ORDERING OF SOME SUBSETS FROM IR^d USED IN THE MULTIDIMENSIONAL INTERPOLATION THEORY

by
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Abstract: The ordering of the elements of a subset of IN^d, $d \geq 1$, with the order relations defined at the beginning of this article and implicitly obtaining some expressions for them, corresponding to the defined order relations, is useful for the multidimensional interpolation theory, especially when it is necessary to exemplify some general theoretical notions. This is the reason for presenting expressions for ordered subsets from IN^2, IN^3 and in general from IN^d.

Next we present the most usual order relations encountered in the interpolation theory and used throughout this article.

1. We say that $\boldsymbol{i}=(i_1,i_2,...,i_d)\in IN^d$ is in the relation "«" with $\boldsymbol{j}=(j_1,j_2,...,j_d)\in IN^d$ and we write $\boldsymbol{i} \ll \boldsymbol{j}$, when $i_1 < j_1$, or $i_1 = j_1$ and $i_2 < j_2$,…, or $i_1 = j_1,\ldots,i_{d-1} = j_{d-1}$ and $i_d \leq j_d$.

2. We say that $\boldsymbol{i}=(i_1,i_2,...,i_d)\in IN^d$ is in the relation "$\prec$" with $\boldsymbol{j}=(j_1,j_2,...,j_d)\in IN^d$ and we write $\boldsymbol{i} \prec \boldsymbol{j}$, when $|\boldsymbol{i}|<|\boldsymbol{j}|$ or $|\boldsymbol{i}|=|\boldsymbol{j}|$ and $\boldsymbol{i} \ll \boldsymbol{j}$.

Let be the set $S=\{\boldsymbol{i}=(i_1,i_2)/0\leq i_1\leq n, 0\leq i_2\leq m_j,\ j=\overline{0,n})\}$ from IN^2, denoted by $S_{n;m_0,...,m_n}$, where n and $0\leq m_0\leq m_1\ldots\leq m_n$ are given natural numbers.

It is obvious that we would rather write out the set $(S_{n;m_0,...,m_n}, \ll)$ than the set $(S_{n;m_0,...,m_n}, \prec)$ whose last terms are difficult to be written. Thus we have:

$$(S_{n;m_0,...,m_n}, \ll)=\bigcup_{j=0}^{n}\bigcup_{k=0}^{m_j}\{(j,k)\}=\bigcup_{j=0}^{n}\{(j,0),(j,1),...,(j,m_j)\}=$$

$$=\{(0,0),(0,1),...,(0,m_0),(1,0),(1,1),...,(1,m_1),...,(n,0),(n,1),...,(n,m_n)\} \tag{1}$$

If in the set $S_{n;m_0,\dots,m_n}$ we take $m_0 = n$, $m_j = n - j$, $j = \overline{1,n}$, and respectively $m_0 = m_1 = \dots = m_n = n_2$, $n = n_1$, we obtain some of the most usual forms of some subsets from $S \subset IN^2$, that is the triangle T_n^2, respectively the rectangle $R_{\boldsymbol{n}}^2$, where $\boldsymbol{n} = (n_1, n_2) \in IN_0^2$. More concrete, these sets are

$$T_n^2 = \left\{ \boldsymbol{i} \in IN_0^2 \ / \ |\boldsymbol{i}| \le n \right\},$$

$$R_{\boldsymbol{n}}^2 = R_{n_1,n_2}^2 = \left\{ \boldsymbol{i} \in IN_0^2 \ / \ 0 \le i_j \le n_j \ , \ j = \overline{1,2} \right\}$$

and they are depicted in Figure 1 *a* and 1 *b*.

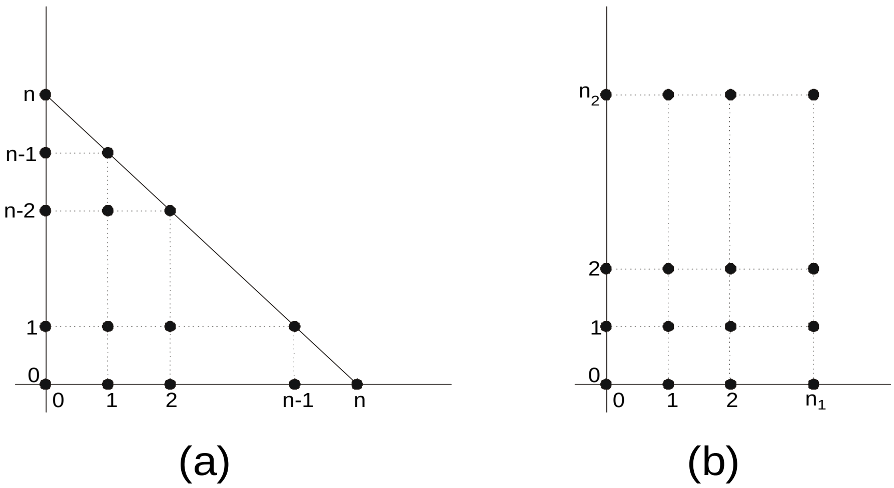

Figure 1

It is quite obvious that the grid of points from Figure 1 *a* is made out of the parallel lines

$$\Delta_k^2 = \bigcup_{j=0}^{k} \{(j, k-j)\}, k = \overline{0,n}.$$

It follows the next explicit expression of triangle T_n^2 ordered by relation "$\prec$"

$$(T_n^2, \prec) = \bigcup_{k=0}^{n} \bigcup_{j=0}^{k} \{(j, k-j)\} = \bigcup_{k=0}^{n} \{(0,k), (1,k-1), \dots, (k,0)\}$$

$$=\{(0,0),(0,1),(1,0),...,(0,k),(1,k-1),...,(k,0),...,(0,n),(1,n-1),...,(n,0)\}. \tag{2}$$

Moreover, if we consider the relation (1) and the fact that $T_n^2 = S_{n;n,n-1,...,0}$, we have:

$$(T_n^2,\text{«}) = \bigcup_{j=0}^{n}\bigcup_{k=0}^{n-j}\{(j,k)\} = \bigcup_{j=0}^{n}\{(j,0),(j,1),...,(j,n-j)\} =$$

$$=\{(0,0),(0,1),...,(0,n),(1,0),(1,1),...,(1,n-1),...,(n-1,0),(n-1,1),...,(n,0)\}. \tag{3}$$

From the same relation (1) and the fact that $T_n^2 = S_{n_1;n_2,n_2,...,n_2}$, we obtain

$$(R_{\boldsymbol{n}}^2,\text{«}) = \bigcup_{j=0}^{n_1}\{(j,0),(j,1),...,(j,n_2)\} = \bigcup_{j=0}^{n_1}\bigcup_{k=0}^{n_2}\{(j,k)\} =$$

$$=\{(0,0),(0,1),...,(0,n_2),(1,0),(1,1),...,(1,n_2),...,(n_1,0),(n_1,1),...,(n_1,n_2)\}. \tag{4}$$

An explicit expression of $(R_{\boldsymbol{n}}^2,\prec)$ is more difficult because it depends on the position of the integers n_1 and n_2 with respect to each other. But for $n_1 = n_2$ we have:

$$(R_{n,n}^2,\prec) = (T_n^2,\prec)\bigcup\{(1,n),(2,n-1),...,(n,1),(2,n),(3,n-1),...,(n,2),...,(n,n)\}.$$

We keep also in mind the following expressions of the ordered subsets from IN^2:

$$(T_n^2 \setminus T_k^2,\prec) = \bigcup_{j=k+1}^{n}\{(0,j).(1,j-1),...,(j,0)\} = \bigcup_{j=k+1}^{n}\bigcup_{i=0}^{j}\{(i,j-i)\} =$$

$$=$$

$$\{(0,k+1),(1,k),...,(k+1,0),(0,k+2),(1,k+1),...,(k+2,0),...,(0,n),(1,n-1),...,(n,0)\}$$

,

$(R_{n,n}^2 \setminus T_n^2,\prec) = ,\{(1,n),(2,n-1),...,(n,1)(2,n),(3,n-1),...,(n,2),...,(n,n)\}$ and

$(R_{n,n}^2 \setminus T_n^2,\text{«}) =$

$\{(1,n),(2,n-1),(2,n),(3,n-2),(3,n-1),(3,n),...,(n,0),(n,1),...,(n,n)\}.$

In general, extensions of the two dimensional case are used as examples in IN^d, $d \geq 3$, among which we exemplify the simplex (Figure 2, in IN_0^3)

$$T_n^d = \left\{ \boldsymbol{i} \in IN_0^d \;/\; |\boldsymbol{i}| \leq n \right\},$$

respectively the hyper-parallelipiped (Figure 4, in IN_0^3)

$$R_{\boldsymbol{n}}^d = R_{n_1,n_2,\ldots,n_d}^d = \left\{ \boldsymbol{i} \in IN_0^d \;/\; 0 \leq i_j \leq n_j \,,\; j = \overline{1,d} \right\}.$$

Proposition: If S is an arbitrary set from IN^d, and $n = \max_{\boldsymbol{i} \in S} |\boldsymbol{i}|$, then $S \subset T_n^d$.

Proof: For any $\boldsymbol{i} \in S$, with $|\boldsymbol{i}| \leq \max_{\boldsymbol{i} \in S} |\boldsymbol{i}| = n$, we have $\boldsymbol{i} \in T_n^d$.

Next we will build successively an expression for the simplex $(T_n^d, \prec)$.

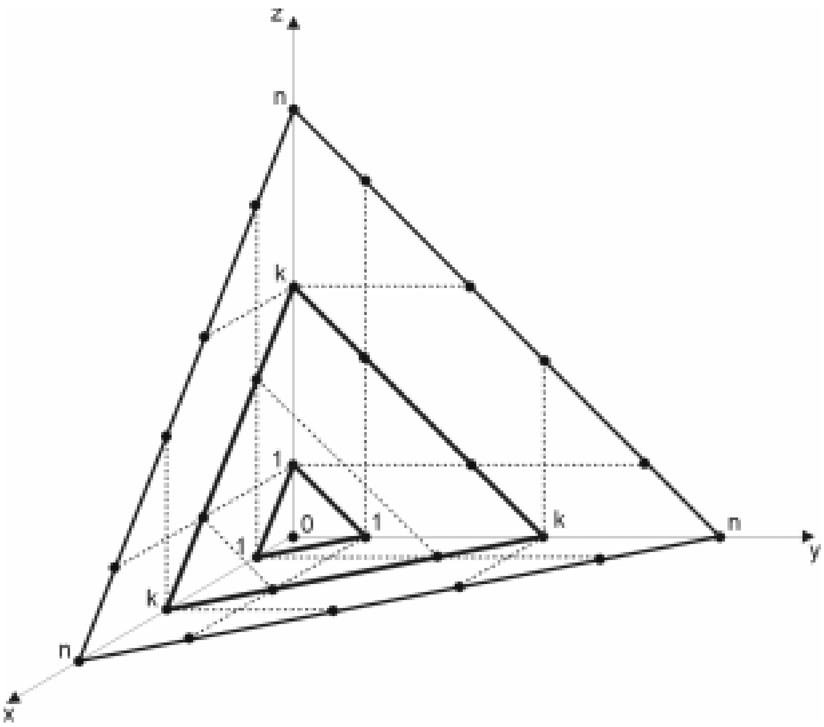

Figure 2

The grid of points from the zero x-coordinate plane, Figure 2 (plane yOz), is made out of the parallel lines

$$\Delta_k^2 = \bigcup_{j=0}^{k} \{(j, k-j)\}, k = \overline{0,n},$$

their traversal order being ($\Delta_k^2, \prec$)=(Δ_k^2,«), and of each of them, from left to right, in descendant order. It follows that Δ_k^2 given in the previous relation is ($\Delta_k^2, \prec$). As each element of the set Δ_k^2 has the sum of indexes k-constant it follows that ($\Delta_k^2, \prec$)=(Δ_k^2,«). From the same relation we have that

$$\Delta_{k-i}^2 = \bigcup_{j=0}^{k-i} \{(j, k-i-j)\}, \ i = \overline{0,k}.$$

If we translate every line Δ_{k-i}^2, $i = \overline{1,k}$ in the $i = \overline{1,k}$ x-coordinate plane, obviously parallel with the initial yOz plane, we obtain the ordered triangle Δ_k^3 from IN_0^3, that is:

$$(\Delta_k^3, \prec) = \bigcup_{i=0}^{k} \Delta_{i,k-i}^2 = \bigcup_{i=0}^{k} \bigcup_{j=0}^{k-i} \{(i, j, k-i-j)\} =$$

$$= \bigcup_{i=0}^{k} \{(i,0,k-i),(i,1,k-i-1),\ldots,(i,0,k-i),(i,k-i,0)\} =$$

$$= \{(0,0,k),(0,1,k-1),\ldots,(0,k,0),(1,0,k-1),(1,1,k-2),\ldots,(1,k-1,0),\ldots,(k,0,0)\},$$

depicted in Figure 2, whose vertices are the points of coordinates $(k,0,0),(0,k,0)$ and $(0,0,k)$. To denote the plane of which Δ_k^2 is a part of after translation, we added the i index to the line Δ_k^2 thus obtaining Δ_{k-i}^2. Actually i becomes the x-coordinate for all points from the translated i x-coordinate plane. The traversal order of the lines $\Delta_{i,k-i}^2$ is $\Delta_{0,k}^2$, $\Delta_{1,k-1}^2$,… , $\Delta_{k,0}^2$. It follows that:

$$(T_n^3, \prec) = \bigcup_{k=0}^{n} \Delta_k^3 = \bigcup_{k=0}^{n} \bigcup_{i=0}^{k} \bigcup_{j=0}^{k-i} \{(i, j, k-i-j)\} \qquad (5)$$

which is the same to $(T_n^3, \prec)$, taking into consideration the traversal order of the triangles Δ_k^3, $k = \overline{0,n}$.

An explicit form of the ordered set $(T_n^3, \prec)$ is:

$$(T_n^3, \prec) =$$
$$\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,0,2),(0,1,1),(0,2,0),(1,0,1),(1,1,0),(2,0,0),...,$$
$$(0,0,n),(0,1,n-1),...,(0,n,0),\ (1,0,n-1),(1,1,n-2),...,(1,n-1,0),...,(n,0,0))\}.$$

Proceeding similarly to the previous situation, we obtain

$$(\Delta_{k-t}^3, \prec) = \bigcup_{i=0}^{k-t} \Delta_{i,k-t-i}^2 = \bigcup_{i=0}^{k-t} \bigcup_{j=0}^{k-t-i} \{(i, j, k-t-i-j)\},$$

respectively

$$(\Delta_{t,k-t}^3, \prec) = \bigcup_{i=0}^{k-t} \bigcup_{j=0}^{k-t-i} \{(t, i, j, k-t-i-j)\},$$

from where we obtain that

$$\Delta_k^4 = \bigcup_{t=0}^{k} \Delta_{t,k-t}^3 = \bigcup_{t=0}^{k} \bigcup_{i=0}^{k-t} \bigcup_{j=0}^{k-t-i} \{(t, i, j, k-t-i-j)\},$$

respectively

$$(T_n^4, \prec) = \bigcup_{k=0}^{n} \Delta_k^4 = \bigcup_{k=0}^{n} \bigcup_{t=0}^{k} \bigcup_{i=0}^{k-t} \bigcup_{j=0}^{k-t-i} \{(t, i, j, k-t-i-j)\}.$$

Now the generalization to the d-dimensional space becomes obvious, namely:

$$\Delta_k^d = \bigcup_{i_1=0}^{k} \bigcup_{i_2=0}^{k-i_1} \bigcup_{i_3=0}^{k-(i_1+i_2)} \cdots \bigcup_{i_{d-1}=0}^{k-(i_1+...+i_{d-2})} \{(i_1, i_2, ..., i_{d-1}, k-(i_1+i_2+...+i_{d-1})\}, \tag{6}$$

respectively

$$(T_n^d, \prec) = \bigcup_{k=0}^{n} \Delta_k^d = \bigcup_{k=0}^{n} \bigcup_{i_1=0}^{k} \bigcup_{i_2=0}^{k-i_1} \bigcup_{i_3=0}^{k-(i_1+i_2)} \cdots \bigcup_{i_{d-1}=0}^{k-(i_1+...+i_{d-2})} \{(i_1, i_2, ..., i_{d-1}, k-(i_1+i_2+...+i_{d-1})\} \tag{7}$$

If in addition we consider also the results of the previous proposition, then we obtain the following theorem:

Theorem: An arbitrary set $(S,\prec)$ from IN^d can be written in the form

$$(S,\prec)=\bigcup_{t=1}^{n}\Delta'^{d}_{k_t}\ ,\tag{8}$$

where $k_n=\max_{\boldsymbol{i}\in S}|\boldsymbol{i}|$, $\Delta'^{d}_{k_t}\subset\Delta^{d}_{k}$ are given by (6).

Example: $S=\{(0,0),(0,2),(1,1),(1,2)\}=\{(0,0)\}\cup\{(0,2),(1,1)\}\cup\{(1,2)\}=$

$=\Delta'^{2}_{k_1}\cup\Delta'^{2}_{k_2}\cup\Delta'^{2}_{k_3}=\bigcup_{t=1}^{3}\Delta'^{2}_{k\,t}$, where $\Delta'^{2}_{k_1}=\Delta^{2}_{0}$, $\Delta'^{2}_{k_2}=\{(0,2),(1,1)\}\subset\Delta^{2}_{2}$, and $\Delta'^{2}_{k_3}=\{(1,2)\}\subset\Delta^{2}_{3}$.

In what follows, we will build in successive steps another expression for the simplex T_n^d using partially the relations "$\prec$" and "«". Eventually we obtain a total order relation defined on T_n^d, which we denote by "$\angle$".

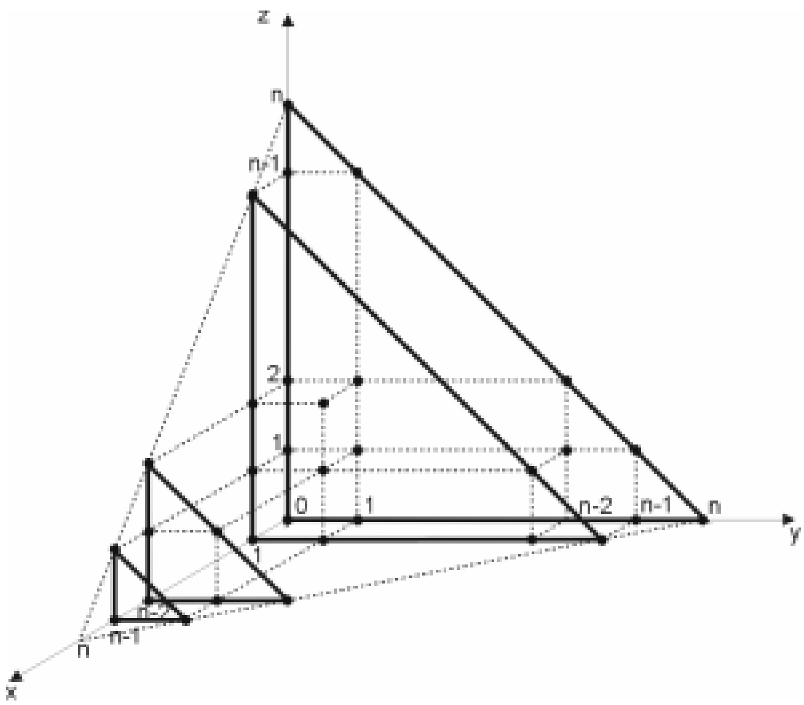

Figure 3

The grid of points from the zero x-coordinate plane, Figure 3 (plane yOz), according to relation (.3), is the triangle $(T_n^2, \prec) = \bigcup_{k=0}^{n} \bigcup_{j=0}^{k} \{(j, k-j)\}$, made out of the parallel lines $\Delta_k^2 = \bigcup_{j=0}^{k} \{(j, k-j)\}, k = \overline{0,n}$, their traversal order being Δ_0^2, Δ_1^2, … , Δ_n^2, and of each of them, from left to right, in descending order. It follows that

$$(T_{n-i}^2, \prec) = \bigcup_{k=0}^{n-i} \bigcup_{j=0}^{k} \{(j, k-j)\}, i = \overline{0,n}.$$

If we translate each triangle $(T_{n-i}^2, \prec)$, with $i = \overline{1,n}$, in the $i = \overline{1,n}$ x-coordinate plane, obviously parallel to the initial plane of the triangle, it results the ordered tetrahedron

$$(T_n^3, \angle) = \bigcup_{i=0}^{n} T_{i,n-i}^2$$

from Figure 3., which is different both from $(T_n^3, \prec)$ and $(T_n^3, «)$. We added the index i to the triangle T_{n-i}^2 to indicate the x-coordinate of the plane of which T_{n-i}^2 is part of after translation. In fact i becomes x-coordinate for all points from the i x-coordinate translated plane. The traversal order of the triangles $T_{i,n-i}^2$ is $T_{0,n}^2$, $T_{1,n-1}^2$, … , $T_{n,0}^2$. It follows that:

$$(T_n^3, \angle) = \bigcup_{i=0}^{n} \bigcup_{k=0}^{n-i} \bigcup_{j=0}^{k} \{(i, j, k-j)\}.$$

An explicit form of the ordered set $(T_n^3, \angle)$ is:

$(T_n^3, \angle)=$
$\{(0,0,0), (0,0,1), (0,1,0), (0,0,2), (0,1,1), (0,2,0),..., (0,0,n), (0,1,n-1),..., (0,n,0),$
$(1,0,0), (1,0,1), (1,1,0), (1,0,2), (1,1,1), (1,2,0),..., (1,0,n-1), (1,1,n-2),...,$
$..., (1,n-1,0,..., (n,0,0))\}$

Continuing the used procedure and generalizing we can obtain rather easily that in d-dimensional space

$$(T_n^d, \angle)=\bigcup_{i_1=0}^{n}\bigcup_{i_2=0}^{n-i_1}\bigcup_{i_3=0}^{n-(i_1+i_2)}\cdots\bigcup_{i_{d-2}=0}^{n-(i_1+\ldots+i_{d-3})}\bigcup_{k=0}^{n-(i_1+\ldots+i_{d-2})}\bigcup_{i_{d-1}=0}^{k}\{(i_1,i_2,\ldots,i_{d-1},k-i_{d-1})\}. \quad (9)$$

An explicit expression for the simplex $(T_n^d, «)$ can be obtained starting from the triangle $(T_n^2, «)=\bigcup_{j=0}^{n}\bigcup_{k=0}^{n-j}\{(j,k)\}$ given by relation (3), lying in the plane yOz from Figure 3. The elements of the triangle $(T_n^2, «)$ are the grid of points lying in the mentioned plane, whose explicit form is obtained if we traverse T_n^2 from bottom to top and from left to right. We can also write that

$$(T_{n-i}^2, «)=\bigcup_{j=0}^{n-i}\bigcup_{k=0}^{n-i-j}\{(j,k)\}.$$

Translating each triangle $(T_{n-i}^2, \prec)$, $i=\overline{1,n}$ in the $i=\overline{1,n}$ x-coordinate plane, we obtain the ordered tetrahedron

$$(T_n^3, «)=\bigcup_{i=0}^{n}T_{i,n-i}^2 .$$

The i index added to T_{n-i}^2, indicates the plane of which T_{n-i}^2 is part of after translation and becomes the x-coordinate for all points from this plane. The traversal order of triangles $T_{i,n-i}^2$ is also $T_{0,n}^2$, $T_{1,n-1}^2, \ldots, T_{n,0}^2$. It follows that:

$$(T_n^3, «)=\bigcup_{i=0}^{n}\bigcup_{j=0}^{n-i}\bigcup_{k=0}^{n-i-j}\{(i,j,k)\}$$

Eventually, for the d-dimensional space we obtain that:

$$(T_n^d, «)=\bigcup_{i_1=0}^{n}\bigcup_{i_2=0}^{n-i_1}\bigcup_{i_3=0}^{n-(i_1+i_2)}\cdots\bigcup_{i_{d-1}=0}^{n-(i_1+\ldots+i_{d-2})}\bigcup_{i_d=0}^{n-(i_1+\ldots+i_{d-1})}\{(i_1,i_2,\ldots,i_{d-1},i_d)\} \quad (10)$$

Next we build an expression for the hyper-parallelipiped $\boldsymbol{R_n^d}$ from the d-dimensional space using the total order relation “«” .

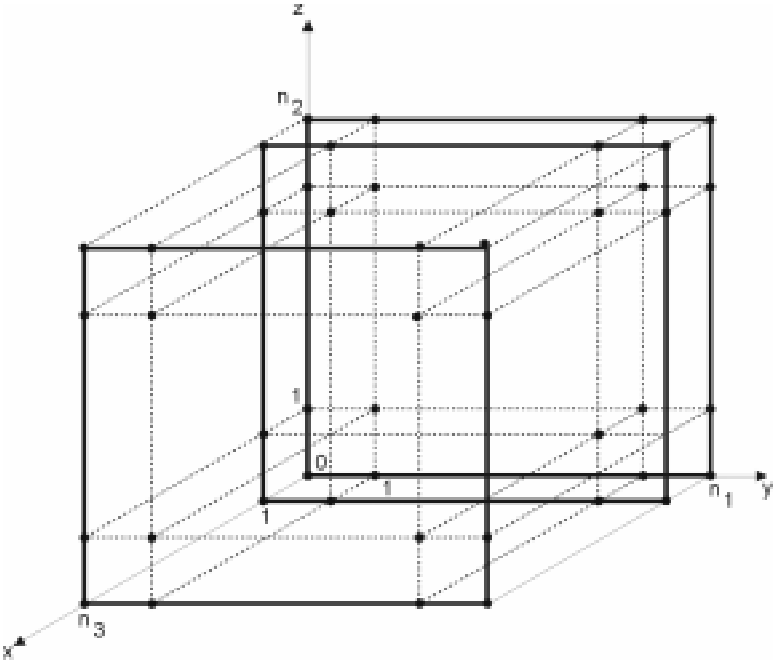

Figure 4

The grid of points from the zero x-coordinate plane, Figure 4 (plane yOz), according to relation (4), is the rectangle $(R^2_{\boldsymbol{n}},«)=(R^2_{n_1,n_2},«)=\bigcup_{j=0}^{n_1}\bigcup_{k=0}^{n_2}\{(j,k)\}$, its traversal order being from bottom to top and from left to right. If we "multiply" the rectangle $R^2_{\boldsymbol{n}}$ by successive translations, in the $i=\overline{1,n_3}$ x-coordinate planes, obviously parallel to the initial plane of the rectangle, we obtain the ordered parallelepiped

$$(R^3_{\boldsymbol{n}},«)=\bigcup_{i=0}^{n_3} R^2_{i,(n_1,n_2)}$$

from Figure 4. The i index added to $R^2_{\boldsymbol{n}}$ indicates the plane of which this is a part of after translation and becomes x-coordinate for all points from the i x-coordinate plane. The traversal order of the rectangles $R^2_{\boldsymbol{n}}$ is $R^2_{0,(n_1,n_2)}$, $R^2_{1,(n_1,n_2)}$, …, $R^2_{n_3,(n_1,n_2)}$. It follows that:

$$(R^3_{n_3,n_1,n_2},«)=\bigcup_{i=0}^{n_3}\bigcup_{j=0}^{n_1}\bigcup_{k=0}^{n_2}\{(i,j,k)\}.$$

Generalizing to the d -dimensional space we obtain:

$$(R^d_{n_1,n_2,\ldots,n_d}, «) = \bigcup_{i_1=0}^{n_1}\bigcup_{i_2=0}^{n_2}\ldots\bigcup_{i_{d-1}=0}^{n_{d-1}}\bigcup_{i_d=0}^{n_d}\{(i_1,i_2,\ldots,i_d)\} \tag{11}$$

Remark: An expression for $(R^d_{n_1,n_2,\ldots,n_d}, \prec)$ can be obtained applying the relation (8) from the previous theorem.

As we pointed out in the beginning, the notions presented in this article are useful in the field of multidimensional interpolation. We illustrate this by two examples.

Example1:

The totally ordered set $S = (R^d_{n_1,n_2,\ldots,n_d}, «)$, given by relation (11), generates the following multidimensional polinom

$$P(z) = \sum_{\boldsymbol{i}\in S} a_{\boldsymbol{i}} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d} = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}\ldots\sum_{i_d=0}^{n_d} a_{i_1,i_2,\ldots,i_d} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d},$$

and in this situation

$$\frac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial x_1^{\alpha_1}\ldots\partial x_2^{\alpha_d}} P(z) =$$

$$= \sum_{i_1=\alpha_1}^{n_1}\sum_{i_2=\alpha_2}^{n_2}\ldots\sum_{i_d=\alpha_d}^{n_d} a_{i_1,i_2,\ldots,i_d} \frac{i_1!}{(i_1-\alpha_1)!}\frac{i_2!}{(i_2-\alpha_2)!}\ldots\frac{i_d!}{(i_d-\alpha_d)!} x_1^{i_1-\alpha_1} x_2^{i_2-\alpha_2}\ldots x_d^{i_d-\alpha_d}.$$

Example 2:

If $S = T_n^2$, then $P(z) = P(x,y) = \sum_{\boldsymbol{i}\in S} a_{\boldsymbol{i}} x^{i_1} y^{i_2}$, and

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial x^{\alpha_1}\partial y^{\alpha_2}} P(z) = \sum_{\boldsymbol{i}\in S} a_{\boldsymbol{i}} \frac{i_1!}{(i_1-\alpha_1)!}\frac{i_2!}{(i_2-\alpha_2)!} x^{i_1-\alpha_1} y^{i_2-\alpha_2}$$, for any $\boldsymbol{\alpha} = (\alpha_1,\alpha_2)\in S$ with $\boldsymbol{\alpha} \le \boldsymbol{i}$.

If $S = (T_n^2, \prec)$ and taking into consideration the relation (2), then $P(x,y)$ $= \sum_{k=0}^{n}\sum_{j=0}^{k} a_{\boldsymbol{j,k-j}} x^j y^{k-j}$, $\frac{\partial^{\alpha_1+\alpha_2}}{\partial x^{\alpha_1}\partial y^{\alpha_2}} P(z) =$

$$= \sum_{k=\alpha_1+\alpha_2}^{n} \sum_{j=\alpha_1}^{k-\alpha_2} a_{j,k-j} \frac{j!}{(j-\alpha_1)!} \frac{(k-j)!}{(k-j-\alpha_2)!} x^{j-\alpha_1} y^{k-j-\alpha_2}$$

If $S = (T_n^2, «)$ and taking into account the relation (3), then $P(x,y) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} a_{j,k} x^j y^k$, and

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} P(x,y) = \sum_{j=\alpha_1}^{n} \sum_{k=\alpha_2}^{n-j} a_{j,k} \frac{j!}{(j-\alpha_1)!} \frac{k!}{(k-\alpha_2)!} x^{j-\alpha_1} y^{k-\alpha_2} .$$

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