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ON IMBALANCE SEQUENCES OF ORIENTED GRAPHS

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Abstract. A necessary and sufficient condition for a sequence of integers to be an irreducible imbalance sequence is obtained. We found bounds for imbalance b_i of a vertex v_i of oriented graphs. Some properties of imbalance sequence of oriented graphs, arranged in lexicographic order, are investigated. In the last we report a result on an imbalance sequence for a self-converse tournament and conjecture that it is true for oriented graphs.

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1. Introduction

An oriented graph is a digraph with no symmetric pair of directed arcs with no loops. The imbalance $b(v_i)$ (or simply b_i)of a vertex v_i in a digraph is defined as $d_i^+ - d_i^-$, where d_i^+ and d_i^- are out-degree and in-degree of vertex v_i respectively.

An oriented graph D is reducible if it is possible to partition its vertices into two nonempty sets V_1 and V_2 in such a way that every vertex of V_2 is adjacent to all vertices of V_1. Let D_1 and D_2 be induced digraphs having vertex sets V_1 and V_2 respectively. Then D consists of all the arcs of D_1, D_2 and every vertex of D_2 is adjacent to all vertices of D_1. We write $D = [D_1, D_2]$. If this is not possible, then the oriented graph D is irreducible. Let $D_1, D_2, \dots, D_k$ be irreducible oriented graphs with disjoint vertex sets. $D = [D_1, D_2, \dots, D_k]$ denotes the oriented graph having all arcs of D_m, $1 \le m \le k$, and every vertex of D_j is adjacent to all vertices of D_i with $1 \le i < j \le k$. $D_1, D_2, \dots, D_k$ are called irreducible components of D. Such decomposition is known as irreducible component decomposition of D and is unique.

An imbalance sequence $B = (b_1, b_2, \dots, b_n)$ with $b_1 \le b_2 \le \dots \le b_n$ is said to be irreducible if all the oriented graphs with the imbalance sequence B are irreducible.

2. Necessary and sufficient condition

A sequence of integers $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \ldots a_n$ is feasible if it has sum zero and satisfies

$$\sum_{i=1}^{k} a_i \leq k(n-k) \text{ for } 1 \leq k < n.$$

The following result gives a condition for a sequence of integers to be the imbalance sequence of a simple directed graph.

Theorem 1. *[10] A sequence is realizable as an imbalance sequence if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \geq b_2 \geq \ldots \geq b_n$ imbalance sequence of a simple directed graph if and only if

$$\sum_{i=1}^{k} b_i \leq k(n-k) \text{ for } 1 \leq k < n \tag{1}$$

with equality when $k = n$.

On arranging the imbalance sequence in nondecreasing order, we obtain the following Corollary 2.

Corollary 2. *A sequence of integers $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of a simple directed graph (without repeated arcs) if and only if*

$$\sum_{i=1}^{k} b_i \geq k(k-n), \text{ for } 1 \leq k < n \tag{2}$$

with equality when $k = n$.

Proof. Let $\bar{b}_i = b_{n-i+1}$. Then the sequence $\bar{B} = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$ satisfies condition

(1). We have

$$\begin{aligned}
\sum_{i=1}^{k} b_i &= \sum_{i=1}^{k} \bar{b}_{n-i+1} \\
&= \sum_{i=1}^{n} \bar{b}_{n-i+1} - \sum_{i=k+1}^{n} \bar{b}_{n-i+1} \\
&= 0 - (\bar{b}_{n-k} + \bar{b}_{n-k+1} + \cdots + \bar{b}_1) \\
&= -\sum_{j=1}^{n-k} \bar{b}_j \\
&\geq -(n-k)\{n-(n-k)\} \text{ (from Condition 1)} \\
&= k(k-n),
\end{aligned}$$

where $1 \leq k \leq n-1$ and equality holds when $k = n$.

3. Construction of an oriented graph with a given imbalance sequence

A sequence of integers is graphic if it is a degree sequence of a simple undirected graph. For characterization of graphic sequences we refer to [2, 3, 6]. Klietman and Wang [7] observed that Havel and Hakimi [3, 6] argument works with the deletion of the any element d_k of the degree sequence $(d_1, d_2, \ldots, d_n)$ with $d_1 \leq d_2 \leq \ldots \leq d_n$, subtracting 1 from the d_k largest other elements.

The analogous statement about imbalance sequence is false. Dhruv et al. [10] considered the imbalance sequence $(3, 1, -1, -3)$ of a transitive tournament. Deleting the element 1 and adding 1 to the smallest imbalance gives $(3, -1, -2)$, which has no realization by a simple digraph.

Theorem 1 provides us an algorithm to construct an oriented graph from a given imbalance sequence. At each stage we form $\hat{B} = (\hat{b}_2, \ldots, \hat{b}_n)$ from $B = (b_1, b_2, \ldots, b_n)$ by deleting the largest imbalance b_1 and adding 1 to b_1 smallest elements of B. Arcs of an oriented graph are defined by $v_1 \to v$ if and only if $\hat{b}_v \neq b_v$. If this procedure applied recursively, then

(i) it tests whether B is an imbalance sequence and if B is an imbalance sequence, then

(ii) an oriented graph D_B with imbalance sequence B is constructed.

Example of algorithm, $n = 5,\ B = (2, 0, 0, 0, -2)$.

Stage	B	Arcs of D_B
1.	(2,0,0,0, -2)	
2.	(-,1,0,0, -1)	$v_1 \to v_2,\ v_5$
3.	(-,-,0,0,0)	$v_2 \to v_5$

4. Irreducible imbalance sequences of oriented graphs

In case of tournaments, the score sequence $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \le s_2 \le \ldots \le s_n$used to decide whether a tournament T having the score sequence S is strong or not [4]. This is not true in case of oriented graphs. For example oriented graphs D_1 and D_2 both have imbalance sequence $(0, 0, 0)$, but D_1 is strong and D_2 is not.

The following Theorem characterizes irreducible imbalance sequences.

Theorem 3. *Let $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$ be an imbalance sequence of oriented graph. Then B is irreducible if and only if*

$$\sum_{i=1}^{k} b_i \ > \ k(k-n), \ \textit{for}\ 1 \le k \le n-1 \tag{3}$$

$$\textit{and} \qquad \sum_{i=1}^{n} b_i \ = \ 0. \tag{4}$$

Proof. Suppose D is an oriented graph with vertex set V, having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$. Equality condition (4) is obvious. To prove inequalities (3.4), let U be the set of k vertices with the smallest imbalances, the arcs within U contribute nothing to $\sum_{i=1}^{k} b_i$, and the ordered pairs $(V \backslash U) \times U$ contributes atmost -1 to each $v \in U$, so

$$\begin{aligned} \sum_{i=1}^{k} b_i \ &\ge \ -k(n-k) \\ &= \ k(k-n), \ \text{for } 1 \le k \le n-1. \end{aligned} \tag{5}$$

Since D is irreducible, there must exist at least one arc from a vertex of U to a vertex of $V \backslash U$.

So condition (5) becomes,

$$\begin{aligned}\sum_{i=1}^{k} b_i &= k(k-n)+2 \\ &= k(k-n), \text{ for } 1 \le k \le n-1.\end{aligned}$$

For the converse, suppose that conditions (3) and (4) hold. Hence from Corollary 2 there exist an oriented graph D having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$.

Suppose that such an oriented graph is reducible. Then there exist a vertex set W with k vertices ($k < n$), such that every vertex of $V \backslash W$ is adjacent to all the vertices of W. Hence

$$\sum_{i=1}^{k} b_i = k(k-n),$$

a contradiction, proving the converse part.

Corollary 4. *Let D be an oriented graph having imbalance sequence $B = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)$ with $\tilde{b}_1 \ge \tilde{b}_2 \ge \ldots \ge \tilde{b}_n$. Then D is irreducible if and only if*

$$\begin{aligned}\sum_{i=1}^{k} \tilde{b}_i &< k(n-k) \textit{ for } 1 \le k \le n \\ \textit{and} \qquad \sum_{i=1}^{n} \tilde{b}_i &= 0\end{aligned}$$

The next result is an extension of Theorem 3.

Theorem 5. *Let D be an oriented graph having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$. Suppose that*

$$\begin{aligned}\sum_{i=1}^{p} b_i &= p(p-n), \\ \sum_{i=1}^{q} b_i &= q(q-n) \\ \textit{and} \qquad \sum_{i=1}^{k} b_i &> k(k-n), \textit{ for } p+1 \le k \le q-1, \textit{ where } 0 \le p < q \le n.\end{aligned}$$

Then subdigraph induced by the vertices $\{v_{p+1}, v_{p+2}, \ldots, v_q\}$ *is an irreducible component of* D *with imbalance sequence*

$$(b_{p+1} + n - p - q, b_{p+2} + n - p - q, \ldots, b_q + n - p - q).$$

Proof. Suppose imbalance of vertex v_i in oriented graph D is $b_i, 1 \le i \le n$. Since $\sum_{i=1}^{q} b_i = q(q-n)$, so clearly each vertex of $W = \{v_{q+1}, v_{q+2}, \ldots, v_n\}$ dominates all vertices of $\{v_1, v_2, \ldots, v_q\}$. Thus the vertices within W contributes $-(n-q)$ to imbalance of every vertex of $\{v_1, v_2, \ldots, v_q\}$. Also $\sum_{i=1}^{p} b_i = p(p-n)$, so each vertex of $V = \{v_{p+1}, v_{p+2}, \ldots, v_q\}$ dominates all vertices of $U = \{v_1, v_2, \ldots, v_p\}$. So vertices within U contribute p to imbalance of every vertex of V. Hence the imbalance sequence of subdigraph induced by vertices $\{v_{p+1}, v_{p+2}, \ldots, v_q\}$ is

$$\begin{aligned} & (b_{p+1} + n - p - q, b_{p+2} + n - p - q, \ldots, b_q + n - p - q) \\ i.e., \quad & (b_{p+1} + n - p - q, b_{p+2} + n - p - q, \ldots, b_q + n - p - q). \end{aligned}$$

Now we have to show that above imbalance sequence is irreducible. We have

$$\begin{aligned} \sum_{i=1}^{k} b_i &> k(k-n) \\ \Rightarrow \sum_{i=1}^{p} b_i + \sum_{i=p+1}^{k} b_i &> k(k-n) \\ \Rightarrow p(p-n) + \sum_{i=p+1}^{k} b_i + (k-p)(n-p-q) &> k(k-n) + (k-p)(n-p-q) \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum_{i=p+1}^{k} (b_i + n - p - q) &> k(k-n) + (k-p)(n-p-q) - p(p-n) \\ &= k^2 - kp - kq + pq \\ &= (k-p)(k-q). \end{aligned}$$

Thus $\sum\limits_{i=p+1}^{k} (b_i + n - p - q) > (k-p)[(k-p) - (q-p)]$, and

$$\begin{aligned}\sum_{i=p+1}^{q}(b_i+n-p-q) &= \sum_{i=p+1}^{q} b_i+(q-p)(n-p-q)\\ &= \sum_{i=1}^{q} b_i-\sum_{i=p+1}^{k} b_i+(q-p)(n-p-q)\\ &= q(q-n)-p(p-n)+(q-p)(n-p-q)\\ &= 0.\end{aligned}$$

Hence by Theorem 3 the imbalance sequence is irreducible.

Theorem 5 shows that the irreducible components of B are determined by the successive values of k for which

$$\sum_{i=1}^{k} b_i = k(k-n) \text{ for } 1 \leq k \leq n. \tag{6}$$

Taking $B=(-6,-5,-4,1,1,1,6,6)$, equation (6) is satisfied for $k=3,6$ and 8. So the irreducible components of B are $(-1,0,1),(0,0,0)$ and $(0,0)$

5. The bounds of imbalances

The converse of an oriented graph D is an oriented graph D', obtained by reversing orientation of all arcs of D. Let $B=(b_1,b_2,\ldots,b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ be imbalance sequence of an oriented graph D. Then

$$B' = (-b_n, -b_{n-1}, \ldots, b_1).$$

Next result gives lower and upper bounds for the imbalance b_i of a vertex v_i of an oriented graph D.

Theorem 6. *If $B=(b_1,b_2,\ldots,b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of an oriented graph D, then for each i,*

$$i-n \leq b_i \leq i-1.$$

Proof. First, we prove that

$$b_i \geq i-n.$$

Suppose that $b_i < i-n$ then, for every $k<i$

$$b_k \leq b_i < i-n.$$

So that,

$$\sum_{k=1}^{i} b_k < \sum_{k=1}^{i} (i-n)$$
$$\Rightarrow \sum_{k=1}^{i} b_k < i(i-n).$$

As $B = (b_1, b_2, \ldots, b_n)$ is an imbalance sequence so, by Corollary 2,

$$\sum_{k=1}^{i} b_k \geq i(i-n).$$

This is a contradiction. Hence

$$(i-n) \leq b_i. \tag{7}$$

The second inequality is dual to the first. In the converse oriented graph D' with imbalance sequence $B' = (b'_1, b'_2, \ldots, b'_n)$. We have

$$b'_{n-i+1} \geq (n-i+1) - n = 1 - i \text{ (using condition 7)}$$

but $b_i = -b'_{n-i+1}$ so,

$$b_i \leq -(1-i) = i - 1.$$

Proving the result.

6. Lexicographic enumeration of imbalance sequences

Let $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ and $C = (c_1, c_2, \ldots, c_n)$ with $c_1 \leq c_2 \leq \ldots \leq c_n$ be sequences of integers of order n. Then B precedes C if there exist a positive integer $k \leq n$ such that $b_i = c_i$ for each $1 \leq i \leq k-1$ and $b_k < c_k$ ($B = C$ if $b_i = c_i$ for $1 \leq i \leq n$).

We write $B \preceq C$ if B precedes C, and we say that C is a successor of B. If $B \preceq C$ and $C \preceq D$, then $B \preceq D$, where $D = (d_1, d_2, \ldots, d_n)$ with $d_1 \leq d_2 \leq \ldots \leq d_n$. We say that C is an immediate successor of B if there is no D such that $B \preceq D \preceq C$. An enumeration of all sequences of a given order with the property that the immediate successor of any sequence follows it in the list is called a lexicographic enumeration.

Let $B = (b_1, b_2, \ldots, b_m)$ with $b_1 \leq b_2 \leq \ldots \leq b_m$ and $C = (c_1, c_2, \ldots, c_n)$ with $c_1 \leq c_2 \leq \ldots \leq c_n$ are two imbalance sequences of order m and n respectively. Then we define

$$B + C = (b_1 - n, b_2 - n, \ldots, b_m - n, c_1 + m, c_2 + m, \ldots, c_n + m).$$

The plus operation defined above is not commutative but it is associative.

Now we establish some results dealing with imbalance sequences that are tournament analogue to Merajuddin [9].

Theorem 7. *Let* $B_1 = (b_1, b_2, \ldots, b_n)$ *with* $b_1 \le b_2 \le \ldots \le b_n$ *and* $B_2 = (-n, b_1 + 1, b_2 + 1, \ldots, b_n + 1)$*. Then* B_1 *is* m^{th} *imbalance sequence of order* n *if and only if* B_2 *is the* m^{th} *imbalance sequence of order* $(n + 1)$*.*

Proof. Suppose D_1 be a realization of B_1.Then $D_2 = [K, D_1]$, where K is an oriented graph of order 1, is a realization of B_2. This shows that B_2 is an imbalance sequence when B_1 is an imbalance sequence. For converse, suppose D be a realization of B_2. We can write $D = [U, W]$, where U is an oriented graph of order 1, Clearly W is a realization of B_1. This shows that B_1 is an imbalance sequence when B_2 is an imbalance sequence. The unique correspondence shows that both are occupying the same position.

Let $b_k(n)$ denotes the number of imbalance sequences of order n, in nondecreasing order, having imbalance k atleast once, for $1 - n \le k \le n - 1$. Then we have the following results.

Theorem 8.

$$
\begin{aligned}
(i) \qquad & b_k(n) = b_{-k}(n) \\
(ii) \qquad & b_{1-n}(n) = b(n-1) \\
(iii) \qquad & b_{n-1} = b(n-1).
\end{aligned}
$$

Proof. (i) This is equivalent to proving that whenever $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$is an imbalance sequence, then $B' = (-b_n, -b_{n-1}, \ldots, b_1)$ is also an imbalance sequence. This always happens, since B is an imbalance sequence of an oriented graph D if and only if B' is an imbalance sequence of oriented graphD', the converse of D.

(ii) Let $B_1 = (b_1, b_2, \ldots, b_{n-1})$ be the last imbalance, i.e., $b(n-1)^{th}$ imbalance sequence of order $n - 1$. By Theorem 7 $B_2 = (-(n-1), b_1 + 1, b_2 + 1, \ldots, b_{n-1} + 1)$ is the $b(n-1)^{th}$ imbalance sequence of order n. Now we show that there does not exist any imbalance sequence $B_3 = (t_1, t_2, \ldots, t_n), B_3 \ne B_2$ such that $t_1 = -(n-1)$ and $B_2 \preceq B_3$.

Suppose that there exists one such B_3. Then by Theorem 7, $B_4 = (t_2 - 1, \ldots, t_n - 1)$ is an imbalance sequence of order $n - 1$ and $B_1 \preceq B_4$, a contradiction as $B_1$is the last imbalance sequence of order $(n - 1)$. Thus B_2 is the last imbalance sequence of order n in which the first entry is $-(n - 1)$.

Hence $b_{1-n}(n) = b(n - 1)$.

(iii) Putting $k = n-1$ in Theorem 8(i), we get

$b_{n-1} = b_{1-n}$

and from Theorem 8(ii),

$b_{1-n} = b(n-1)$

Hence $b_{n-1} = b(n-1)$.

7. Self-converse imbalance sequences

A score sequence $S = (s_1, s_2, \ldots, s_n)$ is said to be self-converse if all the tournaments T, having the score sequence S are self-converse, *i.e.*, $T \cong T'$. If $S = (s_1, s_2, \ldots, s_n)$ is a score sequence of a tournament T, then S' score sequence of T', is given by

$$S' = (n-1-s_1, n-1-s_2, \ldots, n-1-s_n).$$

In 1979, Eplett[1] characterized the self-converse score sequences.

Theorem 9. *[1] A score sequence $S = (s_1, s_2, \ldots, s_n)$ is self-converse if and only if*

$$s_i + s_{n+1-i} = n-1, \ \textit{for}\ 1 \le i \le n. \tag{8}$$

Let $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$ be an imbalance sequence of an oriented graph D. Then the imbalance sequence B' of the oriented graphD', the converse of D, is given by $(-b_n, -b_{n-1}, \ldots, -b_1)$. An oriented graph D is said to be self converse if $D \cong D'$. An imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$is self-converse if all the oriented graph having imbalance sequence B are self-converse.

Next result characterizes self-converse imbalance sequences of tournaments.

Theorem 10. *A sequence of integers $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$ is self-converse if and only if*

$$b_i + b_{n-i+1} = 0, \ \textit{for}\ 1 \le i \le n.$$

Proof. Consider a tournament T having $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$ and $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \le s_2 \le \ldots \le s_n$ as its imbalance and score sequences. As tournament T is self-converse so by Theorem 9[1], we have

$$\begin{aligned}
& s_i + s_{n-i+1} = n-1 \\
\Rightarrow \quad & d_i^+ + d_{n-i+1}^+ = n-1 \\
\Rightarrow \quad & 2d_i^+ + 2d_{n-i+1}^+ = 2(n-1) \\
\Rightarrow \quad & d_i^+ - \left(n-1-d_i^+\right) + d_{n-i+1}^+ - \left(n-1-d_{n-i+1}^+\right) = 0 \\
\Rightarrow \quad & \left(d_i^+ - d_i^-\right) + \left(d_{n-i+1}^+ - d_{n-i+1}^-\right) = 0 \\
\Rightarrow \quad & b_i + b_{n-i+1} = 0.
\end{aligned}$$

This proves the necessity of Theorem. Converse also follows from Theorem 9.

Now we state a conjecture:

Conjecture. The above result is also true for oriented graphs.

Below we obtain following results on self-converse imbalance sequences of tournaments.

Theorem 11. *If $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of a tournament, then $B + B'$ is a self-converse imbalance sequence.*

Proof. Here $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$is an imbalance sequence. By the definition of converse of imbalance sequence,

$$B' = (-b_n, -b_{n-1}, \ldots, -b_1).$$

So, by the definition, we have

$$\begin{aligned} B + B' &= (b_1 - n, b_2 - n, \ldots, b_n - n, -b_n + n, \ldots, -b_1 + n) \\ &= (t_1, t_2, \ldots, t_{2n}), \text{ say} \end{aligned}$$

where

$$t_i = \begin{cases} b_i - n, & \text{for } 1 \leq i \leq n; \\ -b_{2n-i+1} + n, & \text{for } n+1 \leq i \leq 2n. \end{cases}$$

Clearly $t_i + t_{2n-i+1} = 0$, for $1 \leq i \leq 2n$. Hence $B + B'$ is self-converse.

Theorem 12. *Let $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ be a self-converse imbalance sequence and C be any other imbalance sequence in nondecreasing order. Then $C + B + C'$ is a self-converse imbalance sequence.*

Proof. Suppose that $C = (c_1, c_2, \ldots, c_m)$ with $c_1 \leq c_2 \leq \ldots \leq c_m$ is an imbalance sequence of order m. Then by definition of converse

$$C' = (-c_m, -c_{m-1}, \ldots, -c_1)$$

and by definition

$$\begin{aligned} C + B + C' &= (c_1 - m - n, c_2 - m - n, \ldots, c_m - m - n, b_1, b_2, \ldots, \\ &\quad b_n, -c_m + m + n, -c_{m-1} + m + n, \ldots, -c_1 + m + n) \\ &= (r_1, r_2, \ldots, r_{2m+n}), \text{ say} \end{aligned}$$

where

$$r_i = \begin{cases} c_i - m - n, & 1 \le i \le m; \\ b_i - m, & m+1 \le i \le m+n; \\ -c_{2m+m-i+1} + m + n, & m+n+1 \le i \le 2m+n. \end{cases}$$

Case (i). For $1 \le j \le m$,

$$r_j + r_{2m+n-j+1} = c_j - m - n - c_j + m + n$$

{ when $1 \le j \le m$ then $m+n+1 \le 2m+n-j+1 \le 2m+1$ }
$\Rightarrow r_j + r_{2m+n-j+1} = 0$.

Case (ii). For $m+1 \le j \le m+n$,

$$\begin{aligned} r_j + r_{2m+n-j+1} &= b_{j-m} + b_{m+n-j+1} \\ &= b_k + b_{n-k+1} \text{ for } k = j-m \text{ and } 1 \le k \le n \\ &= 0. \end{aligned}$$

As $B = (b_1, b_2, \ldots, b_n)$ is a self-converse imbalance sequence, so $b_i + b_{n-i+1} = 0$, for $1 \le i \le n$. From above we have, $r_j + r_{2m+n-j+1} = 0$, for $1 \le j \le 2m+n$. Hence $C + B + C'$ is a self-converse imbalance sequence.

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