(z)}=\frac{z(Q^{\alpha-2}_{\beta,p}f(z))'}{(Q^{\alpha-2}_{\beta,p}g(z))}-\theta.$$

Let

$$\psi(u,v)=(p-\theta)u+\frac{(p-\theta)v}{\lambda+\alpha+\beta-2+(p-\lambda)q(z)},$$

with $p(z)=u$ and $zp'(z)=v$, then
(i) $\psi(u,v)$ is continuous in $D=(\mathbb{C}\times\mathbb{C})$
(ii) $(1,0)\in D$ and $Re(\psi(1,0))=p-\theta>0$,
(iii) for all $(iu_2,v_1)\in D$ such that $v_1\leq\frac{-n(1+u_2^2)}{2}$,

$$\begin{aligned}Re(\psi(iu_2,v_1))&=Re\left(\frac{(p-\theta)v_1}{\lambda+\alpha+\beta-2+(p-\lambda)q(z)}\right)\\&=Re\left(\frac{(p-\theta)v_1}{\lambda+\alpha+\beta-2+(p-\lambda)(q_1(z)+iq_2(z))}\right),(\ q=q_1+iq_2)\\&=\frac{(p-\theta)v_1(\lambda+\alpha+\beta-2+(p-\lambda)q_1)}{(\lambda+\alpha+\beta-2+(p-\lambda)q_1)^2+(p-\lambda)^2q_2^2}\\&\leq\frac{-n}{2}\quad\frac{(p-\theta)(\lambda+\alpha+\beta-2+(p-\lambda)q_1)(1+u_2^2)}{(\lambda+\alpha+\beta-2+(p-\lambda)q_1)^2+(p-\lambda)^2q_2^2}<0,\end{aligned}$$

provided that $\lambda+\alpha+\beta-2>0$. Therefore the function $\psi(u,v)$ satisfies the conditions of lemma 1. Thus we have $Re(p(z))>0$ that is $f(z)\in K_{\alpha-1}(\beta,\theta,\lambda)$. This completes the proof of Theorem 3(a). In the similar manner Theorem 3(b) can be obtained.

Theorem 5. *(a) Let $\lambda+\alpha+\beta-2>0$. For $f\in\mathbb{A}_{n,p}$, if*

$$Re\left(\frac{z(Q^{\alpha-2}_{\beta,p}f(z))'}{(Q^{\alpha-2}_{\beta,p}g(z)}-\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}g(z)}\right)>0$$

and $\frac{z(Q^{\alpha-1}_{\beta,p}f(z))'}{(Q^{\alpha-1}_{\beta,p}g(z))}$ is analytic function, then $C^*_\alpha(\beta,\theta,\lambda)\subset C^*_{\alpha-1}(\beta,\theta,\lambda)$.

(b) Let $\lambda+\alpha+\beta-1>0$. For $f\in\mathbb{A}_{n,p}$, if $\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{(Q^{\alpha}_{\beta,p}f(z))}$ is analytic function, then $C^*_{\alpha-1}(\beta,\theta,\lambda)\subset C^*_\alpha(\beta,\theta,\lambda)$.

Proof. (a) Let $f(z) \in C^*_\alpha(\beta,\theta,\lambda) \Leftrightarrow Q^\alpha_{\beta,p}f(z) \in C^*(\beta,\theta,\lambda) \Leftrightarrow z(Q^\alpha_{\beta,p}f(z))' \in C(\beta,\theta,\lambda)$ $\Leftrightarrow Q^\alpha_{\beta,p}(zf'(z)) \in C(\beta,\theta,\lambda) \Leftrightarrow zf'(z) \in C_\alpha(\beta,\theta\lambda) \Rightarrow zf'(z) \in C_{\alpha-1}(\beta,\theta,\lambda)$ $\Leftrightarrow Q^{\alpha-1}_{\beta,p}(zf'(z)) \in C(\beta,\theta,\lambda) \Leftrightarrow z(Q^{\alpha-1}_{\beta,p}f(z))' \in C(\beta,\theta,\lambda) \Leftrightarrow Q^{\alpha-1}_{\beta,p}f(z) \in C^*(\beta,\theta,\lambda)$ $\Leftrightarrow f(z) \in C^*_{\alpha-1}(\beta,\theta,\lambda)$.

Proof. (b) Theorem can be proved in similar manner.

4. Integral Operator

For $c > -p$ and $f(z) \in \mathbb{A}_{n,p}$, the integral operator $J_{c,p}f(z) : \mathbb{A}_{n,p} \to \mathbb{A}_{n,p}$ is defined by

$$J_{c,p}f(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt = \left(z^p + \sum_{k=1}^{\infty}\frac{c+p}{c+p+k}z^{p+k}\right)*f(z) \ \ (c > -p, \ \ z \in \mathbb{U}). \tag{4.1}$$

It can be easily verified that

$$z(Q^\alpha_{\beta,p}J_{c,p}f(z))' = (c+p)Q^\alpha_{\beta,p}f(z) - cQ^\alpha_{\beta,p}(J_{c,p}f(z)). \tag{4.2}$$

The operator $J_{c,1}(c \in N)$ was introduced by Bernardi [2]. In particular, the operator $J_{1,1}$ was studied earlier by Libera [5], and Livingston [7]. Some results for the operator $J_{c,p}$ were showed by saitoh [13], and Saitoh *et al.*[14]. Now we are deriving some more results for this integral operator.

Theorem 6. *If $f(z) \in S^*_{n,p}(\alpha,\lambda)$, then $J_{c,p}f(z) \in S^*_{n,p}(\alpha,\lambda)$.*

Proof. Let $f(z) \in S^*_{n,p}(\alpha,\lambda)$. Set

$$\frac{z(Q^\alpha_{\beta,p}J_{c,p}f(z))'}{Q^\alpha_{\beta,p}J_{c,p}f(z)} = \lambda + (p-\lambda)h(z), \tag{4.3}$$

where $h(z) = 1 + c_nz^n + c_{n+1}z^{n+1} + \ldots$. Using the identity (4.2), we have

$$\frac{Q^\alpha_{\beta,p}f(z)}{Q^\alpha_{\beta,p}J_{c,p}f(z)} = \frac{1}{c+p}\{c+\lambda+(p-\lambda)h(z)\}. \tag{4.4}$$

Differentiating (4.4) logarithimically with respect to z ,we obtain

$$\frac{z(Q^\alpha_{\beta,p}f(z))'}{Q^\alpha_{\beta,p}f(z)} - \lambda = (p-\lambda)h(z) + \frac{(p-\lambda)zh'(z)}{(c+\lambda)+(p-\lambda)h(z)}. \tag{4.5}$$

Now, we form the function $\psi(u,v)$ by taking $u = h(z)$ and $v = zh'(z)$ in (4.5) as :

$$\psi(u,v) = (p-\lambda)u + \frac{(p-\lambda)v}{(c+\lambda)+(p-\lambda)u}.$$

It is easy to see that the function $\psi(u,v)$ satisfies the condition (i) and (ii) of lemma 1 for $D = \{\mathbb{C}\backslash\frac{\mathbb{C}+\lambda}{\lambda-p}\} \times \mathbb{C}$. To verify the condition of (iii), we proceed as follows:

$$Re\{\psi(iu_2, v_1)\} = Re\left(\frac{(p-\lambda)v_1}{(c+\lambda)+(p-\lambda)iu_2}\right)$$

$$= \frac{(p-\lambda)(c+\lambda)v_1}{(c+\lambda)^2+(p-\lambda)^2u_2^2} \leq \frac{-n}{2}\frac{(p-\lambda)(c+\lambda)(1+u_2^2)}{((c+\lambda)^2+(p-\lambda)^2u_2^2)} < 0,$$

where $v_1 \leq \frac{-n}{2}(1+u_2^2)$ and $(iu_2, v_1) \in D$. Therefore the function $\psi(u,v)$ satisfies the condition of lemma 1. This shows that if $Re\{\psi h(z), zh'(z)\} > 0 \ \ (z \in \mathbb{U})$, then $Re\{h(z)\} > 0 (z \in U)$, that is, if $f(z) \in S^*_{n,p}(\alpha,\lambda)$, then $J_{c,p}f(z) \in S^*_{n,p}(\alpha,\lambda)$. This completes the proof.

Theorem 7. *Let* $c > -p, \ \ 0 \leq \lambda < p$ *if* $f(z) \in C_\alpha(\lambda)$, *then* $J_{c,p}f(z) \in C_\alpha(\lambda)$.

Proof. $f(z) \in C_\alpha(\lambda) \Leftrightarrow \frac{zf'(z)}{p} \in S^*_{n,p}(\alpha,\lambda) \Rightarrow J_{c,p}\{\frac{zf'(z)}{p}\} \in S^*_{n,p}(\alpha,\lambda) \Leftrightarrow \frac{z}{p}(J_{c,p}f(z))' \in S^*_{n,p}(\alpha,\lambda) \Leftrightarrow J_{c,p}f(z) \in C_\alpha(\lambda)$. this completes the proof of Theorem 6.

Theorem 8. *Let* $c > -\lambda$, $0 \leq \lambda < p$ *if* $f(z) \in K_{n,p}(\alpha,\theta,\lambda)$, *then* $J_{c,p}f(z) \in K_{n,p}(\alpha,\theta,\lambda)$.

Proof. Let $f(z) \in K_{n,p}(\alpha,\theta,\lambda)$.Then there exists a function $g(z) \in S^*_{n,p}(\alpha,\lambda)$ such that

$$Re\left\{\frac{z(Q^\alpha_{\beta,p}f(z))'}{Q^\alpha_{\beta,p}g(z)}\right\} > \theta \qquad (z \in \mathbb{U}).$$

Set

$$\frac{z(Q^\alpha_{\beta,p}J_{c,p}f(z))'}{Q^\alpha_{\beta,p}J_{c,p}g(z)} = \theta + (p-\theta)h(z), \tag{4.6}$$

where $h(z) = 1 + f_n z^n + f_{n+1}z^{n+1}...$. Using (3.2) and (3.6), we have

$$(c+p)z(Q^\alpha_{\beta,p}f(z))' = z(Q^\alpha_{\beta,p}J_{c,p}g(z))'(\theta+(p-\theta)h(z)) +$$

$$+(Q^\alpha_{\beta,p}J_{c,p}g(z))(p-\theta)zh'(z) + cz(Q^\alpha_{\beta,p}J_{c,p}f(z))'. \tag{4.7}$$

Now apply (4.2) for the function $g(z)$ in (4.7), we obtain

$$\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{Q^{\alpha}_{\beta,p}g(z)} = \theta + (p-\theta)h(z) + \frac{(Q^{\alpha}_{\beta,p}J_{c,p}g(z))}{Q^{\alpha}_{\beta,p}g(z)}\frac{(p-\theta)zh'(z)}{(c+p)}. \tag{4.8}$$

Since $g(z) \in S^*_{n,p}(\alpha,\lambda)$, then by Theorem 5, $J_{c,p}g(z) \in S^*_{n,p}(\alpha,\lambda)$. Let

$$\frac{z(Q^{\alpha}_{\beta,p}J_{c,p}g(z))'}{Q^{\alpha}_{\beta,p}J_{c,p}g(z)} = \lambda + (p-\lambda)H(z),$$

where $Re(H(z)) > 0$, $(z \in \mathbb{U})$. Thus (4.8) can be written as

$$\frac{z(Q^{\alpha}_{\beta,p}f(z))'}{Q^{\alpha}_{\beta,p}g(z)} - \theta = (p-\theta)h(z) + \frac{(p-\theta)zh'(z)}{c+\lambda+(p-\lambda)H(z)}. \tag{4.9}$$

Now we form the function $\psi(u,v)$ by setting $u = h(z)$ and $v = zh'(z)$ in (4.9) as :

$$\psi(u,v) = (p-\theta)u + \frac{(p-\theta)v}{c+\lambda+(p-\lambda)H(z)}.$$

It is easy to see that the function $\psi(u,v)$ satisfies the condition (i) and (ii) of lemma 1, for $D = C \times C$. To verify the condition (iii), we proceed as follows

$$Re\{\psi(iu_2,v_1)\} = \frac{(p-\theta)v_1((c+\lambda)+(p-\lambda)h_1(x,y))}{((c+\lambda)+(p-\lambda)h_1(x,y))^2+((p-\lambda)h_2(x,y))^2},$$

where $H(z) = h_1(x,y) + ih_2(x,y)$. $h_1(x,y)$ and $h_2(x,y)$ being function of x and y and $Re\{H(z)\} = h_1(x,y) > 0$. Now by putting $v_1 \le \frac{-n}{2}(1+u_2^2)$, we have

$$Re\psi(iu_2,v_1) \le \frac{-n}{2}\frac{(p-\theta)(1+u_2^2)((c+\lambda)+(p-\lambda)h_1(x,y))}{((c+\lambda)+(p-\lambda)h_1(x,y))^2+((p-\lambda)h_2(x,y))^2} < 0.$$

Which implies $Re(h(z)) > 0$ and hence $J_{c,p}f(z) \in K_{n,p}(\alpha,\theta,\lambda)$. This completes the proof of Theorem 7.

Theorem 9. *Let $c > -\lambda$, $0 \le \lambda < p$ if $f(z) \in C^*_{\alpha}(\beta,\theta,\lambda)$, then $J_{c,p}f(z) \in C^*_{\alpha}(\beta,\theta,\lambda)$.*

Proof. $f(z) \in C^*_{\alpha}(\beta,\theta,\lambda) \Leftrightarrow Q^{\alpha}_{\beta,p}f(z) \in C^*(\beta,\theta,\lambda) \Leftrightarrow z(Q^{\alpha}_{\beta,p}f(z))' \in K(\beta,\theta,\lambda)$
$\Leftrightarrow Q^{\alpha}_{\beta,p}(zf'(z)) \in K(\beta,\theta,\lambda) \Leftrightarrow zf'(z) \in K_{\alpha}(\beta,\theta,\lambda) \Rightarrow J_{c,p}(zf'(z)) \in K_{\alpha}(\beta,\theta,\lambda) \Leftrightarrow$
$z(J_{c,p}f(z))' \in K_{\alpha}(\beta,\theta,\lambda) \Leftrightarrow J_{c,p}f(z) \in C^*_{\alpha}(\beta,\theta,\lambda)$.
This completes the proof of Theorem 8.

References

[1] M.K. Aouf, *On class of p-valent close-to-convex functions,* Int.J.Math sci 11(2) (1988), 259-266.

[2] S.D. Bernardi, *Convex and starlike univalent functions,* Trans.Amer.Math.Soc. 135 (1969), 429-446.

[3] A.W. Goodman, *On Schwartz-Christoffell transformation and p-valent functions,* Trans.Amer.Math.Soc. 68 (1950), 204-223.

[4] T.B. Jung, Y.C. Kim, H.M. Shrivastava,*The Hardy space of analytic functions associated with certain one parameter families of integral operators,* J.Math.Anal.Appl. 176(1993), 138-147.

[5] R.J. Libera, *Some radius of convexity problems,* Duke Math.J.16(1965), 755-758.

[6] J.-L. Liu, S. Owa, *Properties of certain integral operators,* Int.J.Math.Math.Sci. 3(1) (2004), 69-75.

[7] A.E. Livingston, *On the radius of univalence of certain analytic function,* Proc.Amer.Math.Soc.17(1966), 352-357.

[8] S.S. Miller, P.T. Mocanu, *Second differential inequalities in complex plane,* J.Math.Anal.Appl.65(1978), 289-305.

[9] K.I. Noor, *On quasi-convex functions and related topics,* Int.J.Math.Math.Sci. 10 (1987), 241-258.

[10] K.I. Noor, *On some application of the Ruscheweyh derivatives,* Math.J. 18(1978), 53-59.

[11] S. Owa, *On certain subclass of p-valent functions with negative coefficients,* Simon Stevin 59 (1985), 385-402.

[12] D.A. Patil, N.K. Thakare, *On convex hulls and extreme points of p-valent starlike and convex classes with applications,* Bull. Math.Soc.Sci.Math. R.S. Roumaine (N.S.) 27(75) (1983), 145-160.

[13] H. Saitoh, *On certain class of multivalent functions* , Math.Japon 37 (1992), 871-875.

[14] H. Saitoh, S. Owa, T. Sekine, M. Nunokawa, R. Yamakawa, *An application of certain integral operator,* Appl.Math.Lett.5 (1992), 21-24.

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