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DIFFERENTIAL SUBORDINATION RESULTS USING A GENERALIZED SĂLĂGEAN OPERATOR AND RUSCHEWEYH OPERATOR

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Abstract. In the present paper we study the operator using the generalized Sălăgean operator and Ruscheweyh operator, denote by DR_λ^n the Hadamard product of the generalized Sălăgean operator D_λ^n and Ruscheweyh operator R^n, given by $DR_\lambda^n : \mathcal{A} \to \mathcal{A}$, $DR_\lambda^n f(z) = (D_\lambda^n * R^n) f(z)$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. We obtain several differential subordinations regarding the operator DR_λ^n.

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1. Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $K = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$, the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h an univalent function in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \qquad z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Definition 1. *(Al Oboudi [10]) For $f \in \mathcal{A}$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D_λ^n is defined by $D_\lambda^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \\ &... \\ D_\lambda^{n+1} f(z) &= (1-\lambda) D_\lambda^n f(z) + \lambda z \left(D_\lambda^n f(z)\right)' = D_\lambda \left(D_\lambda^n f(z)\right), \quad z \in U. \end{aligned}$$

Remark 1. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\lambda\right]^n a_j z^j$, $z \in U$.

Remark 2. *For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [13].*

Definition 2. *(Ruscheweyh [12]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &... \\ (n+1) R^{n+1} f(z) &= z \left(R^n f(z)\right)' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 3. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.*

Definition 3. *() Let $\lambda \geq 0$ and $n \in \mathbb{N}$. Denote by $DR_\lambda^n : A \to A$ the operator given by the Hadamard product (the convolution product) of the generalized Sălăgean operator D_λ^n and the Ruscheweyh operator R^n:*

$$DR_\lambda^n f(z) = \left(D_\lambda^n * R^n\right) f(z),$$

for any $z \in U$ and each nonnegative integer n.

Remark 4. *If $f \in A$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$DR_\lambda^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} \left[1 + (j-1)\lambda\right]^n a_j^2 z^j$, for $z \in U$.

Remark 5. *The operator DR_λ^n was studied in [2], [3], [8].*

Remark 6. *For $\lambda = 1$ we obtain the Hadamard product SR^n of the Sălăgean operator S^n and Ruscheweyh operator R^n, which was studied in [4], [5], [6], [7].*

Lemma 1. *(Hallenbeck and Ruscheweyh [11, Th. 3.1.6, p. 71]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}\backslash\{0\}$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \qquad z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \qquad z \in U,$$

where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$, $z \in U$.

Lemma 2. *(Miller and Mocanu [11]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, for $z \in U$, where $\alpha > 0$ and n is a positive integer.*

If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots$, $z \in U$, is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z), \qquad z \in U,$$

then

$$p(z) \prec g(z), \qquad z \in U,$$

and this result is sharp.

2. Main results

Theorem 3. *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{z}{\delta} g'(z)$, for $z \in U$.*

If $\lambda, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\frac{DR_\lambda^n f(z)}{z}\right)^{\delta-1} \left(DR_\lambda^n f(z)\right)' \prec h(z), \qquad z \in U, \tag{2}$$

then

$$\left(\frac{DR_\lambda^n f(z)}{z}\right)^{\delta} \prec g(z), \qquad z \in U,$$

and this result is sharp.

Proof. For $f \in A$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have
$DR_\lambda^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} \left[1 + (j-1)\lambda\right]^n a_j^2 z^j$, for $z \in U$.

Consider $p(z) = \left(\frac{DR_\lambda^n f(z)}{z}\right)^\delta = \left(\frac{z+\sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!}[1+(j-1)\lambda]^n a_j^2 z^j}{z}\right)^\delta =$
$=\left(1 + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} \left[1 + (j-1)\lambda\right]^n a_j^2 z^{j-1}\right)^\delta = 1 + p_\delta z^\delta + p_{\delta+1} z^{\delta+1} + \ldots, \quad z \in U$

Differentiating we obtain $\left(\frac{DR_\lambda^n f(z)}{z}\right)^{\delta-1} (DR_\lambda^n f(z))' = p(z) + \frac{1}{\delta} z p'(z)$, $z \in U$.

Then (2) becomes

$$p(z) + \frac{1}{\delta} z p'(z) \prec h(z) = g(z) + \frac{z}{\delta} g'(z), \quad \text{for} \quad z \in U.$$

By using Lemma 2, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad \left(\frac{DR_\lambda^n f(z)}{z}\right)^\delta \prec g(z), \quad z \in U.$$

Theorem 4. *Let h be an holomorphic function which satisfies the inequality $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$.*

If $\lambda, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\frac{DR_\lambda^n f(z))}{z}\right)^{\delta-1} (DR_\lambda^n f(z))' \prec h(z), \quad z \in U, \tag{3}$$

then

$$\left(\frac{DR_\lambda^n f(z))}{z}\right)^\delta \prec q(z), \quad z \in U,$$

where $q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta-1} dt$. The function q is convex and it is the best dominant.

Proof. Let

$$\begin{aligned} p(z) &= \left(\frac{DR_\lambda^n f(z)}{z}\right)^\delta = \left(\frac{z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} \left[1 + (j-1)\lambda\right]^n a_j^2 z^j}{z}\right)^\delta \\ &= \left(1 + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!\,(j-1)!} \left[1 + (j-1)\lambda\right]^n a_j^2 z^{j-1}\right)^\delta = 1 + \sum_{j=\delta+1}^{\infty} p_j z^{j-1}, \end{aligned}$$

for $z \in U$, $p \in \mathcal{H}[1, \delta]$.

Differentiating, we obtain $\left(\frac{DR_\lambda^n f(z)}{z}\right)^{\delta-1}(DR_\lambda^n f(z))' = p(z) + \frac{1}{\delta}zp'(z)$, $z \in U$, and (3) becomes

$$p(z) + \frac{1}{\delta}zp'(z) \prec h(z), \qquad z \in U.$$

Using Lemma 1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \left(\frac{DR_\lambda^n f(z)}{z}\right)^{\delta} \prec q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt, \quad z \in U,$$

and q is the best dominant.

Corollary 5. *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \le \beta < 1$.*
If $\delta, \lambda \ge 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\frac{DR_\lambda^n f(z))}{z}\right)^{\delta-1}(DR_\lambda^n f(z))' \prec h(z), \qquad z \in U, \tag{4}$$

then

$$\left(\frac{DR_\lambda^n f(z)}{z}\right)^{\delta} \prec q(z), \qquad z \in U,$$

where q is given by $q(z) = (2\beta-1) + \frac{2(1-\beta)\delta}{z^\delta}\int_0^z \frac{t^{\delta-1}}{1+t}dt$, $z \in U$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 4 and considering $p(z) = \left(\frac{DR_\lambda^n f(z)}{z}\right)^{\delta}$, the differential subordination (4) becomes

$$p(z) + \frac{z}{\delta}p'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \qquad z \in U.$$

By using Lemma 1 for $\gamma = \delta$, we have $p(z) \prec q(z)$, i.e.

$$\left(\frac{DR_\lambda^n f(z)}{z}\right)^{\delta} \prec q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt =$$

$$\frac{\delta}{z^\delta}\int_0^z t^{\delta-1}\frac{1+(2\beta-1)t}{1+t}dt = \frac{\delta}{z^\delta}\int_0^z \left[(2\beta-1)\,t^{\delta-1} + 2\,(1-\beta)\,\frac{t^{\delta-1}}{1+t}\right]dt$$

$$= (2\beta-1) + \frac{2\,(1-\beta)\,\delta}{z^\delta}\int_0^z \frac{t^{\delta-1}}{1+t}dt, \qquad z \in U.$$

Remark 7. *For $n = 1$, $\lambda = \frac{1}{2}$, $\delta = 1$ we obtain the same example as in [9, Example 7.2.1, p. 273].*

Theorem 6. *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{z}{\delta} g'(z)$, $z \in U$.*

If $\lambda, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and the differential subordination

$$z\frac{\delta+1}{\delta}\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1} f(z)\right)^2}+\frac{z^2}{\delta}\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1} f(z)\right)^2}\left[\frac{(DR_\lambda^n f(z))'}{DR_\lambda^n f(z)}-2\frac{\left(DR_\lambda^{n+1} f(z)\right)'}{DR_\lambda^n f(z)}\right]\prec h(z), \tag{5}$$

$z \in U$, holds, then

$$z\frac{DR_\lambda^n f(z)}{(DR_\lambda^{n+1} f(z))^2}\prec g(z), \quad z \in U,$$

and this result is sharp.

Proof. For $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have $DR_\lambda^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} [1 + (j-1)\lambda]^n a_j^2 z^j$, $z \in U$.

Consider $p(z) = z\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1} f(z)\right)^2}$ and we obtain

$$p(z) + \frac{z}{\delta}p'(z) = z\frac{\delta+1}{\delta}\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1} f(z)\right)^2}+\frac{z^2}{\delta}\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1} f(z)\right)^2}\left[\frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}-2\frac{\left(DR_\lambda^{n+1} f(z)\right)'}{DR_\lambda^{n+1} f(z)}\right].$$

Relation (5) becomes

$$p(z) + \frac{z}{\delta}p'(z) \prec h(z) = g(z) + \frac{z}{\delta}g'(z), \qquad z \in U.$$

By using Lemma 2, we have

$$p(z) \prec g(z), \qquad z \in U, \quad \text{i.e.} \quad z\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1} f(z)\right)^2}\prec g(z), \qquad z \in U.$$

Theorem 7. *Let h be an holomorphic function which satisfies the inequality $Re\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$.*

If $\lambda, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$z\frac{\delta+1}{\delta}\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1} f(z)\right)^2}+\frac{z^2}{\delta}\frac{DR_\lambda^n f(z)}{(DR_\lambda^{n+1} f(z))^2}\left[\frac{(DR_\lambda^n f(z))'}{DR_\lambda^n f(z)}-2\frac{\left(DR_\lambda^{n+1} f(z)\right)'}{DR_\lambda^{n+1} f(z)}\right]\prec h(z), \tag{6}$$

$z \in U$,then

$$z\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1} f(z)\right)^2}\prec q(z), \quad z \in U,$$

where $q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt$. The function q is convex and it is the best dominant.

Proof. Let $p(z)=z\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1}f(z)\right)^2}$, $z\in U$, $p\in\mathcal{H}[1,1]$.

Differentiating, we obtain

$p\left(z\right)+\frac{z}{\delta}p'\left(z\right)=z\frac{\delta+1}{\delta}\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1}f(z)\right)^2}+\frac{z^2}{\delta}\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1}f(z)\right)^2}\left[\frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}-2\frac{\left(DR_\lambda^{n+1}f(z)\right)'}{DR_\lambda^{n+1}f(z)}\right]$,

$z\in U$, and (6) becomes

$$p(z)+\frac{z}{\delta}p'(z)\prec h(z),\qquad z\in U.$$

Using Lemma 1, we have

$$p(z)\prec q(z),\quad z\in U,\quad \text{i.e.}\quad z\frac{DR_\lambda^n f(z)}{\left(DR_\lambda^{n+1}f(z)\right)^2}\prec q(z)=\frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt,\quad z\in U,$$

and q is the best dominant.

Theorem 8. *Let g be a convex function such that $g(0)=1$ and let h be the function $h(z)=g(z)+\frac{z}{\delta}g'(z)$, $z\in U$.*

If $\lambda,\delta\geq 0$, $n\in\mathbb{N}$, $f\in\mathcal{A}$ and the differential subordination

$$z^2\frac{\delta+2}{\delta}\frac{(DR_\lambda^n f(z))'}{DR_\lambda^n f(z)}+\frac{z^3}{\delta}\left[\frac{(DR_\lambda^n f(z))''}{DR_\lambda^n f(z)}-\left(\frac{(DR_\lambda^n f(z))'}{DR_\lambda^n f(z)}\right)^2\right]\prec h(z),\qquad z\in U \quad (7)$$

holds, then

$$z^2\frac{(DR_\lambda^n f(z))'}{DR_\lambda^n f(z)}\prec g(z),\quad z\in U.$$

This result is sharp.

Proof. Let $p(z)=z^2\frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}$. We deduce that $p\in\mathcal{H}[0,1]$.

Differentiating, we obtain

$p\left(z\right)+\frac{z}{\delta}p'\left(z\right)=z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^n f(z)\right)''}{DR_\lambda^n f(z)}-\left(\frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}\right)^2\right]$, $z\in U$.

Using the notation in (7), the differential subordination becomes

$$p(z)+\frac{1}{\delta}zp'(z)\prec h(z)=g(z)+\frac{z}{\delta}g'(z).$$

By using Lemma 2, we have

$$p(z)\prec g(z),\qquad z\in U,\quad \text{i.e.}\quad z^2\frac{(DR_\lambda^n f(z))'}{DR_\lambda^n f(z)}\prec g(z),\qquad z\in U,$$

and this result is sharp.

Theorem 9. *Let h be an holomorphic function which satisfies the inequality* $Re\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *and* $h(0)=1$.

If $\lambda, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ *and satisfies the differential subordination*

$$z^2 \frac{\delta+2}{\delta} \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)} + \frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^n f(z)\right)''}{DR_\lambda^n f(z)} - \left(\frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}\right)^2\right] \prec h(z), \quad z \in U, \tag{8}$$

then

$$z^2 \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt$. *The function q is convex and it is the best dominant.*

Proof. Let $p(z) = z^2 \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}$, $z \in U$, $p \in \mathcal{H}[0,1]$.
Differentiating, we obtain
$p(z) + \frac{z}{\delta}p'(z) = z^2 \frac{\delta+2}{\delta} \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)} + \frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^n f(z)\right)''}{DR_\lambda^n f(z)} - \left(\frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}\right)^2\right]$, $z \in U$, and (8) becomes

$$p(z) + \frac{1}{\delta}zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad z^2 \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)} \prec q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt, \quad z \in U,$$

and q is the best dominant.

Theorem 10. *Let g be a convex function such that $g(0)=1$ and let h be the function* $h(z) = g(z) + zg'(z)$, $z \in U$.

If $\lambda \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ *and the differential subordination*

$$1 - \frac{DR_\lambda^n f(z) \cdot \left(DR_\lambda^n f(z)\right)''}{\left[\left(DR_\lambda^n f(z)\right)'\right]^2} \prec h(z), \quad z \in U \tag{9}$$

holds, then

$$\frac{DR_\lambda^n f(z)}{z\left(DR_\lambda^n f(z)\right)'} \prec g(z), \quad z \in U.$$

This result is sharp.

Proof. Let $p(z) = \frac{DR_\lambda^n f(z)}{z\left(DR_\lambda^n f(z)\right)'}$. We deduce that $p \in \mathcal{H}[1,1]$.

Differentiating, we obtain $1 - \frac{DR_\lambda^n f(z) \cdot \left(DR_\lambda^n f(z)\right)''}{\left[\left(DR_\lambda^n f(z)\right)'\right]^2} = p\left(z\right) + zp'\left(z\right),\, z \in U$.

Using the notation in (9), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 2, we have

$$p(z) \prec g(z), \qquad z \in U, \quad \text{i.e.} \quad \frac{DR_\lambda^n f(z)}{z\left(DR_\lambda^n f(z)\right)'} \prec g(z), \qquad z \in U,$$

and this result is sharp.

Theorem 11. *Let h be an holomorphic function which satisfies the inequality $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$.*

If $\lambda \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$1 - \frac{DR_\lambda^n f(z) \cdot \left(DR_\lambda^n f(z)\right)''}{\left[\left(DR_\lambda^n f(z)\right)'\right]^2} \prec h(z), \qquad z \in U, \tag{10}$$

then

$$\frac{DR_\lambda^n f(z)}{z\left(DR_\lambda^n f(z)\right)'} \prec q(z), \qquad z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the best dominant.

Proof. Let $p(z) = \frac{DR_\lambda^n f(z)}{z\left(DR_\lambda^n f(z)\right)'}$, $z \in U$, $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $1 - \frac{DR_\lambda^n f(z) \cdot \left(DR_\lambda^n f(z)\right)''}{\left[\left(DR_\lambda^n f(z)\right)'\right]^2} = p\left(z\right) + zp'\left(z\right),\, z \in U$, and (10) becomes

$$p(z) + zp'(z) \prec h(z), \qquad z \in U.$$

Using Lemma 1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \frac{DR_\lambda^n f(z)}{z\left(DR_\lambda^n f(z)\right)'} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U,$$

and q is the best dominant.

Corollary 12. *Let* $h(z) = \frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U*, where* $0 \le \beta < 1$.
If $\lambda \ge 0$*,* $n \in \mathbb{N}$*,* $f \in \mathcal{A}$ *and satisfies the differential subordination*

$$1 - \frac{RD^n_{\lambda,\alpha} f(z) \cdot \left(RD^n_{\lambda,\alpha} f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha} f(z)\right)'\right]^2} \prec h(z), \quad z \in U, \tag{11}$$

then

$$\frac{RD^n_{\lambda,\alpha} f(z)}{z\left(RD^n_{\lambda,\alpha} f(z)\right)'} \prec q(z), \quad z \in U,$$

where q *is given by* $q(z) = (2\beta - 1) + 2(1-\beta)\frac{\ln(1+z)}{z}$*,* $z \in U$*. The function* q *is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 11 and considering $p(z) = \frac{RD^n_{\lambda,\alpha} f(z)}{z\left(RD^n_{\lambda,\alpha} f(z)\right)'}$, the differential subordination (11) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \quad z \in U.$$

By using Lemma 1 for $\gamma = 1$, we have $p(z) \prec q(z)$, i.e.

$$\frac{RD^n_{\lambda,\alpha} f(z)}{z\left(RD^n_{\lambda,\alpha} f(z)\right)'} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt =$$

$$\frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = \frac{1}{z}\int_0^z \left[(2\beta-1) + \frac{2(1-\beta)}{1+t}\right]dt$$

$$= (2\beta-1) + 2(1-\beta)\frac{\ln(1+z)}{z}, z \in U.$$

Example 1. *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$ *and* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$.

Let $f(z) = z + z^2$*,* $z \in U$*. For* $n = 1$*,* $\lambda = \frac{1}{2}$*, we obtain* $R^1 f(z) = zf'(z) = z + 2z^2$*,* $D^1_{\frac{1}{2}} f(z) = \frac{1}{2} f(z) + \frac{1}{2} zf'(z) = z + \frac{3}{2}z^2$*,* $DR^1_{\frac{1}{2}} f(z) = z + 3z^2, z \in U$.

Then $\left(DR^1_{\frac{1}{2}} f(z)\right)' = 1 + 6z$*,* $\left(DR^1_{\frac{1}{2}} f(z)\right)'' = 6$*,*

$\frac{DR^1_{\frac{1}{2}} f(z)}{z\left(DR^1_{\frac{1}{2}} f(z)\right)'} = \frac{z+3z^2}{z(1+6z)} = \frac{1+3z}{1+6z}$,

$1-\frac{RDR^1_{\frac{1}{2},2}f(z)\cdot\left(RD^1_{\frac{1}{2},2}f(z)\right)''}{\left[\left(RD^1_{\frac{1}{2}}f(z)\right)'\right]^2}=1-\frac{\left(z+3z^2\right)\cdot 6}{(1+6z)^2}=\frac{18z^2+6z+1}{(1+6z)^2}.$

We have $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}.$

Using Theorem 11 we obtain

$$\frac{18z^2+6z+1}{(1+6z)^2}\prec\frac{1-z}{1+z},\quad z\in U,$$

induce

$$\frac{1+3z}{1+6z}\prec-1+\frac{2\ln(1+z)}{z},\quad z\in U.$$

Theorem 13. *Let g be a convex function such that $g(0)=0$ and let h be the function $h(z)=g(z)+zg'(z)$, $z\in U$.*

If $\lambda\geq 0$, $n\in\mathbb{N}$, $f\in\mathcal{A}$ and the differential subordination

$$\left[\left(DR^n_\lambda f(z)\right)'\right]^2+DR^n_\lambda f(z)\cdot\left(DR^n_\lambda f(z)\right)''\prec h(z),\quad z\in U \tag{12}$$

holds, then

$$\frac{DR^n_\lambda f(z)\cdot\left(DR^n_\lambda f(z)\right)'}{z}\prec g(z),\quad z\in U.$$

This result is sharp.

Proof. Let $p(z)=\frac{DR^n_\lambda f(z)\cdot\left(DR^n_\lambda f(z)\right)'}{z}$. We deduce that $p\in\mathcal{H}[0,1]$.

Differentiating, we obtain $\left[\left(DR^n_\lambda f(z)\right)'\right]^2+DR^n_\lambda f(z)\cdot\left(DR^n_\lambda f(z)\right)''=p(z)+zp'(z)$, $z\in U$.

Using the notation in (12), the differential subordination becomes

$$p(z)+zp'(z)\prec h(z)=g(z)+zg'(z).$$

By using Lemma 2, we have

$$p(z)\prec g(z),\quad z\in U,\quad \text{i.e.}\quad \frac{DR^n_\lambda f(z)\cdot\left(DR^n_\lambda f(z)\right)'}{z}\prec g(z),\quad z\in U,$$

and this result is sharp.

Theorem 14. *Let h be an holomorphic function which satisfies the inequality $Re\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, and $h(0)=0$.*

If $\lambda\geq 0$, $n\in\mathbb{N}$, $f\in\mathcal{A}$ and satisfies the differential subordination

$$\left[\left(DR^n_\lambda f(z)\right)'\right]^2+DR^n_\lambda f(z)\cdot\left(DR^n_\lambda f(z)\right)''\prec h(z),\quad z\in U, \tag{13}$$

then

$$\frac{DR_\lambda^n f(z) \cdot (DR_\lambda^n f(z))'}{z} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. *The function q is convex and it is the best dominant.*

Proof. Let $p(z) = \frac{DR_\lambda^n f(z) \cdot \left(DR_\lambda^n f(z)\right)'}{z}$, $z \in U$, $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $\left[(DR_\lambda^n f(z))'\right]^2 + DR_\lambda^n f(z) \cdot (DR_\lambda^n f(z))'' = p(z) + zp'(z)$, $z \in U$, and (13) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \frac{DR_\lambda^n f(z) \cdot (DR_\lambda^n f(z))'}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U,$$

and q is the best dominant.

Corollary 15. *Let* $h(z) = \frac{1+(2\beta-1)z}{1+z}$ *be a convex function in U, where* $0 \leq \beta < 1$.

If $\lambda \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ *and satisfies the differential subordination*

$$\left[(DR_\lambda^n f(z))'\right]^2 + DR_\lambda^n f(z) \cdot (DR_\lambda^n f(z))'' \prec h(z), \quad z \in U, \tag{14}$$

then

$$\frac{DR_\lambda^n f(z) \cdot (DR_\lambda^n f(z))'}{z} \prec q(z), \quad z \in U,$$

where q is given by $q(z) = (2\beta - 1) + 2(1-\beta)\frac{\ln(1+z)}{z}$, $z \in U$. *The function q is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 14 and considering $p(z) = \frac{DR_\lambda^n f(z) \cdot \left(DR_\lambda^n f(z)\right)'}{z}$, the differential subordination (14) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \quad z \in U.$$

By using Lemma 1 for $\gamma = 1$, we have $p(z) \prec q(z)$, i.e.

$$\frac{DR_\lambda^n f(z) \cdot (DR_\lambda^n f(z))'}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt =$$

$$\frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = \frac{1}{z}\int_0^z \left[(2\beta-1) + \frac{2(1-\beta)}{1+t}\right]dt$$

$$= (2\beta-1) + 2(1-\beta)\frac{\ln(1+z)}{z}, \quad z \in U.$$

Example 2. *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$ *and* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$.

Let $f(z) = z + z^2$, $z \in U$. *For* $n = 1$, $\lambda = \frac{1}{2}$, *we obtain* $DR^1_{\frac{1}{2}} f(z) = z + 3z^2$, $z \in U$.

Then $\left(DR^1_{\frac{1}{2}} f(z)\right)' = 1 + 6z$,

$\frac{DR^n_\lambda f(z) \cdot \left(DR^n_\lambda f(z)\right)'}{z} = \frac{\left(z+3z^2\right)(1+6z)}{z} = 18z^2 + 9z + 1$,

$\left[\left(RD^1_{\frac{1}{2},2} f(z)\right)'\right]^2 + RD^1_{\frac{1}{2},2} f(z) \cdot \left(RD^1_{\frac{1}{2},2} f(z)\right)'' = (1+6z)^2 + \left(z + 3z^2\right) \cdot 6 = 54z^2 + 18z + 1$.

We have $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 14 we obtain

$$54z^2 + 18z + 1 \prec \frac{1-z}{1+z}, \quad z \in U,$$

induce

$$18z^2 + 9z + 1 \prec -1 + \frac{2\ln(1+z)}{z}, \quad z \in U.$$

Theorem 16. *Let* g *be a convex function such that* $g(0) = 0$ *and let* h *be the function* $h(z) = g(z) + \frac{z}{1-\delta} g'(z)$, $z \in U$.

If $\lambda \geq 0$, $\delta \in (0,1)$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ *and the differential subordination*

$$\left(\frac{z}{DR^n_\lambda f(z)}\right)^\delta \frac{DR^{n+1}_\lambda f(z)}{1-\delta}\left(\frac{\left(DR^{n+1}_\lambda f(z)\right)'}{DR^{n+1}_\lambda f(z)} - \delta\frac{\left(DR^n_\lambda f(z)\right)'}{DR^n_\lambda f(z)}\right) \prec h(z), \quad z \in U \tag{15}$$

holds, then

$$\frac{DR^{n+1}_\lambda f(z)}{z} \cdot \left(\frac{z}{DR^n_\lambda f(z)}\right)^\delta \prec g(z), \quad z \in U.$$

This result is sharp.

Proof. Let $p(z) = \frac{DR^{n+1}_\lambda f(z)}{z} \cdot \left(\frac{z}{DR^n_\lambda f(z)}\right)^\delta$. We deduce that $p \in \mathcal{H}[1,1]$.

Differentiating, we obtain $\left(\frac{z}{DR^n_\lambda f(z)}\right)^\delta \frac{DR^{n+1}_\lambda f(z)}{1-\delta}\left(\frac{\left(DR^{n+1}_\lambda f(z)\right)'}{DR^{n+1}_\lambda f(z)} - \delta\frac{\left(DR^n_\lambda f(z)\right)'}{DR^n_\lambda f(z)}\right) = p(z) + \frac{1}{1-\delta} zp'(z)$, $z \in U$.

Using the notation in (15), the differential subordination becomes

$$p(z) + \frac{1}{1-\delta} zp'(z) \prec h(z) = g(z) + \frac{z}{1-\delta} g'(z).$$

By using Lemma 2, we have

$$p(z) \prec g(z), \qquad z \in U, \quad \text{i.e.} \quad \frac{DR_\lambda^{n+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^n f(z)}\right)^\delta \prec g(z), \qquad z \in U,$$

and this result is sharp.

Theorem 17. *Let h be an holomorphic function which satisfies the inequality $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$.*

If $\lambda \geq 0$, $\delta \in (0,1)$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\frac{z}{DR_\lambda^n f(z)}\right)^\delta \frac{DR_\lambda^{n+1} f(z)}{1-\delta} \left(\frac{\left(DR_\lambda^{n+1} f(z)\right)'}{DR_\lambda^{n+1} f(z)} - \delta \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}\right) \prec h(z), \qquad z \in U, \tag{16}$$

then

$$\frac{DR_\lambda^{n+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^n f(z)}\right)^\delta \prec q(z), \qquad z \in U,$$

where $q(z) = \frac{1-\delta}{z^{1-\delta}} \int_0^z h(t) t^{-\delta} dt$. The function q is convex and it is the best dominant.

Proof. Let $p(z) = \frac{DR_\lambda^{n+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^n f(z)}\right)^\delta$, $z \in U$, $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $\left(\frac{z}{DR_\lambda^n f(z)}\right)^\delta \frac{DR_\lambda^{n+1} f(z)}{1-\delta} \left(\frac{\left(DR_\lambda^{n+1} f(z)\right)'}{DR_\lambda^{n+1} f(z)} - \delta \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}\right) = p(z) + \frac{1}{1-\delta} z p'(z)$, $z \in U$, and (16) becomes

$$p(z) + \frac{1}{1-\delta} z p'(z) \prec h(z), \qquad z \in U.$$

Using Lemma 1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \frac{DR_\lambda^{n+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^n f(z)}\right)^\delta \prec q(z) = \frac{1-\delta}{z^{1-\delta}} \int_0^z h(t) t^{-\delta} dt,$$

$z \in U$, and q is the best dominant.

Remark 8. *For $\lambda = 1$ we obtain the same results for the operator SR^n.*

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