Acta Universitatis Apulensis No. 37/2014
ISSN: 1582-5329 pp. 31-44

To Professor ART Solarin on his 60th Birthday Celebration

ON OSBORN LOOPS OF ORDER $4N$

A. O. Isere, S. A. Akinleye and J. O. Adéníran

Abstract. A new method of constructing Osborn loops of order $4n$, $n = 4, 6$ and 12 is presented. The constructed example is found not to satisfy the characteristic identity for universal Osborn loops, like Moufang loops, VD-loops, CC-loops and universal WIPLs. Hence, it is a non-universal Osborn loop. Some existing theorems of product of groups are investigated, and paradigms of them and the conditions for the existence of such theorems in loops are also stated.

2000 *Mathematics Subject Classification*: 20N02, 20N05.

Keywords: Binary operations, loops, construction of loops, product of loops, non-universal Osborn loops

1. Introduction

The desire to construct algebraic structures has been of interest to many authors. Though, it may be challenging but it worth the effort to construct one. A lot has been revealed through construction of examples and counter examples of some algebraic structures [12]. Osborn loop is more or less recent and only few examples are available. The few examples are mostly infinite. The popular finite examples of Osborn loops are mostly of order 16. ([28], [29]) No doubt constructed examples of Osborn loops are spares. Hence, this work is aimed at developing a new method of constructing a finite Osborn loop of order $4n$, $n = 4, 6, 12$.

The origin of Osborn loop can be traced to the work of J.M. Osborn [30] in 1960 on universal WIPLs. He observed that a universal WIPL obeys identity:

$$yx \cdot (zE_y \cdot y) = (y \cdot xz) \cdot y \text{ for all } x, y, z \in G \tag{1}$$

where $E_y = L_y L_{y^\lambda} = R_{y^\rho}^{-1} = L_y R_y L_y^{-1}$.
A loop that necessarily and sufficiently satisfies this identity is called an Osborn loop. Later, in 1968, E.D. Huthnance Jr[16] while carrying out a study on the generalized Moufang loops, named loops that obeys (1) as generalized Moufang loops and later

on in the same thesis, he called them M-loops. Also, he called a universal WIPL an Osborn loop. Basarab[3] in 1979 dubbed a loop $(G, \cdot)$ satisfying the identity.

$$(x^{\lambda}\backslash y) \cdot zx = x(yz \cdot x) \tag{2}$$

Or

$$x(yz \cdot x) = (x \cdot yE_x) \cdot zx \ \forall \ x, y, z \in G \tag{3}$$

an Osborn loop where $E_x = R_x R_{x^{\rho}} = (L_x L_x^{\lambda})^{-1} = R_x L_x R_x^{-1} L_x^{-1}$

It is to be noted that this type of Basarab's Osborn loop is not necessarily a universal WIPL by Huthnance's definition. However, these two definitions are rather complimentary than confusing. Osborn loops generalize Moufang loops, and Moufang loops that are IPLs are universal WIPLs. In other words, a Moufang loop is a variety of Osborn loops that is universal such that the following properties hold: power associative, diassociative and inverse properties etc. Hence, a Moufang that does not obey the properties aforementioned is an Osborn loop. V.O Chiboka [11] In 1990 adopted the Huthnance definition of an Osborn loop. The later deduced some properties of E_X relative to (1) $E_x = E_{x^{\lambda}} = E_{x^{\rho}}$. Jaiyeola and Adeniran in 2009 [26] used these properties to derive two nice identities defining Osborn loop.

$$OS_0 \ : \ x(yz \cdot x) = x(yx^{\lambda} \cdot x) \cdot zx \tag{4}$$

$$OS_1 \ : \ x(yz \cdot x) = [x(yx \cdot x^{\rho})] \cdot zx \tag{5}$$

Using these definitions, they were able to derive two nice identities that characterise a universal Osborn loop-see [26]. To this end, they were able to answer the fundamental part of the question associated with the 2005 open problem of Michael Kinyon-see [27]. In that note also, the authors were able to establish numerous new identities for universal Osborn loops like CC-loops, VD-loops and universal weak inverse property loops[27]. It is to be noted again that the most popularly known varieties of Osborn loops are CC-loops, Moufang loops, VD-loops and universal WIPLs. All these four varieties of Osborn loops are universal [27]. This is what makes non-universal Osborn loops interesting to researchers like Kinyon, Phillips and others [1],[29]. Therefore, it will be a celebrated effort to be able to construct a finite Osborn loop that is non-universal.

2. Preliminaries

Definition 1. *A loop is a set G with binary operation (denoted here simply by juxtaposition) such that*

- *for each a in G, the left multiplication map $L_a : G \to G, x \to ax$ is bijective,*

- *for each a in G, the right multiplication map $R_a : G \to G, x \to xa$ is bijective; and*
- *G has a two-sided identity 1.*

The order of G is its cardinality $|G|$.

Definition 2. *Consider $(G, \cdot)$ and $(H, \circ)$ being two distinct groupoids (quasigroups, loops). Let A, B and C be three distinct non-equal bijective mappings (permutations), that map G onto H. The triple $\alpha = (A, B, C)$ is called an isotopism of $(G, \cdot)$ onto $(H, \circ)$ if and only if*

$$xA \circ yB = (x \cdot y)C \ \forall \ x, y \in G. \tag{6}$$

So, $(H, \circ)$ is called a groupoid(quasigroup, loop) isotope of $(G, \cdot)$.
Similarly, the triple

$$\alpha^{-1} = (A, B, C)^{-1} = (A^{-1}, B^{-1}, C^{-1}) \tag{7}$$

is an isotopism from $(H, \circ)$ onto $(G, \cdot)$ so that $(G, \cdot)$ is also called a groupoid (quasigroup, loop) isotope of $(H, \circ)$. Here, both are said to be isotopic to each other ([8], [9], [15], [27]).

Definition 3. *A property is said to be isotopic invariance if such property is true with a loop as well as its isotopes. ([9]). Such property is called a universal property.*

Isotopic invariance of types and varieties of quasigroups and loops described by one or more equivalent identities, especially those that in the class of Bol- Moufang type loops as first named by Fenyves [13] and [14] in the 1960's and recently considered by Phillips and Vojtechovsky [33], [34], [10],[28] have been studied (see [26]).

Definition 4. *An Osborn loop is said to be universal if every isotope of an Osborn loop is Osborn. Otherwise it is said to be non-universal.*

Theorem 1. *(Kinyon [29]) The smallest order for which proper(non-Moufang and non-CC) Osborn loops with non-trivial nucleus exists is 16.*
There are two of such loops.

- *Each of the two is a G-loop.*
- *Each contains as a subgroup, the dihedral group (D_4) of order 8.*
- *For each loop, the center coincides with the nucleus and has order 2. The quotient by the center is a non-associative CC-loop of order 8.*

- *The second center is $\mathbb{Z}_2 \times \mathbb{Z}$, and the quotient is $\mathbb{Z}_4$.*
- *One loop satisfies $L_x^4 = R_x^4 = I$, the other does not.*

AIP Osborn loops include:

- *commutative Moufang loops and*
- *AIP CC-loops*

Lemma 2. *(Jaiyeola and Adeniran[27]) Let $(Q, \cdot, \backslash, /)$ be a left universal Osborn loop. The following identities are satisfied:*

$$v \cdot vv = v^{\lambda} \backslash v \cdot v \text{ and } vv \cdot vv = v^{\lambda} \backslash (v^{\lambda^2} v) \cdot v$$

Corollary 3. *A simple universal Osborn loop is a Moufang loop.*

A simple loop is a loop that has no non-trivial normal subloop. Vojtechovsky [33],[34], studied simple Moufang loops.

Theorem 4. *(Basarab [5])*
A generalized Moufang loop $(Q, \cdot)$ is a VD-loop if and only if $x^4 \in N(Q)$ $\forall$ $x \in Q$.

Theorem 5. *(Basarab [4])*
An Osborn loop $Q(\cdot)$ in which $x^2 \in N$ for each $x \in Q$ is a G-loop.

Theorem 6. *(Basarab [5])*
Each VD-loop is an Osborn loop.

Theorem 7. *(Basarab)*
Each CC-loop is an Osborn loop.

Some recent studies on universal Osborn loops can be found in Jaiyéọlá [18, 20], Jaiyéọlá and Adéníran [21], Jaiyéọlá et. al. [22, 25, 23, 24].

3. Main Results

We started by investigating the existing theorems in product of groups [6], [7]. The conditions for existence of such theorems in loops are presented.
Suppose G and H are loops. Then, the set $G \times H$ of ordered pair (g, h) with $g \in G$ and $h \in H$ is a loop when equipped with appropriately defined operations.

Lemma 8. *Let G and H be two distinct loops. Then, the set $G \times H$ under the binary operation $'\star'$ defined as $(g_1, h_1) \star (g_2, h_2) = (g_1g_2, h_1h_2)$ $\forall$ $g_1, g_2 \in G$and $h_1, h_2 \in H$ is a loop, called the product of the loops G and H.*

Proof. Suppose $(G,\cdot)$ and $(H,\circ)$ are loops. Then consider $((G\times H),\star)$ given as

$$(g_1,h_1)\star(g_2,h_2)=(g_1\cdot g_2,h_1\circ h_2)=(g^c,h^c)\ \forall\ g^c\in G,h^c\in H.$$

then $G\times H$ is closed with respect to $'\star'$. Next, $(g_1,h_1)\star(g,h)=(g_2,h_2)\star(g,h)$. Then $(g_1\cdot g,h_1\circ h)=(g_2\cdot g,h_2\circ h)$ implies that $g_1\cdot g=g_2\cdot g$ implies that $g_1=g_2$ and $h_1\circ h=h_2\circ h$ implies that $h_1=h_2$. Obviously, $(e_G,e_H)\in((G\times H),\star)$. Hence, $G\times H$ is a loop.

Corollary 9. *Let $G=G_1\times...\times G_n$ be a product of any finite sequence of loops with a binary operation defined as $(g_1,...,g_n)\star(g_1^c,...,g_n^c)=(g_1g_1^c,...,g_ng_n^c)$. Then G is a loop, and will be abelian, if and only if every factor G_i is abelian.*

Proof. Let $((G_1\times...\times G_n),\star)$ be defined as above. From Lemma 3.1, $((G_1\times...\times G_n),\star)$ is closed. now

$$(g_1,...,g_n)\star(d_1,...,d_n)=(g_1^c,...,g_n^c)\star(d_1,...,d_n)$$

$$(g_1d_1,...,g_nd_n)=(g_1^cd_1,...,g_n^cd_n)$$

implies

$$g_1d_1=g_1^cd_1,...,g_nd_n=g_n^cd_n$$

then,

$$g_1=g_1^c,...,g_n=g_n^c$$

and the presence of $(e_1,...,e_n)$ in $((G_1\times...\times G_n),\star)$ makes it a loop. And $((G_1\times...\times G_n),\star)$ is abelian iff every G_i is abelian. Consider

$$(g_1,...,g_n)\star(g_1^c,...,g_n^c)=(g_1g_1^c,...,g_ng_n^c)=(g_1^cg_1,...,g_n^cg_n)=(g_1^c,...,g_n^c)\star(g_1,...,g_n)$$

Since every G_i is abelian

$$(g_1g_1^c,...,g_ng_n^c)=(g_1^cg_1,...,g_n^cg_n)=(g_1^c,...,g_n^c)\star(g_1,...,g_n)$$

The proof is complete.

Lemma 10. *Let G and H be two distinct power associative loops. Suppose m and n are relatively prime, then the order of (g,h) in $(G\times H)$ is the least common multiple of m and n, the orders $g\in G$ and $h\in H$ respectively.*

Proof. The identity element e of $G\times H$ is given as (e_G,e_H). where $e_G\in G$ and $e_H\in H$ (the identity elements of G and H). Suppose $g^m=e_G$ and $h^n=e_H$, if m,n are relatively prime, then $(g,h)^{mn} = e$, mn being the least common multiple of m and n.

$$(g,h)^{mn} = (\underbrace{g...g}_{mn-times}, \underbrace{h...h}_{mn-times}) = (\underbrace{\underbrace{gg...g}_{m-times} ... \underbrace{gg...g}_{m-times}}_{n-times}, \underbrace{\underbrace{hh...h}_{n-times} ... \underbrace{hh...h}_{n-times}}_{m-times})$$
$$= (\underbrace{e_G...e_G}_{m-times} ... \underbrace{e_G...e_G}_{n-times}) = (e_G, e_G).$$

The lemma follows.

Proposition 1. *Let G be a (qusaigroup, loop). Suppose $g_1, ..., g_r$ are elements from the center of G of orders $n_1, ..., n_r$ respectively. Let $Z_{n_1} \times ... \times Z_{n_r}$ be defined by binary operation*

$$< k_1, ..., k_r > \star < l_1, ..., l_r > = < k_1 + l_1, ..., k_r + l_r > .$$

Then the map

$$\phi : Z_{n_1} \times ... \times Z_{n_r} \to G^1 \text{ given by } \phi(< k_1, ..., k_r >) = g_1^{k_1} ... g_r^{k_r}$$

is an isomorphism of G onto the G^1 (subloop of G) generated by $g_1, ..., g_r$.

Proof. If the order of g is n, $g^k g^1 = g^{k+1}$ where the addition in the exponent is performed modulo n. Thus

$$\phi(< k_1, ..., k_r > \star < l_1, ..., l_r >) =$$
$$(< k_1 + l_1, ..., k_r + l_r >) = g_1^{k_1+l_1} ... g_r^{k_r+l_r} = g_1^{k_1} g_1^{l_1} ... g_r^{k_r} g_r^{l_r} =$$
$$(g_1^{k_1} ... g_r^{k_r})(g_1^{l_1} ... g_r^{l_r}) = \phi(< k_1, ..., k_r >) \cdot \phi(< l_1, ..., l_r >)$$

Therefore, ϕ is a homomorphism. Since $\phi(< 0, ..., 1, ..., 0 >) = g_i$ when the 1 is in the ith place, the image of ϕ contains each g_i and thus is G^1. Implies $\phi(g_i) = G^1 \ \forall \ i = 1, 2, ..., r$. Therefore, ϕ is onto
. Next, let

$$\phi(< k_1, ..., k_r >) = \phi(< l_1, ..., l_r >), \text{implies } (g_1^{k_1} ... g_r^{k_r}) = (g_1^{l_1} ... g_r^{l_r}).$$

Then,

$$(g_1^{k_1} ... g_r^{k_r})(g_1^{l_1} ... g_r^{l_r})^{-1} = e = (g_1^{k_1-l_1} ... g_r^{k_r-l_r}) = (g^0 ... g^0).$$

Thus, $k_1 - l_1 = 0, ..., k_r - l_r = 0$. So, $k_1 = l_1, ..., k_r = l_r$. Therefore, $< k_1, ..., k_r > = < l_1, ..., l_r >$. ϕ is injective, and so ϕ is an isomorphism.

4. Osborn loops

The class of Bol-Moufang type of loops play an important role in the theory of quasigroups and in their applications in other branches of mathematics [13]. In what follows, we give 14 possible non-trivial identities, each defining an Osborn loop. Most of these identities are stated by various authors as will be indicated. Effort is being made here to outline the relationship of some of these identities and also state these as equivalent identities for Osborn loops.

(i) $(y \cdot xz) \cdot y = yx \cdot (zE_y \cdot y)$ [30]

(ii) $y(zx \cdot y) = (y \cdot zE_y) \cdot xy$

$$E_y = L_y L_{y^\lambda} = R_{y^\rho}^{-1} R_y^{-1} = L_y R_y L_y^{-1} R_y^{-1}$$

(iii) $x(yz \cdot x) = (x \cdot yE_y) \cdot zx$ [3]

(iv) $(x \cdot zy)x = xz \cdot (yE_y \cdot x)$

$$E_x = R_x R_{x^\rho} = (L_x L_{x^\lambda})^{-1} = R_x L_x R_x^{-1} L_x^{-1}$$

(v) $(x \cdot yz)x = xy \cdot (zE_x^{-1} \cdot x)$ [27]

(vi) $(x^\lambda \backslash y) \cdot zx = x(yz \cdot x)$ [3], [27], [29]

(vii) $xy \cdot (z/x^\rho) = (x \cdot yz)x$ [27]

(viii) $x(yx^\lambda \cdot x) \cdot zx = x(yz \cdot x)$ [27]

(ix) $x(yx \cdot x^\rho) \cdot zx = x(yz \cdot x)$ [26]

(x) $x[(x^\lambda y)z \cdot x] = y \cdot zx$ [26]

(xi) $[x \cdot y(zx^\rho)]x = xy \cdot z$ [27]

(xii) $x[(x^\lambda y)z \cdot x] = y \cdot zx$ [26]

(xiii) $(x \cdot yz)x = xy \cdot [(x^\lambda \cdot xz) \cdot x]$ [26]

(xiv) $(x \cdot yz)x = xy \cdot [(x \cdot x^\rho z) \cdot x]$ [27]

5. Construction of non-universal Osborn loop

The binary operations as defined in the construction below hold between two active(non-arbitrary) variables $'a'$ and $'b'$. Whereas, the combination $'b+c'$ or $'a+c'$ is peculiar and unique to Osborn loops as defined in the construction below.

Example 1. *Let $I(\cdot) = C_{2n} \times C_2$ that is $I = \{(x^\alpha, y^\beta), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ and the binary operation is defined as follows:*

$$(x^a, e) \cdot (x^b, y^\beta) = (x^{a+b}, y^\beta)$$

$$(x^a, y^\alpha) \cdot (x^b, e) = (x^{a+b}, y^\alpha)$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+b}, y^{\alpha+\beta}) \text{ if } b \equiv 0(mod\ 2)$$

$$= (x^{a+b+ab^2}, y^{\alpha+\beta}) \text{ if } b \equiv 1(mod\ 2)$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+c}, y^{\alpha+\delta}) \text{ if } b \equiv 0(mod\ 2)$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+c+ab^2}, y^{\alpha+\delta}) \text{ if } b \equiv 1(mod\ 2)$$

Then $I(\cdot)$ is an Osborn loop of order $4n$, where $n = 2, 3, 4, 6$ and 12.

Proof. We first show that $I(\cdot)$ satisfies Osborn identity (vi):

$$(X^\lambda \backslash Y) \cdot ZX = X(YZ \cdot X)$$

(a) Let $X = (x^a, e)$; $Y = (x^b, e)$; $Z = (x^c, e)$, then by direct computations, we have

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c}, e)$$

$$X(YZ \cdot X) = (x^{2a+b+c}, e)$$

(b) Let $X = (x^a, e)$; $Y = (x^b, e)$; $Z = (x^c, y^\gamma)$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c}, y^\gamma)$$

$$X(YZ \cdot X) = (x^{2a+b+c}, y^\gamma)$$

(c) Let $X = (x^a, e)$; $Y = (x^b, y^\beta)$; $Z = (x^c, e)$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c}, y^\beta)\ b = even$$

$$X(YZ \cdot X) = (x^{2a+b+c}, y^\beta)\ b = even$$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c+ab^2}, y^\beta) b = odd$$

$$X(YZ \cdot X) = (x^{2a+b+c+ab^2}, y^\beta)\ b = odd$$

(d) Let $X = (x^a, e); Y = (x^b, y^\beta); Z = (x^c, y^\gamma)$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c}, y^{\beta+\gamma}) \ b = even$$
$$X(YZ \cdot X) = (x^{2a+b+c}, y^{\beta+\gamma}) \ b = even$$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c+ab^2}, y^{\beta+\gamma}) \ b = odd$$
$$X(YZ \cdot X) = (x^{2a+b+c+ab^2}, y^{\beta+\gamma}) \ b = odd$$

(e) Let $X = (x^a, y^\alpha); Y = (x^b, e); Z = (x^c, e)$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c}, y^{2\alpha}) \ a = even$$
$$X(YZ \cdot X) = (x^{2a+b+c}, y^{2\alpha}) \ a = even$$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c+a^2c}, y^{2\alpha}) \ a = odd$$
$$X(YZ \cdot X) = (x^{2a+b+c+(b+c)a^2}, y^{2\alpha}) \ a = odd$$

(f) Let $X = (x^a, y^\alpha); Y = (x^b, e); Z = (x^c, y^\gamma)$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c}, y^{2\alpha+\gamma}) \ a = even$$
$$X(YZ \cdot X) = (x^{2a+b+c}, y^{2\alpha+\gamma}) \ a = even$$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+a^2c+b+c+}, y^{2\alpha+\gamma}) \ a = odd$$
$$X(YZ \cdot X) = (x^{2a+b+c+(b+c)a^2}, y^{2\alpha+\gamma}) \ a = odd$$

(g) Let $X = (x^a, y^\alpha);\ Y = (x^b, y^\beta);\ Z = (x^c, e)$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c}, y^{2\alpha+\beta}) \ a = even, b = even$$
$$X(YZ \cdot X) = (x^{2a+b+c}, y^{2\alpha+\beta}) \ a = even, b = even$$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+ab^2+b+c}, y^{2\alpha+\beta}) \ a = even, b = odd$$
$$X(YZ \cdot X) = (x^{2a+ab^2+b+c+}, y^{2\alpha+\beta}) \ a = even, b = odd$$

$$(X^\lambda \backslash Y) \cdot ZX = (x^{2a+b+c+a^2c}, y^\beta) \ a = odd, b = even$$

$$X(YZ \cdot X) = (x^{2a+b+c+(b+c)a^2}, y^{\beta}) \ a = odd, b = even$$

$$(X^{\lambda}\backslash Y) \cdot ZX = (x^{2a+b+c+a^2c+ab^2}, y^{2\alpha+\beta}) \ a = odd, b = odd$$
$$X(YZ \cdot X) = (x^{2a+b+c+(b+c)a^2+a^2c+ab^2}, y^{2\alpha+\beta}) \ a = odd, b = odd$$

(h) Let $X = (x^a, y^\alpha)$; $Y = (x^b, y^\beta)$; $Z = (x^c, y^\gamma)$

$$(X^{\lambda}\backslash Y) \cdot ZX = (x^{2a+b+c}, y^{2\alpha+\beta+\gamma}) \ a = even, b = even$$
$$X(YZ \cdot X) = (x^{2a+b+c}, y^{2\alpha+\beta+\gamma}) \ a = even, b = even$$

$$(X^{\lambda}\backslash Y) \cdot ZX = (x^{2a+b+c+ab^2}, y^{2\alpha+\beta+\gamma}) \ a = even, b = odd$$
$$X(YZ \cdot X) = (x^{2a+b+c+ab^2}, y^{2\alpha+\beta+\gamma}) \ a = even, b = odd$$

$$(X^{\lambda}\backslash Y) \cdot ZX = (x^{2a+b+c+a^2c}, y^{\beta+\gamma}) \ a = odd, b = even$$
$$X(YZ \cdot X) = (x^{2a+b+c+(b+c)a^2}, y^{\beta+\gamma}) \ a = odd, b = even$$

$$(X^{\lambda}\backslash Y) \cdot ZX = (x^{2a+b+c+a^2c+ab^2}, y^{2\alpha+\beta+\gamma}) \ a = odd, b = odd$$
$$X(YZ \cdot X) = (x^{2a+b+c+(b+c)a^2+ab^2}, y^{2\alpha+\beta+\gamma}) \ a = odd, b = odd$$

Since $(X^{\lambda}\backslash Y) \cdot ZX = X(YZ \cdot X)$ holds in cases whenever $25 \equiv 1(mod\ 2n)$, that is $n = 2, 3, 4, 6$ and 12, hence, the example is an Osborn loop of order $4n$ where $n = 2, 3, 4, 6$ and 12- see Solarin and Sharma [35].

Also (e, e) is the two sided identity. Moreover, if $X = (x^a, e)$, then $X^{-1} = (x^{-a}, e)$. If $X = (x^a, y^a)$ then

$$X^{-1} = (x^{-a}, y^{-\alpha}) \text{ if a = even}$$

And

$$X^{-1} = (x^{-(a+a^2b)}, y^{-\alpha}) \text{ if a=odd.}$$

Therefore, the inverses are defined.

Also for non-associativity
Let

$$X = (x^a, y^\alpha); Y = (x^b, y^\beta); Z = (x^c, y^\gamma)$$

where a is an even integer and b an odd integer.

$$(XY)Z = (x^{a+b+c+ab^2}, y^{\alpha+\beta+\gamma})$$
$$X(YZ) = (x^{a+b+c}, y^{\alpha+\beta+\gamma})b = odd$$

Thus

$$(XY)Z \neq X(YZ)$$

whenever 4,6 are not congruence to 0 mod $2n$.

Next, we verify that the example above is non-universal using [26] and [27]. Jaiyeola and Adeniran [27] showed that an Osborn loop $(H, \star)$ should be universal Osborn loop, if it should obey the identity $Y \star YY = Y^{\lambda} \backslash Y \star Y$ [27]-see lemma 2 above.

Applying the above, the constructed example 1 defined as above should be a universal Osborn loop, if it should obey the the identity: $Y \star YY = Y^{\lambda} \backslash Y \star Y$.
Let $Y = (x^b, y^{\beta})$, b being an odd integer. Then, by direct computation, we have:

$$Y \star YY = (x^{3b+b^3}, y^{\beta})\ b = odd$$

$$Y^{\lambda} \backslash Y \star Y = (x^b, y^{\beta})^{\lambda} \backslash (x^b, y^{\beta}) \cdot (x^b, y^{\beta})$$
$$Y^{\lambda} \backslash Y \star Y = (x^{3b+3b^3+b^5}, y^3\beta)\ b = odd$$

Thus, $Y \star YY \neq Y^{\lambda} \backslash Y \star Y$. Thus, $I(\cdot) = C_{2n} \times C_2$ is not a universal Osborn loop.

6. Concluding Remarks

Example 1 above is a new method of constructing proper non-associative Osborn loops of order 16, 24 and 48. The example above is found not to be flexible and does not have the left(right) alternative properties [LAP(RAP)] or the left(right) inverse properties [LIP(RIP)] or the anti-automorphic inverse properties (AAIP). Consequently, it is not Moufang. For detail about those properties, see [8], [10],[17], [31], [32].

Since the smallest order possible for Osborn is 16 [29], implies that $I(\cdot)$ is an Osborn loop when $n = 4, 6, 12$. At $n = 2$, we have a group of order 8 with a single normal subgroup of order 2. Thence, we hypothesised that at order 12, the construction is either a Moufang loop or the dihedral group on six elements ($C_6 \times C_2$) [2]. We reach this conclusion since the smallest Moufang loop which is not a group is of order 12, and Osborn loops simultaneously generalize Moufang and CC-loops [29].

The hypothesis above is yet to be proven. By and large, the constructed example is of order 16, 24 and 48.

References

[1] A.A. Albert, *Quasigroups I*, Trans-Amer. Math. Soc., 54 (1943), 507-519.

[2] A.M. Asiru,*A study of the classification of finite Bol loops*, Ph.D thesis university of Agriculture (2008), Abeokuta.

[3] A. S. Basarab and A.I. Belioglo, *UAI Osborn loops*, Quasigroups and loops, Mat. Issled. 51 (1979), 8-16.

[4] A. S. Basarab, *Osborn's $\mathcal{G}$-loop*, Quasigroups and Related Systems 1 (1994), 51-56.

[5] A. S. Basarab , *Generalised Moufang G-loops*, Quasigroups and Related Systems 3 (1996), 1-6.

[6] R. Bear, *Nets and Groups, I and II*, Trans. Amer. Math. Soc., 46 (1940), 110-141.

[7] E.D. Bolker, *Elementary Number Theory an algebraic approach*, W.A. Benjamin, Inc. New York, (1970).

[8] R. H. Bruck, *A survey of binary systems*,Springer-Verlag, Berlin and New York, (1958).

[9] B.F. Bryant and H. Schneider, *Principal loop-isotopes of quasiqroups*, Canad. J. Math. 18 (1966), 120-125

[10] O. Chein, H. O. Pflugfelder and J. D. H. Smith, *Quasigroups and loops : Theory and applications*, Heldermann Verlag (1990), 568pp.

[11] V. O. Chiboka, *The study of properties and construction of certain finite order G-loops*, Ph.D thesis, Obafemi Awolowo University, Ile-Ife, (1990).

[12] P. Csörgő *loops and groups*, Ph.D. Thesis, Budapest 2009.

[13] F. Fenyves, *Extra loops I*, Publ. Math. Debrecen, 15 (1968), 235–238.

[14] F. Fenyves, *Extra loops II*, Publ. Math. Debrecen, 16 (1969), 187–192.

[15] E. G. Goodaire and D. A. Robinson, *A class of loops which are isomorphic to all loop isotopes*, Can. J. Math. 34 (1982), 662–672.

[16] E. D. Huthnance Jr., *A theory of generalised Moufang loops*, Ph.D. thesis, Georgia Institute of Technology, (1968).

[17] A. O. Isere, J. O. Adeniran and A. R. T. Solarin,*Somes Examples Of Finite Osborn Loops*, Journal of Nigerian Mathematical Society, 31 (2012), 91-106.

[18] T. G. Jaiyéọlá , *On three cryptographic identities in left universal Osborn loops*, Journal of Discrete Mathematical Sciences and Cryptography, 14, 1 (2011), 33-50. (DOI:10.1080/09720529.2011.10698322)

[19] T. G. Jaiyéọlá, *Osborn loops and their universality*, Scientific Annals of "Al.I. Cuza" University of Iasi. 58, 2 (2012), 437-452.

[20] T. G. Jaiyéọlá, *On two cryptographic identities in universal Osborn loops*, Journal of Discrete Mathematical Sciences and Cryptography, 16, 2-3 (2013), 95116. (DOI:10.1080/09720529.2013.821371)

[21] T. G. Jaiyéọlá and J. O. Adéníran, *On another two cryptographic identities in universal Osborn loops*, Surveys in Mathematics and its Applications, 5 (2010), 17–34.

[22] T. G. Jaiyéọlá and J. O. Adéníran, *Loops that are isomorphic to their Osborn loop isotopes(G-Osborn loops)*, Octogon Mathematical Magazine, 19, 2 (2011), 328-348.

[23] T. G. Jaiyéọlá , J. O. Adéníran and A. A. A. Agboọlá, *On the second Bryant Schneider group of universal Osborn loops*, Societatea Română de Matematică Aplicată si Industrială Journal, 9, 1 (2013), , 37-50.

[24] T. G. Jaiyéọlá , J. O. Adéníran and A. R. T. Sòlárìn, *The universality of Osborn loops*, Acta Universitatis Apulensis Mathematics-Informatics, 26 (2011), 301-320.

[25] T. G. Jaiyéọlá , J. O. Adéníran and A. R. T. Sòlárìn , *Some necessary conditions for the existence of a finite Osborn loop with trivial nucleus*, Algebras, Groups and Geometries, 28, 4 (2011), 363-380.

[26] T.G. Jaiyeola And J.O. Adeniran, *New identities in universal Osborn loops, Quasigroups and Related Systems*, Moldova 17, 1 (2009), 55-76.

[27] T.G. Jaiyeola And J.O. Adeniran , *Not Every Osborn loop is Universal*, Acta Math. Acad. Paed. Nviregvhaziensis 25 (2009), 189-190.

[28] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský , *Loops of Bol-Moufang type with a subgroup of index two*, Bul. Acad. Stiinte Repub. Mold. Mat. 49, 3(2005), 71-87.

[29] M. K. Kinyon, *A survey of Osborn loops*, Milehigh conference on loops, quasigroups and non-associative systems, University of Denver, Denver, Colorado, 2005.

[30] J. M. Osborn, *Loops with the weak inverse property*, Pac. J. Math. 10 (1961), 295-304.

[31] H. O. Pflugfelder, *special class of Moufang loops*, Proc. Amer. Math. Soc. 26 (1970), 583-586.

[32] H. O. Pflugfelder, *Quasigroups and loops : Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin (1990), 147pp.

[33] J. D. Phillips and P. Vojtěchovský, *The varieties of loops of Bol-Moufang type*, Alg. Univer. 54, 3(2005), 259-383..

[34] J. D. Phillips and P. Vojtěchovský , *The varieties of quasigroups of Bol-Moufang type : An equational approach*, J. Alg. 293 (2005), 17-33.

[35] A. R. T. Solarin and B.L. Sharma , *Some Examples of Bol loops*, ACTA universitatis carolinae- Mathematica et Physics, 25, 1(1983), 59-68.

A. O. Isere
Department of Mathematics,
Ambrose Alli University,
Ekpoma 310001, Nigeria
email: *abednis@yahoo.co.uk*

S. A. Akinleye
Department of Mathematics,
Federal University of Agriculture,
Abeokuta 110101, Nigeria.
email: *akinleyesa@yahoo.com*

J. O. Adeniran
Department of Mathematics,
Federal University of Agriculture,
Abeokuta 110101, Nigeria.
email: *adeniranoj@unaab.edu.ng*