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ON UPPER AND LOWER C-γ-CONTINUOUS MULTIFUNCTIONS

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Abstract. The aim of this paper is to introduce a new class of multifunctions namely C-γ-continuous multifunctions and we obtain some characterizations of it.

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1. Introduction

One of the most important and basic topics in the theory of classical point set topology and several branches of Mathematics, which has been investigated by many authors, is the continuity of functions. This concept has been extended to the setting of multifunctions. Multifunction or multivalued mapping has many applications in Mathematical Programming, Probability, Statistics, Differential Inclusions, Fixed point theorems and even in Economics. There are several various types of continuous functions and some of them have been extended to the multifunctions. The aim of this paper is to introduce a new class of multifunctions namely C-γ-continuous multifunctions and we obtain some characterizations and properties of it.

2. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) means topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset A of X, the closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of (X,τ) is said to be α-open [5] (resp. γ-open [3]($=b$-open [1]) if $A\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A\subset \mathrm{Int}(\mathrm{Cl}(A))\cup \mathrm{Cl}(\mathrm{Int}(A))$)). The complement of a γ-open set is called γ-closed [3] ($=b$-closed [1]). While, the family of all γ-open (resp. γ-closed) subsets of (X,τ) is denoted by $\gamma O(X)$ (resp. $\gamma C(X)$). We set $\gamma O(X,x)=\{A : A\in \gamma O(X) \text{ and } x\in A\}$. The intersection (resp. union) of all γ-closed (resp. γ-open) sets of (X,τ) containing (resp. contained in) A is called the

γ-closure [3] (resp. γ-interior [3]) of A and is denoted by $\gamma\,\mathrm{Cl}(A)$ (resp. $\gamma\,\mathrm{Int}(A)$). By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset A of Y by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F^+(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$.

Lemma 1. *[1, 3] Let A be a subset of a topological space X. Then we have the following*

1. *A is γ-closed if and only if $\gamma\,\mathrm{Cl}(A) = A$.*
2. *$\gamma\,\mathrm{Cl}(A) = A \cup (\mathrm{Cl}(\mathrm{Int}(A)) \cap (\mathrm{Int}(\mathrm{Cl}(A))))$.*
3. *$\gamma\,\mathrm{Cl}(\gamma\,\mathrm{Cl}(A)) = \gamma\,\mathrm{Cl}(A)$.*

Definition 1. *[2] A multifunction $F : X \to Y$ is said to be*

1. *upper γ-continuous if $F^+(V) \in \gamma O(X)$ for each open set V of Y,*
2. *lower γ-continuous if $F^-(V) \in \gamma O(X)$ for each open set V of Y.*

3. C-γ-continuous Multifunctions

Definition 2. *A multifunction $F : X \to Y$ is said to be*

1. *upper C-γ-continuous at a point $x \in X$ if each open set V containing $F(x)$ and having compact complement, there exists $U \in \gamma O(X, x)$ such that $F(U) \subset V$,*
2. *lower C-γ-continuous at a point $x \in X$ if each open set V having compact complement such that $F(x) \cap V \neq \emptyset$, there exists $U \in \gamma O(X, x)$ such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,*
3. *upper (lower) C-γ-continuous in X if it has this property at every point of X.*

Theorem 2. *The following are equivalent for a multifunction $F : X \to Y$:*

1. *F is upper C-γ-continuous;*
2. *$F^+(V) \in \gamma O(X)$ for each open set V of Y having compact complement;*
3. *$F^-(K)$ is γ-closed in X for every compact closed set K of Y;*
4. *$\mathrm{Cl}(\mathrm{Int}(F^-(B))) \cap \mathrm{Int}(\mathrm{Cl}(F^-(B))) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y having the compact closure;*

5. *$\gamma \operatorname{Cl}(F^-(B)) \subset F^-(\operatorname{Cl}(B))$ for every subset B of Y having the compact closure;*

6. *$F^+(\operatorname{Int}(B)) \subset \gamma \operatorname{Int}(F^+(B))$ for every subset B of Y such that $Y \setminus \operatorname{Int}(B)$ is compact.*

Proof. (1)⇒(2): Let V be any open set of Y having compact complement and $x \in F^+(V)$. There exists $U \in \gamma O(X, x)$ such that $F(U) \subset V$. Therefore, we have $x \in U \subset \operatorname{Int}(\operatorname{Cl}(F^+(V))) \cup \operatorname{Int}(\operatorname{Cl}(F^+(V)))$. Thus $F^+(V) \subset \operatorname{Int}(\operatorname{Cl}(F^+(V))) \cup \operatorname{Int}(\operatorname{Cl}(F^+(V)))$ and hence $F^+(V) \in \gamma O(X)$.
(2)⇒(3): The proof follows immediately from the fact that $F^+(Y \setminus B) = X \setminus F^-(B)$ for every subset B of Y.
(3)⇒(4): Let B be a subset of Y having the compact closure. Then $F^-(\operatorname{Cl}(B))$ is a γ-closed set of X. By Lemma 1, we have $\operatorname{Cl}(\operatorname{Int}(F^-(B))) \cap \operatorname{Int}(\operatorname{Cl}(F^-(B))) \subset \operatorname{Cl}(\operatorname{Int}(F^-(\operatorname{Cl}(B)))) \cap \operatorname{Int}(\operatorname{Cl}(F^-(\operatorname{Cl}(B)))) \subset \gamma \operatorname{Cl}(F^-(\operatorname{Cl}(B))) = F^-(\operatorname{Cl}(B))$. Therefore, we obtain $\operatorname{Cl}(\operatorname{Int}(F^-(B))) \cap \operatorname{Int}(\operatorname{Cl}(F^-(B))) = F^-(\operatorname{Cl}(B))$.
(4)⇒(5): Let B be a subset of Y having the compact closure. It follows that Lemma 1 that $\gamma \operatorname{Cl}(F^-(B)) = F^-(B) \cup (\operatorname{Cl}(\operatorname{Int}(F^-(B))) \cap \operatorname{Int}(\operatorname{Cl}(F^-(B)))) \subset F^-(\operatorname{Cl}(B))$.
(5)⇒(6): Let B be a subset of Y such that $Y \setminus \operatorname{Int}(B)$ is compact. Then we have $X \setminus \gamma \operatorname{Int}(F^+(B)) = \gamma \operatorname{Cl}(X \setminus F^+(B)) = \gamma \operatorname{Cl}(F^-(Y \setminus B)) \subset F^-(\operatorname{Cl}(Y \setminus B)) = F^-(Y \setminus \operatorname{Int}(B)) = X \setminus F^+(\operatorname{Int}(B))$. Therefore, we obtain $F^+(\operatorname{Int}(B)) \subset \gamma \operatorname{Int}(F^+(B))$.
(6)⇒(1): Let $x \in X$ and V be any open set of Y containing $F(x)$ and having compact complement. Then $F^+(V) = F^+(\operatorname{Int}(V)) \subset \gamma \operatorname{Int}(F^+(V))$. Put $U = \gamma \operatorname{Int}(F^+(V))$, then $U \in \gamma O(X, x)$ and $F(U) \subset V$. This shows that F is upper C-γ-continuous.

Theorem 3. *The following are equivalent for a multifunction $F : X \to Y$:*

1. *F is lower C-γ-continuous;*

2. *$F^-(V) \in \gamma O(X)$ for each open set V of Y having compact complement;*

3. *$F^+(K)$ is γ-closed in X for every compact closed set K of Y;*

4. *$\operatorname{Cl}(\operatorname{Int}(F^+(B))) \cap \operatorname{Int}(\operatorname{Cl}(F^+(B))) \subset F^+(\operatorname{Cl}(B))$ for every subset B of Y having the compact closure;*

5. *$\gamma \operatorname{Cl}(F^+(B)) \subset F^+(\operatorname{Cl}(B))$ for every subset B of Y having the compact closure;*

6. *$F^-(\operatorname{Int}(B)) \subset \gamma \operatorname{Int}(F^-(B))$ for every subset B of Y such that $Y \setminus \operatorname{Int}(B)$ is compact.*

Proof. The proof is similar to that of Theorem 2.

Corollary 4. *A multifunction $F : X \to Y$ is upper C-γ-continuous (resp. lower C-γ-continuous) if $F^-(G)$ (resp. $F^+(G)$) is γ-closed in X for every compact set G of Y.*

Proof. Let G be an open set of Y having compact complement. Then $Y \backslash G$ is compact and $F^-(Y \backslash G)$ is γ-closed in X. Therefore, $F^+(G) \in \gamma O(X)$ and by Theorem 2 F is upper C-γ-continuous. The proof for F lower C-γ-continuous is entirely similar.

For a multifunction $F : X \to Y$, by $\operatorname{Cl} F : X \to Y$ we denote a multifunction defined as follows: $(\operatorname{Cl} F)(x) = \operatorname{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\gamma \operatorname{Cl} F : X \to Y$.

Definition 3. *A subset A of a topological space (X, τ) is said to be:*

(i) α-regular [4] if for each $a \in A$ and any open set U of X containing a, there exists an open set G of X such that $a \in G \subset \operatorname{Cl}(G) \subset U$;

(ii) α-paracompact [8] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 5. *[4] If A is an α-paracompact α-regular set of a topological space (X, τ) and U an open neighbourhood of A, then there exists an open set G of X such that $A \subset G \subset \operatorname{Cl}(G) \subset U$.*

Lemma 6. *[2] If $F : X \to Y$ be a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then for each open set V of Y $(\operatorname{Cl} F)^+(V) = (\gamma \operatorname{Cl} F)^+(V) = F^+(V)$.*

Theorem 7. *If $F : X \to Y$ be a multifunction such that $F(x)$ is α-regular α-paracompact for each $x \in X$. Then the following are equivalent:*

1. *F is upper C-γ-continuous.*
2. *$\gamma \operatorname{Cl} F$ is upper C-γ-continuous;*
3. *$\operatorname{Cl} F$ is upper C-γ-continuous.*

Proof. We set $G = \gamma \operatorname{Cl} F$ or $\operatorname{Cl} F$. Suppose that F is upper C-γ-continuous. Let V be any open set of Y containing $G(x)$ and having compact complement By Lemma 6, we have $G^+(V) = F^+(V)$ and hence there exists $U \in \gamma O(X, x)$ such that $F(U) \subset V$. Since $F(u)$ is α-paracompact and α-regular for each $u \in U$, by Lemma 5 there exists an open set H such that $F(u) \subset H \subset \operatorname{Cl}(H) \subset V$; hence $G(u) \subset \operatorname{Cl}(H) \subset V$ for every $u \in U$. Therefore, we obtain $G(U) \subset V$. This shows that G is upper C-γ-continuous. Conversely, suppose that G is upper C-γ-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$ and having compact complement. By Lemma 6, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. There exists $U \in \gamma O(X, x)$ such that $G(U) \subset V$. Therefore, we obtain $U \subset G^+(V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is upper C-γ-continuous.

Lemma 8. *[2] If $F : X \to Y$ is a multifunction, then $(\mathrm{Cl}\, F)^{-}(V) = (\gamma\, \mathrm{Cl}\, F)^{-}(V) = F^{-}(V)$ for each open set V of Y.*

Theorem 9. *If $F : X \to Y$, the following are equivalent:*

1. *F is lower C-γ-continuous;*
2. *$\gamma\, \mathrm{Cl}\, F$ is lower C-γ-continuous;*
3. *$\mathrm{Cl}\, F$ is lower C-γ-continuous.*

Proof. By using Lemma 8 this is shown similarly as in Theorem 7.

Lemma 10. *[1, 3] Let U and X_0 be subsets of a topological space (X, τ).*

1. *If $U \in \gamma O(X)$ and $X_0 \in \alpha O(X)$, then $U \cap X_0 \in \gamma O(X_0)$.*
2. *If $U \subset X_0 \subset X$, $U \in \gamma O(X_0)$ and $X_0 \in \gamma O(X)$, then $U \in \gamma O(X)$.*

Theorem 11. *If a multifunction $F : X \to Y$ is upper C-γ-continuous and $X_0 \in \alpha O(X)$, then the restriction $F|_{X_0} : X_0 \to Y$ is upper C-γ-continuous.*

Proof. Let $x \in X_0$ and V be an open set of Y having compact complement such that $(F|_{X_0})(x) \subset V$. Since F is upper C-γ-continuous and $(F|_{X_0})(x) = F(x)$, there exists $U \in \gamma O(X, x)$ such that $F(U) \subset V$. Set $U_0 = U \cap X_0$, then by Lemma 10 we have $U_0 \in \gamma O(X_0, x)$ and $(F|_{X_0})(U_0) = F(U_0) \subset V$. This shows that $F|_{X_0}$ is upper C-γ-continuous.

Theorem 12. *A multifunction $F : X \to Y$ is upper C-γ-continuous if for each $x \in X$ there exists $X_0 \in \gamma O(X, x)$ such that the restriction $F|_{X_0} : X_0 \to Y$ is upper C-γ-continuous.*

Proof. Let $x \in X$ and V be an open set containing $F(x)$ and having compact complement. There exists $X_0 \in \gamma O(X, x)$ such that $F|_{X_0} : X_0 \to Y$ is upper C-γ-continuous. Therefore, there exists $U_0 \in \gamma O(X_0, x)$ such that $(F|_{X_0})(U_0) \subset V$. By Lemma 10, $U_0 \in \gamma O(X, x)$ and $F(u) = F|_{X_0}(u)$ for every $u \in U_0$. This shows that F is upper C-γ-continuous.

Theorem 13. *If a multifunction $F : X \to Y$ is lower C-γ-continuous and $X_0 \in \alpha O(X)$, then the restriction $F|_{X_0} : X_0 \to Y$ is lower C-γ-continuous.*

Theorem 14. *A multifunction $F : X \to Y$ is lower C-γ-continuous if for each $x \in X$ there exists $X_0 \in \gamma O(X, x)$ such that the restriction $F|_{X_0} : X_0 \to Y$ is lower C-γ-continuous.*

Corollary 15. *Let $\{U_i : i \in \Gamma\}$ be an α-open cover of X. A multifunction $F : X \to Y$ is upper C-γ-continuous (resp. lower C-γ-continuous) if and only if the restriction $F|_{U_i} : U_i \to Y$ is upper C-γ-continuous (resp. lower C-γ-continuous) for each $i \in \Gamma$.*

Proof. This is immediate consequence of Theorems 11 and 12 (resp. Theorems 13 and 14).

Theorem 16. *If $F : X \to Y$ is lower C-γ-continuous multifunction and $F(A)$ is compact for every subset A of X, then F is lower γ-continuous.*

Proof. Let A be any subset of X. Since $\mathrm{Cl}(F(A))$ is closed and compact by Theorem 3 $F^+(\mathrm{Cl}(F(A)))$ is γ-closed in X and $A \subset F^+(F(A)) \subset F^+(\mathrm{Cl}(F(A)))$. Thus, we have $\gamma\,\mathrm{Cl}(A) \subset F^+(\mathrm{Cl}(F(A)))$ and $F(\gamma\,\mathrm{Cl}(A)) \subset \mathrm{Cl}(F(A))$. It follows that, F is lower γ-continuous (see [2]).

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined $G_F(x) = \{x\} \times F(x)$ for each $x \in X$.

Lemma 17. *[6] The following hold for a multifunction $F : X \to Y$:*

1. *$G_F^+(A \times B) = A \cap F^+(B)$ and*
2. *$G_F^-(A \times B) = A \cap F^-(B)$*

for every subsets $A \subset X$ and $B \subset Y$.

Theorem 18. *Let $F : X \to Y$ be a multifunction and X be compact. If $G_F : X \to X \times Y$ is upper C-γ-continuous (resp. lower C-γ-continuous), then F is upper C-γ-continuous (resp. lower C-γ-continuous).*

Proof. Suppose that G_F is upper C-γ-continuous. Let $x \in X$ and V be an open set of Y containing $F(x)$ and having the compact complement. Then $X \times V$ is an open set of $X \times Y$ and has compact complement. Since $G_F(x) \subset X \times V$, there exists $U \in \gamma O(X, x)$ such that $G_F(U) \subset X \times V$. Therefore, by Lemma 17 we obtain $U \subset G_F^+(X \times V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is upper C-γ-continuous. The case for lower C-γ-continuous is similar.

For a multifunction $F : X \to Y$, the graph $G(F) = \{(x, F(x)) : x \in X\}$ is said to be strongly γ-closed if for each $(x, y) \in (X \times Y)\backslash G(F)$, there exist $U \in \gamma O(X, x)$ and an open set V of Y containing y such that $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$.

Lemma 19. *A multifunction $F : X \to Y$ has a strongly γ-closed graph if and only if for each $(x, y) \in (X \times Y)\backslash G(F)$, there exist $U \in \gamma O(X, x)$ and an open set V of Y containing y such that $F(U) \cap \mathrm{Cl}(V) = \emptyset$.*

Proof. This proof is obvious.

Theorem 20. *Let Y be a regular locally compact space. If $F : X \to Y$ is an upper C-γ-continuous multifunction such that $F(x)$ is closed for each $x \in X$, then $G(F)$ is strongly γ-closed.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(F)$, then $y \in Y\backslash F(x)$. Since Y is regular, there exist disjoint open sets V_1 and V_2 of Y such that $F(x) \subset V_1$ and $y \in V_2$. Moreover, since Y is locally compact and regular, there exists an open compact set V such that $y \in \mathrm{Cl}(V) \subset V_2$. Since F is upper C-γ-continuous and $Y\backslash \mathrm{Cl}(V)$ is an open set having compact complement, there exists $U \in \gamma O(X, x)$ such that $F(U) \subset Y\backslash \mathrm{Cl}(V)$. Therefore, we have $F(U) \cap \mathrm{Cl}(V) = \emptyset$ and by Lemma 19 $G(F)$ is strongly γ-closed.

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, we define $D^+_{c\gamma}(F)$ and $D^-_{c\gamma}(F)$ as follows:
$D^+_{c\gamma}(F) = \{x \in X : F$ is not upper C-γ-continuous at $x\}$.
$D^-_{c\gamma}(F) = \{x \in X : F$ is not lower C-γ-continuous at $x\}$.

Theorem 21. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D^+_{c\gamma} &= \bigcup_{G\in\sigma cc}\{F^+(G)\backslash\gamma\,\mathrm{Int}(F^+(G))\} \\ &= \bigcup_{B\in icc}\{F^+(\mathrm{Int}(B))\backslash\gamma\,\mathrm{Int}(F^+(B))\} \\ &= \bigcup_{B\in cc}\{\gamma\,\mathrm{Cl}(F^-(B))\backslash F^-(\mathrm{Cl}(B))\} \\ &= \bigcup_{H\in\mathcal{F}}\{\gamma\,\mathrm{Cl}(F^-(H))\backslash F^-(H)\}, \textit{where} \end{aligned}$$

σcc is the family of all σ-open sets of Y having compact complement,
icc is the family of all subsets B of Y such that $Y\backslash \mathrm{Int}(B)$ is compact,
cc is the family of all subsets B of Y having the compact closure,
$\mathcal{F}$ is the family of all closed and compact sets of (Y, σ).

Proof. We shall only the first equality and the last equality since the proofs of other are similar to the first.
Let $x \in D^+_{c\gamma}(F)$. Then, by Theorem 2, there exists an open set V of Y containing $F(x)$ and having compact complement such that $x \in \gamma\,\mathrm{Int}(F^+(V))$. Therefore, $x \in F^+(V)\backslash\gamma\,\mathrm{Int}(F^+(V)) \subset \bigcup_{G\in\sigma cc}\{F^+(G)\backslash\gamma\,\mathrm{Int}(F^+(G))\}$. Conversely, let $x \in \bigcup_{G\in\sigma cc}\{F^+(G)\backslash\gamma\,\mathrm{Int}(F^+(G))\}$. Then there exists an open set V of Y having compact complement such that $x \in F^+(V)\backslash\gamma\,\mathrm{Int}(F^+(V))$. By Theorem 2, $x \in D^+_{c\gamma}(F)$. We prove the last equality. $\bigcup_{H\in\mathcal{F}}\{\gamma\,\mathrm{Cl}(F^-(H))\backslash F^-(H)\} \subset \bigcup_{B\in cc}\{\gamma\,\mathrm{Cl}(F^-(B))\backslash F^-(\mathrm{Cl}(B))\} = D^+_{c\gamma}(F)$. Conversely, we have $D^+_{c\gamma}(F) = \bigcup_{B\in cc}\{\gamma\,\mathrm{Cl}(F^-(B)) \bigcup \bigcup_{H\in\mathcal{F}}\{\gamma\,\mathrm{Cl}(F^-(H))\backslash F^-(H)\}$.

Theorem 22. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties hold:*

$$\begin{aligned} D^-_{c\gamma} &= \bigcup_{G\in\sigma cc}\{F^-(G)\backslash\gamma\,\mathrm{Int}(F^-(G))\} \\ &= \bigcup_{B\in icc}\{F^-(\mathrm{Int}(B))\backslash\gamma\,\mathrm{Int}(F^-(B))\} \\ &= \bigcup_{B\in cc}\{\gamma\,\mathrm{Cl}(F^+(B))\backslash F^+(\mathrm{Cl}(B))\} \\ &= \bigcup_{H\in\mathcal{F}}\{\gamma\,\mathrm{Cl}(F^+(H))\backslash F^+(H)\}. \end{aligned}$$

Proof. The proof is similar to that of Theorem 21

Definition 4. *A topological space (X,σ) is called a KC-space [7] if every compact set of Y is closed.*

Definition 5. *A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be bounded at the point $p \in X$ if there exists a γ-open set U containing p and a compact set C of Y such that $F(x) \subset C$ for each $x \in U$.*

Theorem 23. *If a multifunction $F : (X,\tau) \to (Y,\sigma)$ is upper γ-continuous (resp. lower γ-continuous) at a point $p \in X$ and (Y,σ) is a KC space, then $F : (X,\tau) \to (Y,\sigma)$ is bounded at $p \in X$ and upper c-γ-continuous (resp. lower c-γ-continuous) at $p \in X$.*

Proof. We prove only the first case, the proof of the second being entirely similar. Let U be a γ-open set containing p and C a compact set of Y such that $F(x) \subset C$ for each $x \in U$. Let V be any open set of Y such that $F(p) \subset V$. Put $G = V \cup (Y\backslash C)$. Then G is open and $Y\backslash G$ is compact. By hypothesis, there exists a γ-open set W containing p such that $F(x) \subset G$ for every $x \in W$. Put $H = W \cap U$, then H is a γ-open set containing p such that $F(x) \subset G \cap C$ for any $x \in H$. Then $F(x) \subset V$ for any $x \in H$. Therefore, $F : (X,\tau) \to (Y,\sigma)$ is upper c-γ-continuous at $p \in X$.

Definition 6. *A topological space (X,τ) is said to be γ-saturated if for any $x \in X$ the intersection of all γ-open sets containing x is γ-open.*

Theorem 24. *Let (X,τ) be a γ-saturated topological space and (Y,σ) a T_1-space. If is upper C-γ-continuous, then F is upper γ-continuous.*

Theorem 25. *Let (X,τ) be a γ-saturated space and (Y,σ) a locally compact Hausdorff space. If is an upper C-γ-continuous and closed valued multifunction, then F is upper c-γ-continuous.*

Proof. Suppose that F is not upper γ-continuous at $x_0 \in X$. Then, there exists an open set V of Y such that $F(x_0) \subset V$ and $F(U) \cap (Y\backslash V) \neq \emptyset$ for every γ-open set U containing x_0. Let U_0 be the intersection of all γ-open sets containing x_0. Then U_0

is γ-open and there exists $z_1 \in U_0$ such that $F(Z_1) \cap (Y \backslash V) \neq \emptyset$. Hence there exists $y \in F(Z_1) \cap (Y \backslash V)$. Since (Y, σ) is locally compact Hausdorff, (Y, σ) is regular. Since $F(x_0)$ is a closed set and $y \notin F(x_0)$, there exists an open set W containing y such that $\mathrm{Cl}(W)$ is a compact set and $\mathrm{Cl}(W) \subset Y \backslash F(x_0)$. Since $F(x_0) \subset Y \backslash \mathrm{Cl}(W)$ and F is upper C-γ-continuous at x_0, there exists a γ-open set G containing x_0 and $F(x) \subset Y \backslash \mathrm{Cl}(W)$ for each $x \in G$. This is a contradiction. Since $z_1 \in U_0 \subset G$, $F(z_1) \subset Y \backslash \mathrm{Cl}(W)$. This contradicts that $F(z_1) \cap \mathrm{Cl}(W) \neq \emptyset$.

Theorem 26. *Let (X, τ) be a γ-saturated space and (Y, σ) a KC-space. If $F : (X, \tau) \to (Y, \sigma)$ is lower C-γ-continuous and for each $x \in X$ there exits a compact set C_x such that $F(x) \subset C_x$, then F is lower c-γ-continuous.*

Proof. Suppose that F is not lower c-γ-continuous at $x_0 \in X$. Then, there exists an open set V of Y such that $F(x_0) \cap V \neq \emptyset$ and for each γ-open set U containing x_0 there exists $u \in U$ such that $F(u) \cap V = \emptyset$. Let U_0 be the intersection of all γ-open sets containing x_0. Then U_0 is γ-open and there exists $x \in U_0$ such that $F(x) \cap V = \emptyset$. By the hypothesis, there exists a compact set C_x such that $F(x) \subset C_x$. Therefore, we have $F(x) \subset C_x \backslash V$ and $C_x \backslash V$ is a compact set. The set $Y \backslash (C_x \backslash V)$ is open and $F(x_0) \cap (Y \backslash (C_x \backslash V)) \neq \emptyset$. Since F is lower C-γ-continuous at x_0, there exists a γ-open set G containing x_0 such that for any $z \in G$ we have $F(z) \cap (Y \backslash (C_x \backslash V)) \neq \emptyset$. This is a contradiction because $x \in U_0 \subset G$ and $F(x) \subset C_x \backslash V$.

Definition 7. *The γ-frontier [2] of a subset A of a space X, denoted by $\gamma Fr(A)$, is defined by $\gamma Fr(A) = \gamma \,\mathrm{Cl}(A) \cap \gamma \,\mathrm{Cl}(X \backslash A) = \gamma \,\mathrm{Cl}(A) \backslash \gamma \,\mathrm{Int}(A)$.*

Theorem 27. *The set of all points x of X at which a multifunction $F : X \to Y$ is not upper C-γ-continuous (resp. lower C-γ-continuous) is identical with the union of the γ-frontier of the upper (resp. lower) inverse images of open sets containing (resp. meeting) $F(x)$ and havening compact complement.*

Proof. Let x be a point of X at which F is not upper C-γ-continuous. Then, there exists an open set V of Y containing $F(x)$ and having the compact complement such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every $U \in \gamma O(X, x)$. Therefore, we have $x \in X \backslash \gamma \,\mathrm{Int}(F^+(V))$ and hence $x \in \gamma Fr(F^+(V))$ since $x \in F^+(V) \subset \gamma \,\mathrm{Cl}(F^+(V))$. Conversely, suppose that V is an open set of Y containing $F(x)$ and having the compact complement such that $x \in \gamma \,\mathrm{Cl}(F^+(V))$. If F is upper C-γ-continuous at x, then there exists $U \in \gamma O(X, x)$ such that $U \subset F^+(V)$; hence $x \in \gamma \,\mathrm{Int}(F^+(V))$. This is a contradiction and hence F is not upper C-γ-continuous at x. The case for lower C-γ-continuous is similar.

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