INFERIOR AND SUPERIOR SETS WITH RESPECT TO AN ARBITRARY SET S FROM IN^d USED IN THE MULTIDIMENSIONAL INTERPOLATION THEORY

by
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Abstract: Among the necessary and useful notions for the elaboration of some multidimensional interpolation schemes, there is the one of inferior (superior) set with respect to an arbitrary set S from IN^d, $d \geq 1$. That is why we proposed in the current paper to approach some notions related to these sets, by definitions, remarks, theorems and examples. After the definition of inferior (superior) sets, we show a disjoint decomposition of the set $S \subset IN^d$ with inferior (superior) sets, which by coalescences and under some conditions (theorem 1), leads to inferior (superior) sets too. Next, we define two total order relations on $S \subset IN^d$ which are more often used in the interpolation theory and we study the relation between the first (last) elements of S and the inferior (superior) sets of this set (theorem 2).

Notations: Throughout this paper we use the notation IN^d to denote the set $\{\boldsymbol{i}=(i_1,i_2,...,i_d)/i_k \geq 0, i_k \in IN, k=\overline{1,d}\}$, we put $|\boldsymbol{i}|=i_1+i_2+...+i_d$ for any $i \in IN^d$, and the pairs (S,ρ) stand for subsets $S \subset IN^d$ together with relations "ρ".

Definition 1: Given $\boldsymbol{i}=(i_1,i_2,...,i_d)\in IN^d$, $\boldsymbol{j}=(j_1,j_2,...,j_d)\in IN^d$, we say that $\boldsymbol{i} \leq \boldsymbol{j}$, if $0 \leq i_k \leq j_k$, for any $k=\overline{1,d}$.

Remarks:

1.1 The relation "$\leq$" is reflexive, anti- symmetric and transitive, i.e. it is an order relation on IN^d. We use the same symbol "$\leq$" to denote the relation induced on (arbitrary) subsets $S \subset IN^d$. Unless otherwise specified, we will consider such subsets S to be endowed with the relation "$\leq$".

1.2 We will also denote by $S_{\boldsymbol{\alpha}}$ the set

$$S_{\boldsymbol{\alpha}}=\{\boldsymbol{i}=(i_1,i_2,...,i_d)\in S/\boldsymbol{\alpha} \leq \boldsymbol{i}\}$$

with $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,...,\alpha_d)\in S$. It is obvious that $S_{\boldsymbol{0}}=S$.

Definition 2: We say that $S \subset IN^d$ is an *inferior set* with respect to IN^d if, for any $\boldsymbol{i} \in S$, $\boldsymbol{j} \in IN^d$ with $\boldsymbol{j} \le \boldsymbol{i}$, we have $\boldsymbol{j} \in S$ too. Similarly, we say that S is a *superior set* with respect to IN^d if for any $\boldsymbol{i} \in IN^d$, $\boldsymbol{j} \in S$ with $\boldsymbol{j} \le \boldsymbol{i}$, we have $\boldsymbol{i} \in S$.

Remarks:

2.1. Similarly, one can define the notion of inferior (superior) sets with respect to arbitrary subsets $S \subset IN^d$.

2.2 Often we simply say inferior, respectively superior set, without indicating the set to which it is an inferior, respectively superior, this fact being implied. In genera, in these situations, it is implied that it is with respect to either IN^d , or the set itself.

Proposition 1: If $S \subset IN^d$ and $X \subset S$ is inferior (superior) with respect to S , then $S \setminus X$ is superior (inferior) with respect to S .

Proof: Let X be an inferior set with respect to S , $\boldsymbol{i} \in S \setminus X$, $\boldsymbol{j} \in S$ and $\boldsymbol{i} \le \boldsymbol{j}$. If $\boldsymbol{j} \notin S \setminus X$, then $\boldsymbol{j} \in X$, hence, since X is inferior and $\boldsymbol{i} \le \boldsymbol{j}$, $\boldsymbol{i} \in S \setminus X \subset S$, it follows that $\boldsymbol{j} \in S \setminus X$. The second part of the proposition can be proven similarly.

Remarks:

3.1 For any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d) \in S$, the $S_{\boldsymbol{\alpha}}$ is superior with respect to S , and $S \setminus S_{\boldsymbol{\alpha}}$ is inferior with respect to S .

3.2 The empty set is inferior, respectively superior with respect to any set $S \subset IN^d$.

3.3 Any set S from IN^d is inferior, respectively superior with respect to itself.

3.4 The relation "is inferior with respect to" defines an order on the set of subsets of IN^d , while the relation "is superior with respect to" does not (because it is not transitive).

3.5 If $X_i \subset IN^d$ and X_i is inferior (respectively superior) with respect to X_{i+1} for any $i = \overline{1, n-1}$, then $X_1 \subset X_2 \subset ... \subset X_n$, (respectively $X_1 \supset X_2 \supset ... \supset X_n$).

The remarks 3.1 and 3.2 are obvious from the definitions, while the remark 3.3 results from 3.1 for $S^{'}$=Ø. The reflexivity of the relation "inferior with respect to", results from 3.3 too. All other remarks result rather easily from the previous definitions.

Theorem 1: If $S_1 \cup S_2 \cup ... \cup S_n$ is a disjoint cover of $S \subset IN^d$, so that S_i is inferior (superior) with respect to $S_i \cup ... \cup S_n$ for any $i = \overline{1,n}$, then

1. $S_k \cup S_l$ is inferior (superior) with respect to $S_k \cup S_l \cup ... \cup S_n$

2. $S_k \cup ... \cup S_l$ is inferior (superior) with respect to $S_k \cup ... \cup S_n$, for any $1 \le k \le l \le n$.

Proof: **1.** The decomposition of $S \subset IN^2$ for the inferior sets case is shown in Figure 1 a, while Figure 1b depicts the case for superior sets where each pair of curves limits points of natural coordinates from S.

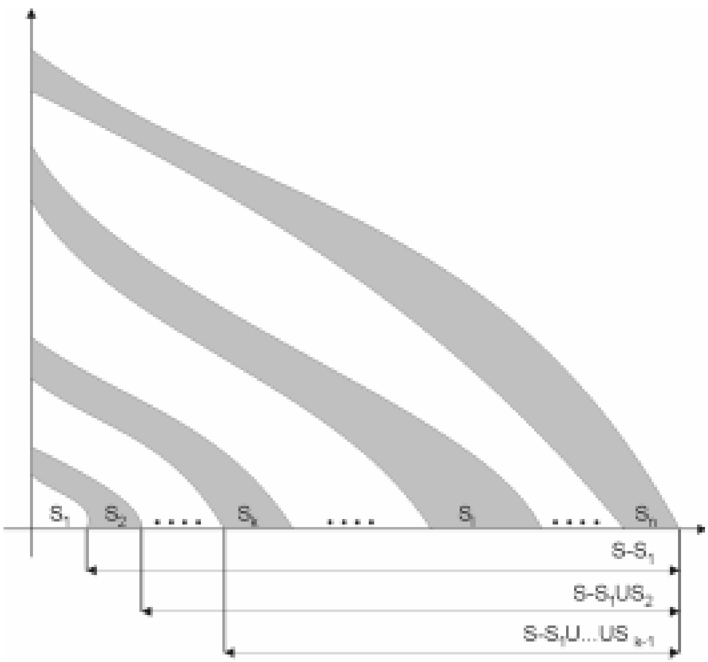

Figure 1a

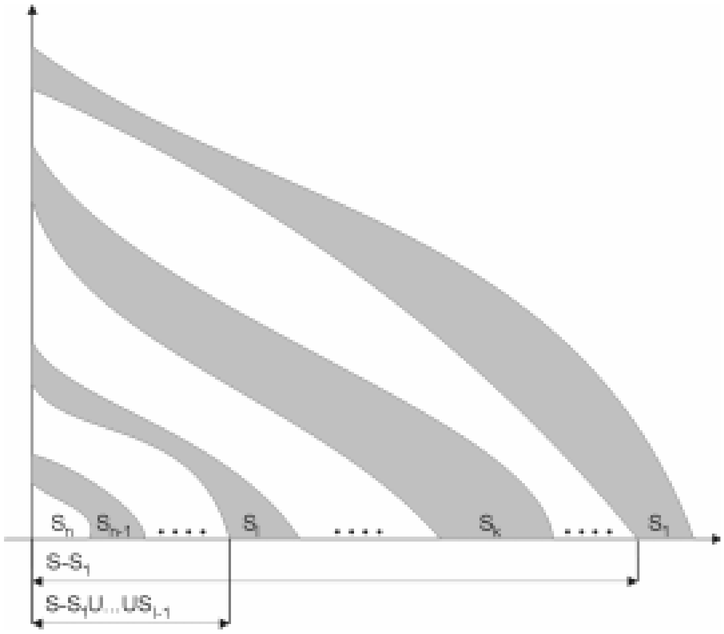

Figure 1*b*

Given $\boldsymbol{i} \in S_k \cup S_l$, $\boldsymbol{j} \in S_k \cup S_l \cup \ldots \cup S_n$ with $\boldsymbol{j} \le \boldsymbol{i}$, it follows that ($\boldsymbol{i} \in S_k$, $\boldsymbol{j} \in S_k \cup S_l \cup \ldots \cup S_n$ with $\boldsymbol{j} \le \boldsymbol{i}$) or ($\boldsymbol{i} \in S_l$, $\boldsymbol{j} \in S_k \cup S_l \cup \ldots \cup S_n$ with $\boldsymbol{j} \le \boldsymbol{i}$).

If ($\boldsymbol{i} \in S_k$, $\boldsymbol{j} \in S_k \cup S_l \cup \ldots \cup S_n$ with $\boldsymbol{j} \le \boldsymbol{i}$), it follows that ($\boldsymbol{i} \in S_k$, $\boldsymbol{j} \in S_k \cup S_l \cup \ldots \cup S_n \subset S_k \cup S_{k+1} \cup \ldots \cup S_n$ with $\boldsymbol{j} \le \boldsymbol{i}$) and because according to the hypothesis S_k is inferior with respect to $S_k \cup \ldots \cup S_n$, it results that $\boldsymbol{j} \in S_k$, i.e. $\boldsymbol{j} \in S_k \cup S_l$.

Given ($\boldsymbol{i} \in S_l$, $\boldsymbol{j} \in S_k \cup S_l \cup \ldots \cup S_n$ with $\boldsymbol{j} \le \boldsymbol{i}$), it follows that ($\boldsymbol{i} \in S_l$, $\boldsymbol{j} \in S_k$ with $\boldsymbol{j} \le \boldsymbol{i}$), or ($\boldsymbol{i} \in S_l$, $\boldsymbol{j} \in S_l \cup \ldots \cup S_n$ with $\boldsymbol{j} \le \boldsymbol{i}$), and thus $\boldsymbol{j} \in S_k$ or $\boldsymbol{j} \in S_l$, i.e. $\boldsymbol{j} \in S_k \cup S_l$. We relied also on the hypothesis that S_l is inferior with respect to $S_l \cup \ldots \cup S_n$.

In both possible cases discussed previously it results that $\boldsymbol{j} \in S_k \cup S_l$. Also according to the definition 2, it follows that $S_k \cup S_l$ is inferior with respect to $S_k \cup S_l \cup \ldots \cup S_n$.

2. The second point of the theorem results applying successively the first one. Also the situation regarding the superior set can be proven either similarly to the previous one, using the hypothesis and the definition 2, or using the proposition 1.

Remarks:

4.1 The point 1 of the theorem can be extended to more then two "isolated" set (not necessary consecutive)

4.2 The statement "S_i is inferior (superior) with respect to $S_k \cup S_l \cup ... \cup S_n$", is equivalent to "$S_i$ is inferior (superior) with respect to $S \setminus (S_1 \cup ... \cup S_{i-1})$", which is more often used in practice.

It is worthwhile to notice that the relation "$\leq$" defined between the elements of the set IN^d (definition 1) and used when defining an inferior (superior) subset of $S \subset IN^d$, is not a complete order relation on this set. This means that the relation "$\leq$" is not sufficient for ordering all elements of the set IN^d or $S \subset IN^d$. More concrete, if we use the relation "$\leq$", $(0,0) \in IN^2$ can be followed by $(0,1) \in IN^2$ and by $(1,0) \in IN^2$ without any way of comparing these two elements between them.Next we will define two total order relations on the set $S \subset IN^d$ and the first discussed one will be the lexicographic one.

Definition 3: Given $\boldsymbol{i} = (i_1, i_2, ..., i_d) \in IN^d$, $\boldsymbol{j} = (j_1, j_2, ..., j_d) \in IN^d$ we say that $\boldsymbol{i} \ll \boldsymbol{j}$, if $i_1 < j_1$, or $i_1 = j_1$ and $i_2 < j_2 \dots$ or $i_1 = j_1 \dots i_{d-1} = j_{d-1}$ and $i_d \leq j_d$.

Definition 4: Given $\boldsymbol{i} = (i_1, i_2, ..., i_d) \in IN^d$, $\boldsymbol{j} = (j_1, j_2, ..., j_d) \in IN^d$, we say that $\boldsymbol{i} \prec \boldsymbol{j}$, if $|\boldsymbol{i}| < |\boldsymbol{j}|$ or $|\boldsymbol{i}| = |\boldsymbol{j}|$ and $\boldsymbol{i} \ll \boldsymbol{j}$.

Remarks

5.1. On IN^1, the relations from definitions 3 and 4 coincide with the relation from the definition 1.

5.2. The relations defined by 3 and 4 are total order relations on IN^d and on any $S \subset IN^d$. In other words, with any of these two relations, the elements of IN^d or of any of its subsets can be enumerated, that is these sets are completely ordered.

5.3. For the same $|\boldsymbol{i}|$, the relations "$\prec$" and "$\ll$" coincide.

5.4. The elements of $X \subset S$ are the first $|X|$ elements of (S,ρ), with respect to the relation $\rho \in \{"\prec", "«"\}$, if for any $\boldsymbol{i} \in X$, $\boldsymbol{j} \in S \setminus X$, it follows that $\boldsymbol{i}\rho\,\boldsymbol{j}$, and the elements of $Y \subset S$ are the last $|Y|$ elements of (S,ρ) with respect to the relation $\rho \in \{"\prec", "«"\}$, if for any $\boldsymbol{i} \in Y$, $\boldsymbol{j} \in S \setminus Y$, it results $\boldsymbol{j}\rho\,\boldsymbol{i}$.

5.5. It is easy to notice that we can also define other total relations on IN^d

The usage of one of the two previous relations depends on $S \subset IN^d$ and the problem to be solved.

Proposition 2: Let $\boldsymbol{i}, \boldsymbol{j} \in S$ and $\rho \in \{"\prec", "«"\}$.

2.1. If $\boldsymbol{i} \le \boldsymbol{j}$, then $\boldsymbol{i}\rho\,\boldsymbol{j}$, but not conversely. If $\boldsymbol{i} \prec \boldsymbol{j}$, this does not imply $\boldsymbol{i} « \boldsymbol{j}$, and conversely.

2.2. The statement "elements of $X \subset S$ are the first $|X|$ elements of (S,ρ) with respect to the relation $\rho \in \{"\prec", "«"\}$" is equivalent to "for any $\boldsymbol{i} \in X$, $\boldsymbol{j} \in S$ with $\boldsymbol{j}\rho\,\boldsymbol{i}$, then $\boldsymbol{j} \in X$". The assertion "the elements $Y \subset S$ are the last $|Y|$ elements of (S,ρ), with respect to the relation $\rho \in \{"\prec", "«"\}$" is equivalent to "for any $\boldsymbol{i} \in Y$, $\boldsymbol{j} \in S$ with $\boldsymbol{i}\rho\,\boldsymbol{j}$, then $\boldsymbol{j} \in Y$".

Proof: **1.** The first part of this proposition results easily by combining the definitions 1, 3 and 4. The other sentences are justified by counter examples. Thus, for $(1,3), (2,1) \in IN^2$, $(1,3) « (2,1)$ and $(1,3)$ is not in the relation $"\le"$, respectively $"\prec"$ with $(2,1)$. Also, if (3,1), (2,4)$\in IN^2$, $(3,1) \prec (2,4)$ and (3,1) are not in the relation $"\le"$, respectively "«" with (2,4).

2. Assume that the elements of $X \subset S$ are the first $|X|$ elements of (S,ρ), in the relation $\rho \in \{"\prec", "«"\}$, and the assertion "for any $\boldsymbol{i} \in X$, $\boldsymbol{j} \in S$ with $\boldsymbol{j}\rho\,\boldsymbol{i}$, then $\boldsymbol{j} \in X$" is false. Then we would have $\boldsymbol{i} \in X$, $\boldsymbol{j} \in S$ with $\boldsymbol{j}\rho\,\boldsymbol{i}$, and $\boldsymbol{j} \notin X$, i.e. we would have $\boldsymbol{i} \in X$, $\boldsymbol{j} \in S \setminus X$ with $\boldsymbol{j}\rho\,\boldsymbol{i}$. In addition, we would have that $\boldsymbol{i} \in X$, $\boldsymbol{j} \in S \setminus X$ implies $\boldsymbol{i}\rho\,\boldsymbol{j}$ (according to the observation 5.4), that is $\boldsymbol{i} = \boldsymbol{j}$ ($"\rho"$ being anti-symmetric). This is impossible because this would mean that X and $S \setminus X$ have $\boldsymbol{i}$ in common, hence the assertion can not be false. The other assertions can be similarly justified.

The useful information for the multidimensional interpolation presented so far can be grouped in the next theorem.

Theorem 2: If $S=(S,\rho)$ is a total ordered set from IN^d with the relation $\rho \in \{"\prec","«"\}$ and if the elements of X are the first $|X|$ elements of (S,ρ), then X is inferior with respect to S, and if the elements of Y are the last $|Y|$ elements of (S,ρ), then Y is a superior set with respect to S.

Proof: Let the elements of X be the first elements of S in the relation $\rho \in \{"\prec","«"\}$. Let $\boldsymbol{i} \in X$, $\boldsymbol{j} \in S$ with $\boldsymbol{j} \le \boldsymbol{i}$. Since, according to proposition 2, if $\boldsymbol{j} \le \boldsymbol{i}$ then $\boldsymbol{j}\rho\boldsymbol{i}$, then from "$\boldsymbol{i} \in X$, $\boldsymbol{j} \in S$ with $\boldsymbol{j} \le \boldsymbol{i}$" it follows that "$\boldsymbol{i} \in X$, $\boldsymbol{j} \in S$ with $\boldsymbol{j}\rho\boldsymbol{i}$". From this deduction according to proposition 2, we infer that $\boldsymbol{j} \in X$ (the elements of X are the first elements of S). We have that, if $\boldsymbol{i} \in X$, $\boldsymbol{j} \in S$ with $\boldsymbol{j} \le \boldsymbol{i}$ then $\boldsymbol{j} \in X$, which according to the definition 2, is the same with the fact that X is an inferior set with respect to S.

The last part of the theorem can be proved by analogy with the first one. Its proof can be made using the proposition 1, as well as the results of the first part. Thus, if the elements of Y are the last $|Y|$ elements of (S,ρ), and because $|Y|+|S\setminus Y|=|S|$, it results that the elements of $S\setminus Y$ are the first $|S\setminus Y|$ elements of (S,ρ). Thus, according to the first part of the theorem, $S\setminus Y$ is inferior with respect to S, which according to proposition 1, implies that $S\setminus(S\setminus Y)=Y$ is superior with respect to S.

Remarks

6.1. Given $\rho \in \{"\prec","«"\}$, $\boldsymbol{\alpha} \in (S,\rho)$, the set $\{\boldsymbol{i} \in (S,\rho) / \boldsymbol{i}\rho\boldsymbol{\alpha}\}$ is inferior with respect to S, but the set $\{\boldsymbol{i} \in (S,\rho) / \boldsymbol{\alpha}\rho\boldsymbol{i}\}$ is superior with respect to S, because the elements of these sets are the first, respectively the last elements of (S,ρ).

6.2. In general, the previous theorem has no reciprocity. Thus, the set $X=\{(0,0),(0,1),(1,0),(2,0)\}$ is inferior with respect to $S=(S,\prec)=(T_2^2,\prec)=\{(0,0),(0,1),(1,0),(0,2),(1,1),(2,0)\}$, and the four elements of X are not also the first four elements of $(T_2^2,\prec)$. Also $X=\{(0,0),(0,1),(1,0)\}$ is inferior with respect to $S=(S,«)=(T_2^2,«)=\{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)\}$, and the three elements of X are not also the first ones of $(T_2^2,«)$. The set $Y=\{(1,0),(1,1),(2,0)\}$ is also superior with respect to $S=T_2^2$ and the three elements of Y are the last three elements of $(T_2^2,«)$, while the three elements of Y are not the last three elements of $(T_2^2,\prec)$.

6.3. According to observation 5.4, proposition 2.2 and theorem 2, if $S=(S,\rho)$ is a total ordered set from IN^d and if $S_1 \cup S_2 \cup ... \cup S_n$ is a disjoint cover of $S \subset IN^d$ so that the elements of S_i are the first (last) $|S_i|$ elements of $S_i \cup ... \cup S_n$ for any $i=\overline{1,n}$, then it follows that $S_k \cup S_l$ is inferior (superior) with respect to $S_k \cup S_l \cup ... \cup S_n$ and that $S_k \cup ... \cup S_l$ is inferior (superior) with respect to $S_k \cup ... \cup S_n$, for any $1 \le k \le l \le n$. In the same conditions, we have that the elements of the set $S_k \cup S_l$ are the first (last) $|S_k \cup S_l|$ elements of the set $S_k \cup S_l \cup ... \cup S_n$, and the elements of the set $S_k \cup ... \cup S_l$ are the first (last) $|S_k \cup ... \cup S_l|$ elements of the set $S_k \cup ... \cup S_n$ for any $1 \le k \le l \le n$.

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