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$\delta - b-$OPEN SETS AND $\delta - b-$CONTINUOUS FUNCTIONS

Uday Shankar Chakraborty

Abstract. The aim of this paper is to introduce the concept of $\delta - b-$open set together with its corresponding operators $\delta - b-$interior and $\delta - b-$closure. A few relations between these operators and the operators defined before are established. In this paper, the concept of $\delta - b-$continuity has been introduced with the aid of $\delta - b-$open sets. Some basic properties of this mapping have also been studied.

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1. Introduction

Veličko [1] introduced the concept of $\delta-$open sets as a generalization of open sets. After him many others like Raychoudhury and Mukherjee [2], Noiri [3], Hatir and Noiri [4] further generalised the concept and introduced the notion of $\delta-$prepen, $\delta-$semiopen and $\delta - \beta-$open sets. In this paper we have introduced the notions of $\delta - b-$open sets and $\delta - b-$continuity. The class of $\delta - b-$continuous functions contains both the classes of $\delta-$precontinuous and $\delta-$semicontinuous functions and is contained in the class of all $\delta - \beta-$continuous functions. We obtained characterizations of $\delta - b-$continuous functions and analysed some of the basic properties of the function. The relationships between $\delta - b-$continuity and separation axioms have also been investigated.

2. Preliminaries

Throughout the present paper, X and Y are always topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X. The interior of A, the closure of A, the $\delta-$interior of A, the $\delta-$closure, the semi-interior, the semi-closure, the pre-interior and the pre-closure of A are denoted by $\text{int}(A)$, $\text{cl}(A)$, $\text{int}_\delta(A)$, $\text{cl}_\delta(A)$, $\text{sint}(A)$, $\text{scl}(A)$, $\text{pint}(A)$ and $\text{pcl}(A)$ respectively. A subset

A of X is said to be regular open (resp. regular closed) [5] if $A =$int(cl(A)) (resp. $A =$cl(int(A))). The $\delta-$interior [2] of a subset A of X is the union of all regular open sets of X contained in A. A subset A is called $\delta-$open [1] if $A =\text{int}_\delta(A)$, i.e., a set is $\delta-$open if it is the union of regular open sets. The complement of a $\delta-$open set is called $\delta-$closed, alternatively, a subset A of X is called $\delta-$closed [1] if $A =\text{cl}_\delta(A)$, where $\text{cl}_\delta(A) = \{x \in X : A\cap \text{int}(\text{cl}(U))\neq \emptyset, U \text{ is open in } X \text{ and } x \in U\}$.

Definition 1. *A subset A of X is called*
(a) Preopen [6] if $A \subset int(cl(A))$,
(b) Semiopen [7] if $A \subset cl(int(A))$,
(c) $\beta-$*open [8] if* $A \subset cl(int(cl((A))))$,
(d) $\delta-$*preopen [2] if* $A \subset int(cl_\delta(A))$,
(e) $\delta - \beta-$*open [4] if* $A \subset cl(int(cl_\delta(A)))$,
(f) $\delta-$*semiopen [3] if* $A \subset cl(int_\delta(A))$,
(g) $b-$*open [9] if* $A \subset int(cl(A)) \cup cl(int(A))$,
(h) $\delta - b-$*open if* $A \subset int(cl_\delta(A)) \cup cl(int_\delta(A))$

The family of all $\delta - b-$open (resp. $\delta-$preopen, $\delta-$semiopen, $\delta - \beta-$open) sets of X is denoted by $\delta BO(X)$ (resp. $\delta PO(X)$, $\delta SO(X)$, $\delta\beta O(X)$).

Definition 2. *Let A be a subset of a topological space X.*

(a) The complement of a $\delta - b-$*open (resp.* $\delta-$*preopen,* $\delta-$*semiopen,* $\delta - \beta-$*open) set is called* $\delta - b-$*closed (resp.* $\delta-$*preclosed [2],* $\delta-$*semclosed [3],* $\delta - \beta-$*closed [4]).*

(b) The union of all $\delta - b-$*open (resp.* $\delta-$*preopen,* $\delta-$*semiopen,* $\delta - \beta-$*open) sets contained in A is called the* $\delta - b-$*interior (resp.* $\delta-$*preinterior [10],* $\delta-$*seminterior [3],* $\delta - \beta-$*interior [4]) of A and is denoted by* $bint_\delta(A)$ *(resp.* $pint_\delta(A)$, $sint_\delta(A)$, $\beta int_\delta(A)$*).*

(c) The intersection of all $\delta - b-$*closed (resp.* $\delta -$ *preclosed,* $\delta-$*semiclosed,* $\delta - \beta-$*closed) sets containing A is called the* $\delta - b-$*closure (resp.* $\delta-$*preclosure [2],* $\delta-$*semclosure [3],* $\delta - \beta$*-closure [4]) of A and is denoted by* $bcl_\delta(A)$ *(resp.* $pcl_\delta(A)$, $scl_\delta(A)$, $\beta cl_\delta(A)$*).*

Lemma 1. *[4] For a subset A of a topological space X, the following properties hold:*

(a) $pint_\delta(A) = A \cap int(cl_\delta(A))$; $pcl_\delta(A) = A \cup cl(int_\delta(A))$,

(b) $sint_\delta(A) = A \cap cl(int_\delta(A))$; $scl_\delta(A) = A \cup int(cl_\delta(A))$,

(c) $\beta int_\delta(A) = A \cap cl(int(cl_\delta(A)))$; $\beta cl_\delta(A) = A \cup int(cl(int_\delta(A)))$.

3. $\delta - b-$open sets

Theorem 2. *$\delta PO(X) \cup \delta SO(X) \subset \delta BO(X) \subset \delta\beta O(X)$.*

Remark 1. *The inclusions can not be replaced with equalities as shown by the following examples.*

Example 1. *Let $X = \{a, b, c, d, e\}$ and let $\tau = \{X, \emptyset, \{a\}, \{e\}, \{a, e\}, \{c, d\}, \{a, c, d\}, \{c, d, e\}, \{a, c, d, e\}\}$. Then $A = \{a, b, d\}$ is $\delta - b-$open but neither $\delta-$preopen nor $\delta-$semiopen.*

Example 2. *Let $X = \{a, b, c, d\}$ and let $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}$. Then $A = \{a, d\}$ is $\delta - \beta-$open but not $\delta - b-$open.*

Theorem 3. *Let $A \in \delta BO(X)$ such that $int_\delta(A) = \emptyset$, then $A \in \delta PO(X)$.*

Theorem 4. *A subset A of X is $\delta-b-$closed if and only if $cl(int_\delta(A)) \cap int(cl_\delta(A)) \subset A$.*

Theorem 5. *Arbitrary union (intersection) of $\delta - b-$open (resp. $\delta - b-$closed) sets is $\delta - b-$open (resp. $\delta - b-$closed).*

Remark 2. *The intersection of two $\delta - b-$open sets may not be $\delta - b-$open. This can be shown by the following example.*

Example 3. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $A = \{a, c\}$ and $B = \{b, c\}$ are both $\delta - b-$open sets, but $A \cap B = \{c\}$ is not $\delta - b-$open.*

Theorem 6. *The following properties hold for the $\delta - b-$closures of subsets A, B of X.*

(a) A is $\delta - b-$closed in X if and only if $A = bcl_\delta(A)$,

(b) $bcl_\delta(A) \subset bcl_\delta(B)$ whenever $A \subset B \subset X$,

(c) $bcl_\delta(A)$ is $\delta - b-$closed in X,

(d) $bcl_\delta(bcl_\delta(A)) = bcl_\delta(A)$,

(e) $x \in cl_\delta(A)$ if $A \cap U \neq \emptyset$ for every $\delta - b-$open set U containing x.

Theorem 7. *The following properties hold for the $\delta - b-$interiors of subsets A, B of X.*

(a) $bint_\delta(A) \subset bint_\delta(B)$ whenever $A \subset B \subset X$,

(b) $bint_{\delta}(B)$ *is* $b-$*open in* X,

(c) $bint_{\delta}(bint_{\delta}(A)) = bint_{\delta}(A)$.

Theorem 8. *Let* $A, B \subset X$ *be such that* A *is* $\delta - b-$*open and* B *is* $\delta - b-$*closed. Then there exist a* $\delta - b-$*open set* H *and a* $\delta - b-$*closed set* K *such that* $A \cap B \subset K$ *and* $H \subset A \cup B$.

Proof. Let $K = \mathrm{bcl}_{\delta}(A) \cap B$ and $H = A \cup \mathrm{bint}_{\delta}(B)$. Then K is $\delta - b-$closed and H is $\delta - b-$open. Also $A \cap B \subset \mathrm{bcl}_{\delta}(A) \cap B = K$ and $H = A \cup \mathrm{bint}_{\delta}(B) \subset A \cup B$.

Theorem 9. *For a subset* A *of a space* X, *the following are equivalent.*

(a) A *is* $\delta - b-$*open.*

(b) $A = pint_{\delta}(A) \cup sint_{\delta}(A)$.

(c) $A \subset pcl_{\delta}(pint_{\delta}(A))$.

Proof.

(a)$\Rightarrow$ (b): Let A be $\delta - b-$open. Then $A \subset \mathrm{cl}(\mathrm{int}_{\delta}(A)) \cup \mathrm{int}(\mathrm{cl}_{\delta}(A))$. Now $\mathrm{pint}_{\delta}(A) \cup \mathrm{sint}_{\delta}(A) = [A \cap \mathrm{int}(\mathrm{cl}_{\delta}(A))] \cup [A \cap \mathrm{cl}(\mathrm{int}_{\delta}(A))] = A \cap [\mathrm{int}(\mathrm{cl}_{\delta}(A)) \cup \mathrm{cl}(\mathrm{int}_{\delta}(A))] = A$.

(b)$\Rightarrow$ (c): $A = \mathrm{pint}_{\delta}(A) \cup \mathrm{sint}_{\delta}(A) = \mathrm{pint}_{\delta}(A) \cup [A \cap \mathrm{cl}(\mathrm{int}_{\delta}(A))] \subset \mathrm{pint}_{\delta}(A) \cup \mathrm{cl}(\mathrm{int}_{\delta}(A)) = \mathrm{pcl}_{\delta}(\mathrm{pint}_{\delta}(A))$

(c)$\Rightarrow$(a): $A \subset \mathrm{pcl}_{\delta}(\mathrm{pint}_{\delta}(A)) = \mathrm{pint}_{\delta}(A) \cup \mathrm{cl}(\mathrm{int}_{\delta}(A)) \subset (A \cap \mathrm{cl}(\mathrm{int}_{\delta}(A)) \cup \mathrm{int}(\mathrm{cl}_{\delta}(A))$. Therefore, A is $\delta - b-$open.

Lemma 10. *[2] Let* A *be a subset of a space* X. *Then*

(a) $cl_{\delta}(A) \cap G \subset cl_{\delta}(A \cap G)$, *for any* δ-*open set* G *in* X,

(b) $int_{\delta}(A \cup F) \subset int_{\delta}(A) \cup F$, *for any* $\delta-$*closed set* F *in* X.

Theorem 11. *For a subset* A *of a space* X, *the following properties hold.*

(a) $bcl_{\delta}(A) = scl_{\delta}(A) \cap pcl_{\delta}(A)$,

(b) $bint_{\delta}(A) = sint_{\delta}(A) \cup pint_{\delta}(A)$,

(c) $bcl_{\delta}(X \setminus A) = X \setminus bint_{\delta}(A)$,

(d) $x \in bcl_{\delta}(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in \delta BO(X)$ *containing* x.

Proof. (a) Since $\mathrm{bcl}_\delta(A)$ is $\delta-b-$closed, therefore, $\mathrm{int}(\mathrm{cl}_\delta(\mathrm{bcl}_\delta(A)))\cap \mathrm{cl}(\mathrm{int}_\delta(\mathrm{bcl}_\delta(A))) \subset \mathrm{bcl}_\delta(A)$. This implies that $A\cup[\mathrm{int}(\mathrm{cl}_\delta(A))\cap \mathrm{cl}(\mathrm{int}_\delta(A))] \subset \mathrm{bcl}_\delta(A)$.
Thus $\mathrm{scl}_\delta(A)\cap\mathrm{pcl}_\delta(A) \subset \mathrm{bcl}_\delta(A)$.
To prove the reverse inclusion, we have, $\mathrm{scl}_\delta(A)\cap\mathrm{pcl}_\delta(A)$ is a $\delta - b-$closed set containing A. Hence, $\mathrm{bcl}_\delta(A) \subset\mathrm{scl}_\delta(A)\cap\mathrm{pcl}_\delta(A)$.

(b) Since $\mathrm{bint}_\delta(A)$ is $\delta - b-$open, therefore, $\mathrm{int}(\mathrm{cl}_\delta(\mathrm{bint}_\delta(A)))\cup \mathrm{cl}(\mathrm{int}_\delta(\mathrm{bint}_\delta(A))) \supset \mathrm{bint}_\delta(A)$. This implies that $A\cap[\mathrm{int}(\mathrm{cl}_\delta(A))\cap \mathrm{cl}(\mathrm{int}_\delta(A))] \supset \mathrm{bint}_\delta(A)$. Thus $\mathrm{sint}_\delta(A)\cup\mathrm{pint}_\delta(A) \supset \mathrm{bint}_\delta(A)$.
To prove the reverse inclusion, we have, $\mathrm{sint}_\delta(A)\cup\mathrm{pint}_\delta(A)$ is a $\delta - b-$open set containing A. Hence, $\mathrm{sint}_\delta(A)\cup\mathrm{pint}_\delta(A) \subset \mathrm{bint}_\delta(A)$.

(c) We have, $\mathrm{bcl}_\delta(X\backslash A) =\mathrm{scl}_\delta(X\backslash A)\cap \mathrm{pcl}_\delta(X\backslash A) = [X\backslash \mathrm{sint}_\delta(A)]\cap[X\backslash \mathrm{pint}_\delta(A)] = X\backslash[\mathrm{sint}_\delta(A)\cup\mathrm{pint}_\delta(A)] = X\backslash \mathrm{bint}_\delta(A)$

(d) Let $x \in \mathrm{bcl}_\delta(A)$. Thus $x \in\mathrm{cl}_\delta(A)$. Hence, $A \cap U \neq \emptyset$ for every $U \in \delta BO(X)$ containing x.
Conversely, let $A \cap U \neq \emptyset$ for every $U \in \delta BO(X)$ containing x. Let $x \in X\backslash \mathrm{bcl}_\delta(A) =\mathrm{bint}_\delta(X \setminus A)$. Therefore, there exists $U \in \delta BO(X)$ containing x such that $U \subset X \setminus A$. This implies, $A \cap U = \emptyset$. Which is a contradiction. Hence, $x \in \mathrm{bcl}_\delta(A)$.

Theorem 12. *A set A in X is $\delta - b-$open if and only if $U \cap A \in \delta BO(X)$, for every regular open (equivalently $\delta-$open)set U of X.*

Proof. Let $A \in \delta BO(X)$. Therefore, $U \cap A \subset U\cap[\mathrm{int}(\mathrm{cl}_\delta(A))\cup\mathrm{cl}(\mathrm{int}_\delta(A))]$
$\subset[\mathrm{int}(U)\cap\mathrm{int}(\mathrm{cl}_\delta(A))]\cup \mathrm{cl}(U\cap \mathrm{int}_\delta(A)) = \mathrm{int}(U\cap \mathrm{cl}_\delta(A))\cup \mathrm{cl}(\mathrm{int}_\delta(U)\cap \mathrm{int}_\delta(A)) \subset \mathrm{int}(\mathrm{cl}_\delta(U \cap A))\cup \mathrm{cl}(\mathrm{int}_\delta(U \cap A))$. Thus, $U \cap A \in \delta BO(X)$.
Conversely, let $U \cap A \in \delta BO(X)$, for every regular open set U of X. Since, X is regular open, therefore, $X \cap A = A \in \delta BO(X)$.

Definition 3. *A subset A of a space X is called a $\delta - b-$neighbourhood of x in X if there exists $U \in \delta BO(X)$ such that $x \in U \subset A$.*

Theorem 13. *If U is a $\delta-$open subset of a space X and $V \in \delta BO(X)$, then $U \cap V \in \delta BO(U)$.*

Proof. We have, $U \cap V \subset U\cap[\mathrm{int}(\mathrm{cl}_\delta(A))\cup \mathrm{cl}(\mathrm{int}_\delta(A))]$
$= [U\cap\mathrm{int}(\mathrm{cl}_\delta(V))] \cup [U\cap \mathrm{cl}(\mathrm{int}_\delta(V))] \subset \mathrm{int}_U(U\cap\mathrm{cl}_\delta(V)) \cup [U\cap\mathrm{cl}(U\cap\mathrm{int}_\delta(V))] \subset$
$\mathrm{int}_U(U\cap\mathrm{cl}_\delta(U \cap V))\cup \mathrm{cl}_U(U\cap\mathrm{int}_\delta(V)) =\mathrm{int}_U(\mathrm{cl}_{\delta U}(U \cap V))\cup \mathrm{cl}_U(\mathrm{int}_\delta(U \cap V)) \subset$
$\mathrm{int}_U(\mathrm{cl}_{\delta U}(U \cap V))\cup \mathrm{cl}_U(\mathrm{int}_{\delta U}(U \cap V))$. Therefore, $U \cap V \in \delta BO(U)$.

4. $\delta - b-$CONTINUOUS FUNCTION

Definition 4. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\delta - b-$continuous (resp. $\delta-$precontinuous, $\delta-$semicontinuous [3], $\delta - \beta-$continuous [4]) if for each $V \in \sigma$, $f^{-1}(V)$ is $\delta - b-$open (resp. $\delta-$preopen, $\delta-$semiopen, $\delta - \beta-$open) in X.*

Remark 3. *Every $\delta-$precontinuous as well as every $\delta-$semicontinuous function is $\delta - b-$continuous function also every $\delta - b-$continuous function is $\delta - \beta-$continuous. But none of these relations can be reversed as given by the following examples.*

Example 4. *The function $f : (X, \tau) \to (Y, \sigma)$, where (X, τ) is the topological space given in Example 1 and $Y = \{a, b\}, \sigma = \{Y, \emptyset, \{a\}\}$, defined by $f(a) = f(b) = f(d) = a, f(c) = f(e) = b$ is $\delta - b-$continuous but it is neither $\delta-$precontinuous nor $\delta-$semicontinuous.*

Example 5. *The function $f : (X, \tau) \to (Y, \sigma)$, where (X, τ) is the topological space given in Example 3.2. and $Y = \{a, b\}, \sigma = \{Y, \emptyset, \{a\}\}$, defined by $f(a) = f(d) = a, f(c) = f(d) = b$ is $\delta - \beta-$continuous but not $\delta - b-$continuous.*

Definition 5. *[11] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\delta-$continuous if for each $x \in X$ and for each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau$ containing x such that $f(int(cl(U))) \subset int(cl(V))$.*

Remark 4. *$\delta-$continuity and $\delta - b-$continuity are independent of each other as given by the following example.*

Example 6. *Let $X = Y = \{a, b, c\}$ and let $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{Y, \emptyset, \{b, c\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = c, f(b) = b, f(c) = a$ is $\delta-$continuous but not $\delta - b-$continuous whereas the identity function $i : (Y, \sigma) \to (X, \tau)$ defined by $i(x) = x$ for all $x \in Y$ is $\delta - b-$continuous but not $\delta-$continuous.*

Definition 6. *Let A be a subset of a space X. Then $\delta - b$-frontier of A is defined by $bfr_\delta(A) = bcl_\delta(A) \cap bcl_\delta(X \setminus A) = bcl_\delta(A) \setminus bint_\delta(A)$.*

Theorem 14. *The following statements are equivalent for a function $f : X \to Y$.*

(a) f is $\delta - b-$continuous,

(b) For each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \delta BO(X)$ containing x such that $f(U) \subset V$,

(c) For each closed subset W of Y, $f^{-1}(W)$ is $\delta - b$-closed.

(d) For each subset A of X, $f(bcl_\delta(A)) \subset cl(f(A))$.

(e) For each subset B of Y, $bcl_\delta(f^{-1}(B)) \subset f^{-1}(cl(B))$.

(f) For each subset B of Y, $f^{-1}(int(B)) \subset bint_\delta(f^{-1}(B))$.

(g) For each subset B of Y, $bfr_\delta(f^{-1}(B)) \subset f^{-1}(fr(B))$.

Proof. (a)⇔(b) and (a)⇔(c) are straightforward.

(c)⇒(d): For any subset A of X, $f^{-1}(\mathrm{cl}(f(A)))$ is $\delta - b-$closed and contains A. Thus $\mathrm{bcl}_\delta(A) \subset f^{-1}(\mathrm{cl}(f(A)))$, so that $f(\mathrm{bcl}_\delta(A)) \subset f(f^{-1}(\mathrm{cl}(f(A)))) \subset \mathrm{cl}(f(A))$.

(d)⇒ (e): Let B be any subset of Y. Then $f(bcl_\delta(f^{-1}(B))) \subset \mathrm{cl}(f(f^{-1}(B))) \subset \mathrm{cl}(B)$. Hence $bcl_\delta(f^{-1}(B)) \subset f^{-1}(\mathrm{cl}(B))$.

(e)⇒ (c): Let W be a closed subset of Y.
Then $\mathrm{bcl}_\delta(f^{-1}(B)) \subset f^{-1}(\mathrm{cl}(B) = f^{-1}(B)$. Thus $f^{-1}(B)$ is $\delta - b-$closed in X. Hence, f is $\delta - b-$continuous.

(a)⇒ (f): Let B be any subset of Y. Then $f^{-1}(\mathrm{int}(B)) \in \delta BO(X)$.
Thus $f^{-1}(\mathrm{int}(B)) = b\mathrm{int}_\delta(f^{-1}(\mathrm{int}(B))) \subset b\mathrm{int}_\delta(f^{-1}(B)))$.

(f)⇒ (a): Let $V \in \sigma$. Then $f^{-1}(V) = f^{-1}(\mathrm{int}(V)) \subset \mathrm{bint}_\delta(f^{-1}(V)))$. Therefore, $f^{-1}(V) \in \delta BO(X)$. Hence, f is $\delta - b-$continuous.

(d)⇒ (g): Let B be a subset of Y.
We have, $b\mathrm{fr}_\delta(f^{-1}(B)) = b\mathrm{cl}_\delta(f^{-1}(B)) \cap b\mathrm{cl}_\delta(X \setminus f^{-1}(B))$
$\subset f^{-1}(\mathrm{cl}(B)) \cap b\mathrm{cl}_\delta(f^{-1}(Y \setminus B)) \subset f^{-1}(\mathrm{cl}(B)) \cap f^{-1}(\mathrm{cl}(Y \setminus B)) = f^{-1}(\mathrm{fr}(B))$.

(g)⇒(c): Let W be a closed subset of Y. Then $b\mathrm{fr}_\delta(f^{-1}(W)) \subset f^{-1}(\mathrm{fr}(W)) \subset f^{-1}(W)$. Thus $f^{-1}(W)$ is $\delta - b-$closed in X. Hence, f is $\delta - b-$continuous.

Theorem 15. *The set of all points of X at which a function $f : X \to Y$ is not $\delta - b-$continuous is identical with the union of the $\delta - b-$frontiers of the inverse images of the open sets containing $f(x)$.*

Proof. Let $x \in X$ and let f be not $\delta - b-$continuous at x. Therefore, there exists an open set V in Y containing $f(x)$ such that $U \cap (X \setminus f^{-1}(V)) \neq \emptyset$ for every $U \in \delta BO(X)$ containing x. This implies $x \in \mathrm{bcl}_\delta(X \setminus f^{-1}(V))$ and $x \in f^{-1}(V)$. Thus $x \in b\mathrm{fr}_\delta(f^{-1}(V))$.
Conversely, suppose that f is $\delta - b-$continuous at $x \in X$ and let V be an open set containing $f(x)$. Therefore, there exists $U \in \delta BO(X)$ containing x such that $U \subset f^{-1}(V)$. This implies that $x \in \mathrm{bint}_\delta(f^{-1}(V))$ and hence, $x \in X \setminus \mathrm{bfr}_\delta(f^{-1}(V))$.

Remark 5. *The composition of two $\delta - b-$continuous functions may not be $\delta - b-$continuous as shown by the following example.*

Example 7. *Let $X = Y = Z = \{a, b, c\}$ and let $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{Y, \emptyset, \{a\}, \{b, c\}\}$, $\gamma = \{Z, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, c\}\}$. Then the functions $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ defined by $f(a) = g(a) = c, f(b) = g(b) = c, f(c) = g(c) = b$ are $\delta - b-$continuous but their composition $g \circ f$ is not $\delta - b-$continuous.*

Theorem 16. *Let $f : X \to Y$ and $g : Y \to Z$. If $f : X \to Z$ is $\delta - b-$continuous and g is continuous, then $g \circ f$ is $\delta - b-$continuous.*

Proof. Straightforward.

Definition 7. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\delta - b-$open if $f(V) \in \delta BO(Y)$ for each $V \in \delta BO(X)$.*

Theorem 17. *A function $f : X \to Y$ is $\delta - b-$open if and only if $f^{-1}(bcl_\delta(B)) \subset bcl_\delta(f^{-1}(B))$ for each subset B of Y.*

Proof. Let f be $\delta - b-$open and let $x \in f^{-1}(\text{bcl}_\delta(B))$. Let G be a $\delta - b-$open set in X containing x. Therefore, $f(G)$ is $\delta - b-$open in Y containing $f(x)$. As a result $B \cap f(G) \neq \emptyset$ and so $f^{-1}(B) \cap G \neq \emptyset$. Thus $x \in \text{bcl}_\delta(f^{-1}(B))$. Hence, $f^{-1}(\text{bcl}_\delta(B)) \subset \text{bcl}_\delta(f^{-1}(B))$.
Conversely, let $f^{-1}(bcl_\delta(B)) \subset bcl_\delta(f^{-1}(B))$ for each subset B of Y. Let A be $\delta - b-$open in X and let $C = Y \setminus f(A)$. Now, $A \cap f^{-1}(\text{bcl}_\delta(C) \cap f(A)) \subset A \cap f^{-1}(\text{bcl}_\delta(C) \subset A \cap \text{bcl}_\delta(f^{-1}(C)) \subset A \cap (X \setminus A) = \emptyset$. This implies that $f^{-1}(\text{bcl}_\delta(C) \cap f(A)) = \emptyset$. As a result $\text{bcl}_\delta(C) \cap f(A) = \emptyset$ and so $\text{bcl}_\delta(C) \subset C$. Therefore, C is $\delta - b-$closed and hence, $f(A) \in \delta BO(Y)$.

Theorem 18. *Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f : X \to Z$ is $\delta - b-$continuous and f is a $\delta - b-$open surjection, then g is $\delta - b-$continuous.*

Proof. Let V be open in Z. Since $g \circ f$ is $\delta - b-$continuous, therefore, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \in \delta BO(X)$. Since, f is a $\delta - b-$open surjection, therefore, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V) \in \delta BO(Y)$. Hence, g is $\delta - b-$continuous.

Remark 6. *The term "surjection" can not be dropped from the above theorem as shown by the following example.*

Example 8. *Let $X = Y = Z = \{a, b, c\}$ and let $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$, $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\gamma = \{Z, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = a, f(b) = f(c) = b$ is $\delta - b-$open whereas $g : (Y, \sigma) \to (Z, \gamma)$ defined by $g(a) = b, g(b) = c, g(c) = a$ is not $\delta - b-$continuous. But $g_o f$ is $\delta - b-$continuous.*

Definition 8 and Definition 9 can be given as in [12, 13].

Definition 8. *A net $\{x_\lambda : \lambda \in \Lambda\}$ in a topological space X is said to $\delta - b-$converge to $x \in X$ if for every $\delta - b$-neighbourhood U of x, there is some $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ when $\lambda \geq \lambda_0$.*

Theorem 19. *A function $f : (X, \tau) \to (Y, \sigma)$ is $\delta - b-$continuous at $x \in X$ if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which $\delta - b-$converges to x, the net $\{f(x_\lambda) : \lambda \in \Lambda\}$ converges to $f(x)$.*

Proof. Let f be $\delta - b-$continuous at x and let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X such that it $\delta - b-$converges to x. Let $V \in \sigma$ contain $f(x)$. Therefore, there exists a $\delta - b-$open set U in X containing x such that $f(U) \subset V$. Now, $\{x_\lambda : \lambda \in \Lambda\}$ $\delta - b$-converges to x implies that there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$. This implies that $f(x_\lambda) \in f(U) \subset V$ for all $\lambda \geq \lambda_0$. Thus $\{f(x_\lambda) : \lambda \in \Lambda\}$ converges to $f(x)$.

Conversely, let f be not $\delta - b-$continuous at $x \in X$. Therefore, there exists an open neighbourhood V of $f(x)$ such that $f(U)$ is not a subset of V for every $U \in \delta BO(X)$ containing x. Thus for every $\delta - b-$open neighbourhood U of x we can find $x_U \in U$ such that $f(x_U) \notin V$. Let $N(x)$ be the set of all $\delta - b$-neighbourhoods U of x in X. The set $N(x)$ with the relation $U_1 \leqslant U_2$ if and only if $U_2 \subset U_1$, form a directed set. Therefore, the net $\{x_U : U \in N(x)\}$ $\delta - b$-converges to x but $\{f(x_U) : U \in N(x)\}$ does not converge to $f(x)$ in Y. Which is a contradiction. Hence f is $\delta - b-$continuous at $x \in X$.

Definition 9. *A net $\{f_\alpha : \alpha \in \Delta\}$ in $\delta BO(X, Y)$ is said to $\delta - b-$continuously converge to $f \in \delta BO(X, Y)$ if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which $\delta - b$-converges to $x \in X$, the net $\{f_\alpha(x_\lambda) : (\alpha, \lambda) \in \Delta \times \Lambda\}$ converges to $f(x)$ in Y, where $\delta BO(X, Y)$ denotes the set of all $\delta - b-$continuous functions of X to Y.*

Theorem 20. *A net $\{f_\alpha : \alpha \in \Delta\}$ in $\delta BO(X, Y)$ $\delta - b-$continuously converges to $f \in \delta BO(X, Y)$ if and only if for every $x \in X$ and for every open neighbourhood V of $f(x)$ in Y, there exists an element $\alpha_0 \in \Delta$ and a $\delta - b-$open neighbourhood U of x in X such that $f_\alpha(U) \subset V$ for every $\alpha \geq \alpha_0, \alpha \in \Delta$.*

Proof. Let $x \in X$ and let V be an open neighbourhood of $f(x)$ in Y such that for every $\alpha \in \Delta$ and for every $\delta - b-$open neighbourhood U of $x \in X$, there exists $\alpha' \geq \alpha, \alpha \in \Delta$ such that $f_{\alpha'}(U)$ is not a subset of V. Then for every $\delta - b-$open neighbourhood U of x in X we can choose a point $x_U \in U$ such that $f_{\alpha'}(x_U) \notin V$. Therefore, the net $\{x_U : U \in N(x)\}$ $\delta - b$-converges to x, but the net $\{f_\alpha(x_U) : (\alpha, U) \in \Delta \times N\}$ does not converge to $f(x)$ in Y.

Conversely, let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in $\delta BO(X, Y)$ which $\delta - b-$converges to x in X and an element $\alpha_0 \in \Lambda$ such that $f_\alpha(U) \subset V$ for all $\alpha \geq \alpha_0, \alpha \in \Delta$. Since the net $\{x_\lambda : \lambda \in \Lambda\}$ $\delta - b$-converges to x in X, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \in \Lambda, \lambda \geq \lambda_0$. Let $(\lambda_0, \alpha_0) \in \Lambda \times \Delta$. Then for every $(\lambda, \alpha) \in \Lambda \times \Delta$, $\lambda \geq \lambda_0, \alpha \geq \alpha_0$, we have, $f_\alpha(x_\lambda) \in f_\alpha(U) \subset V$. Thus the net $\{f_\alpha(x_\lambda) : (\alpha, \lambda) \in \Delta \times \Lambda\}$ converges to $f(x)$ in Y.

Theorem 21. *If $f : X \to Y$ is $\delta - b-$continuous and U is $\delta-$open in X, then $f|_U : U \to Y$ is $\delta - b-$continuous.*

Proof. Let V be an open subset of Y. Since f is $\delta - b-$continuous, $f^{-1}(V) \in \delta BO(X)$. Now $(f|U)^{-1}(V) = f^{-1}(V) \cap U \in \delta BO(U)$. Hence $f|_U : U \to Y$ is $\delta - b-$continuous.

Let $\{X_\alpha : \alpha \in \Lambda\}$ and $\{Y_\alpha : \alpha \in \Lambda\}$ be any two families of spaces with the same index set Λ. Let$f_\alpha : X_\alpha \to Y_\alpha$ be a function for each $\alpha \in \Lambda$. The product space $\Pi\{X_\alpha : \alpha \in \Lambda\}$ will be denoted by ΠX_α and $f : \Pi X_\alpha \to \Pi Y_\alpha$ denotes the product function defined by $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\} \in \Pi X_\alpha$.

Theorem 22. *If a function $f : X \to \Pi Y_\alpha$ is $\delta - b-$continuous, then $p_\alpha \circ f : X \to Y_\alpha$ is $\delta - b-$continuous for each $\alpha \in \Lambda$, where p_α is the projection of ΠY_α onto Y_α.*

Proof. Let V_α be an open set in Y_α. Since p_α is continuous, therefore, $p_\alpha^{-1}(V_\alpha)$ is open in ΠY_α and hence, $f^{-1}(p_\alpha^{-1}(V_\alpha)) = (p_\alpha \circ f)^{-1}(V_\alpha) \in \delta BO(X)$. This shows that $p_\alpha \circ f$ is $\delta - b-$continuous for each $\alpha \in \Lambda$.

Definition 10. *A topological space X is said to be $\delta-$Hausdorff if for any $x, y(\neq x) \in X$, there exist disjoint $\delta-$open sets G, H such that $x \in G$ and $y \in H$.*

Lemma 23. *If $U \in \delta BO(X)$ and V is $\delta-$open in Y, then $U \times V \in \delta BO(X \times Y)$.*

Theorem 24. *If $f : X \to Y$ is $\delta - b-$continuous, $g : X \to Y$ is $\delta-$continuous and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is $\delta - b-$closed in X.*

Proof. Let $A = \{x \in X : f(x) = g(x)\}$ and let $x \in X \setminus A$. Thus $f(x) \neq g(x)$. Since, Y is Hausdorff, therefore, there exists open sets G and H in Y such that $f(x) \in G, g(x) \in H$ and $G \cap H = \emptyset$. This implies $G \cap \text{int}(\text{cl}(H)) = \emptyset$. Since, f is $\delta - b-$continuous, there exists $U \in \delta BO(X)$ containing x such that $f(U) \subset G$. Since, g is $\delta-$continuous, there exists an open set V in X containing x such that $f(\text{int}(\text{cl}(V))) \subset \text{int}(\text{cl}(H))$. Let $W = U \cap \text{int}(\text{cl}(V))$. Now, $W \in \delta BO(X)$ and $f(W) \cap g(W) \subset f(U) \cap g(\text{int}(\text{cl}(V))) \subset G \cap \text{int}(\text{cl}(H)) = \emptyset$. This implies $W \cap A = \emptyset$. Thus $x \in X \setminus b\text{cl}_\delta(A)$. Hence, A is $\delta - b-$closed.

Theorem 25. *If $f : X \to Y$ is $\delta - b-$continuous and Y is a $\delta-$Hausdorff space, then the graph $G_f = \{(x, f(x)) : x \in X\}$ is $\delta - b-$closed.*

Proof. Let $(x, y) \notin G_f$. Then $f(x) \neq y$. Since, Y is a $\delta-$Hausdorff space, therefore, there exist disjoint $\delta-$open sets G and H such that $f(x) \in G$ and $y \in H$. Since f is $\delta - b-$continuous, there is a $\delta - b-$open set U containing x such that $f(U) \subset G$. Thus $(x, y) \in U \times H \subset X \times Y \setminus G_f$. As a result of which $X \times Y \setminus G_f \in \delta BO(X \times Y)$ since, by Lemma 22, $U \times H \in \delta BO(X \times Y)$. Hence, G_f is $\delta - b-$closed.

Definition 11. *A space X is called $\delta - b-$connected if X is not the union of two disjoint non-empty $\delta - b-$open sets.*

Theorem 26. *Let $f : X \to Y$ be a $\delta - b-$surjection. If X is $\delta - b-$connected, then Y is connected.*

Proof. Suppose that Y is not connected. Therefore, there exist disjoint open sets G and H such that $Y = G \cup H$. Since, f is $\delta - b-$continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are $\delta - b-$open in X. On the other hand, $f^{-1}(G)$ and $f^{-1}(H)$ are non-empty disjoint sets and $X = f^{-1}(G) \cup f^{-1}(H)$. This shows that X is not $\delta - b-$connected which is a contradiction.

Definition 12. *A space X is said to be*

(a) $\delta - b - T_1$ if for each pair of distinct points x and y in X, there exist $\delta - b-$open sets G and H containing x and y, respectively, such that $y \notin G$ and $x \notin H$.

(b) $\delta - b-$Hausdorff if for each pair of distinct points x and y in X, there exist disjoint $\delta - b-$open sets G and H containing x and y, respectively.

Theorem 27. *The following properties hold for a $\delta - b-$continuous injection $f : X \to Y$.*

(a) If Y is a Hausdorff space, then X $\delta - b-$Hausdorff.

(b) If Y is a T_1-space, then X is a $\delta - b - T_1$ space.

Proof. (a) Let $x, y(\neq x) \in X$. Since, f is injective, therefore, $f(x) \neq f(y)$ in Y. Since, Y is Hausdorff, therefore, there exist disjoint open sets G and H such that $f(x) \in G$ and $f(y) \in H$. This implies that $f^{-1}(G), f^{-1}(H)$ are disjoint $\delta - b-$open sets in X containing x, y respectively. Hence, X is $\delta - b-$Hausdorff.

(b) Let $x, y(\neq x) \in X$. Since, f is injective, therefore, $f(x) \neq f(y)$ in Y. Since, Y is T_1, therefore, there exist open sets G and H containing x and y respectively such that $f(x) \notin G$ and $f(y) \notin H$. This implies that $f^{-1}(G), f^{-1}(H)$ are $\delta - b-$open sets in X containing x, y respectively such that $x \notin f^{-1}(G)$ and $y \notin f^{-1}(H)$. Hence, X is a $\delta - b - T_1$ space.

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Uday Shankar Chakraborty
Department of Mathematics,
Albert Einstein School of Physical Science,
Assam University, Silchar
Assam, India
email: *udayhkd@gmail.com, uday.shankar.chakraborty@aus.ac.in*