GENERALIZATIONS OF DURRMEYER TYPE

by
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Abstract. In this paper are presented first the operators of Durrmeyer which have been introduced by J.L. Durrmeyer in 1967 and investigated by M.M. Derrienic in 1981 and their properties.Then are presented the generalizations of Durrmeyer type for the operators of Szász and Baskakov. We study a generalization of Durrmeyer type for the operators of Schurer and we estimate the values of this operators for the test functions. By means of the modulus of continuity of the function used one gives evaluations of the orders of approximation by the considered operators.

Keywords: Durrmeyer-type operators, Bohman-Korovkin theorem, modulus of continuity, operators of Schurer, operators of Szász and Baskakov.

1. THE OPERATORS OF DURRMEYER

Let m 0 ⎤. The operators $D_m : L_1([0,1]) \to C([0,1])$, defined by

$$\left(D_m f\right)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(t) f(t)dt \qquad (1.1)$$

where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$, $k = \overline{0,m}$, x 0 [0, 1], are the operators of Durrmeyer.

Theorem [3] . The operators of Durrmeyer have the properties:

i) $\left(D_m e_0\right)(x) = 1$.

ii) Turn every polynomial of the order p, p # m, in the polynomial of the order p .

iii) $\lim_{m\to\infty} D_m f = f$, uniformly on [0, 1], (□) f 0 C([0, 1]).

iv) $\left|\left(D_m f\right)(x) - f(x)\right| \le 2\omega\left(f; \frac{1}{\sqrt{2m+6}}\right)$, (□) f 0 C([0, 1]), (□) m ∃3.

Theorem [10] .The operators of Durrmeyer satisfy the following properties:

i) Let f be a limited and integral function on [0, 1]. If for x 0 [0, 1] the function are the derivatives of the orders two , then

$$\lim_{m\to\infty} m\left[\left(D_m f\right)(x) - f(x)\right] = (1-2x)f'(x) + x(1-x)f''(x) .$$

The limit is uniformly if f"and f' 0 C([0, 1]).

ii)Let f be a limited and integral function on [0, 1]. If the function are the derivatives of the orders r on [0, 1] , then

$$\lim_{m\to\infty} \frac{d^r}{dx^r}\left(D_m f\right)(x) = \frac{d^r}{dx^r} f(x) .$$

iii) Let $f \in C^r([0,1])$. The sequence $\left(\frac{d^r}{dx^r}(D_m f)\right)_{m\geq 1}$ converges to $f^{(r)}$ uniformly and we have

$$\sup_{x\in[0,1]}\left|\frac{(m+r+1)!(m-r)!}{(m+1)!m!}\frac{d^r}{dx^r}(D_m f)(x) - f^{(r)}(x)\right| \leq K_r \omega(f^{(r)};\frac{1}{\sqrt{m}})$$

where K_r is an independent constant by f and m .

2. GENERALIZATIONS OF DURRMEYER TYPE FOR THE OPERATORS OF SZÁSZ AND BASKAKOV

In 1985 S.M. Mazhar and V. Totik [5] have been introduced the operators Szász modified in the following form

$$\left(S_n^{**} f\right)(x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t) f(t)dt, \quad \text{n 0 N} \tag{2.1}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$, x ∃ 0, $f \in R^{[0,\infty)}$ such that the integrals exists and the series is converges.

Theorem[9]. The operators defined by (2.1) have the properties:

i) For $f \in C_B([0,\infty))$

$$\left|\left(S_n^{**} f\right)(x) - f(x)\right| \leq K\omega_1\left(f;\frac{\sqrt{nx+1}}{n}\right),$$

where $\omega_1(f;t) = \sup\limits_{0<h\le t}\sup\limits_{x\ge h/2}\left| f(x+h/2) - f(x-h/2)\right|$.

ii) For $f \in C_B([0,\infty))$ and $0 < \forall < 1$

$$\left|\left(S_n^{**} f\right)(x) - f(x)\right| \le M\left(\frac{x}{n} + \frac{1}{n^2}\right)^{\alpha/2}$$

$(x \exists 0,\ n\ 0\ N)$ $]\ \omega_1(f;t) = O(t^{\alpha}) \quad (t \to 0^+)$.

iii) For $f \in C([0,\infty)) \cap L_\infty([0,\infty)),\ 0 < \forall < 1$

$$\omega_1(f;t) = O(t^{\alpha}) \Leftrightarrow \left|\left(S_n^{**} f\right)'(x)\right| \le M(\min\{n^2, n/x\})^{(1-\alpha)/2}.$$

iv) For $f \in C([0,\infty)) \cap L_\infty([0,\infty)),\ 0 < \forall < 2$

$$\omega_2(f;t) = O(t^{\alpha}) \Leftrightarrow \left|\left(S_n^{**} f\right)''(x)\right| \le M(\min\{n^2, n/x\})^{(2-\alpha)/2},$$

where $\omega_2(f;t) = \sup\limits_{0<h\le t}\sup\limits_{x\ge 0}\left| f(x) - 2f(x+h) + f(x+2h)\right|$.

v) If $f \in C_B([0,\infty))$ then

$$\left\|S_n^{**} f - f\right\| \le C(\omega_{2,\varphi}(f;n^{-1/2}) + \omega_1(f;n^{-1}) + n^{-1}\|f\|),$$

where C is an independent constant by n, $\varphi(x) = \sqrt{x}$ and

$$\omega_{2,\varphi}(f;t) = \sup_{0<h\le t}\left\|\delta_{h\varphi}^2 f\right\| = \sup_{0<h\le t}\sup_{x\ge h^2}\left| f(x - h\sqrt{x}) - 2f(x) + f(x + h\sqrt{x})\right|$$

In 1985 Ashok Sahai and Govind Prasad [7] defined the operators modified into form integral

$$\left(V_n^{**} f\right)(x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} v_{n,k}(t) f(t)dt\,,\ (n\ 0\ N) \qquad (2.2)$$

where $v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-(n+k)}$, $x \exists 0$,

$f \in R^{[0,\infty)}$ such that the integrals exists and the series is converges.

Theorem[1]. Let be the multitude D** made by all the measure Lebesgue functions such that

$\int_0^\infty \frac{|f(t)|}{(1+t)^n}dt < \infty$ for n 0 N, and let be f 0 D** a limited on any limited subinterval from [0,4) such that

$f(t) = O(t^\alpha)$ (t 6 4) where ∀ > 0.

i) If f have derivates by the order r + 2 , $r \in N_0$, in a fixed point x > 0 then

$$\lim_{n\to\infty} n\left(\left(V_n^{**} f\right)^{(r)}(x) - f^{(r)}(x)\right) =$$

$$= r(r+1) f^{(r)}(x) + (r+1)(1+2x) f^{(r+1)}(x) + x(1+x) f^{(r+2)}(x).$$

ii) If exist $f^{(r+1)}, r \in N_0$, and it is continue on an opened interval J δ [a,b], 0 < a, then for n enough bigger we have

$$\left\|\left(V_n^{**} f\right)^{(r)} - f^{(r)}\right\| \le C_1\left(\left\|f^{(r)}\right\| + \left\|f^{(r+1)}\right\|\right)n^{-1} + \\ + C_2 n^{-1/2}\omega\left(f^{(r+1)}; n^{-1/2}\right) + O(n^{-s}),$$

for all s > 0, where C_1, C_2 are the independent constants by n and f.

Theorem[7].Let f be a function with limited variation on any limited subinterval from [0,4). If

$f(t) = O(t^\alpha)$ (t 6 4) where ∀ > 2 then for n enough bigger we have

$$\left|\left(V_n^{**} f\right)(x) - \frac{1}{2}(f(x+) - f(x-))\right| \le \frac{6+7x}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \\ + \frac{6(1+9x(1+x))^{1/2} + 2x+1}{4\sqrt{nx(1+x)}}\left|f(x+) - f(x-)\right| + \frac{1+x}{nx^{\alpha+1}}O(1)$$

where

$g_x \in R^{[0,\infty)}$,

$g_x(t) = f(t) - f(x-)$ for 0 # t < x,

$g_x(x) = 0$ si $g_x(t) = f(t) - f(x+)$ for t > x.

3. A GENERALIZATION OF DURRMEYER TYPE FOR THE OPERATORS OF SCHURER

We consider the operators of Schurer modified into integral form

$$B^{**}_{m,p} : C([0,1+p]) \to C([0,1]),$$

$$(B^{**}_{m,p} f)(x) = (m+p+1)\sum_{k=0}^{m+p} \binom{m+p}{k} q^k_{m,p}(x) \int_0^1 q^k_{m,p}(t) f(t) dt \quad (3.1)$$

where $q^k_{m,p}(x) = \binom{m+p}{k} t^k (1-t)^{m+p-k}$, (□) f 0 C([0,1+p]), (□) x 0 [0, 1].

Theorem . The operators defined by (3.1) have the properties:

i) $\left(B^{**}_{m,p} e_0\right)(x) = 1$;

ii) $\left(B^{**}_{m,p} e_1\right)(x) = \dfrac{(m+p)x+1}{m+p+2}$;

iii) $\left(B^{**}_{m,p} e_2\right)(x) = \dfrac{(m+p)(m+p-1)x^2 + 4(m+p)x + 2}{(m+p+2)(m+p+3)}$.

Proof :

i) $\left(B^{**}_{m,p} e_0\right)(x) = (m+p+1)\sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} \dfrac{1}{m+p+1} = 1$

if on used the relations : $\sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} = 1$ and

$$\int_0^1 \binom{m+p}{k} t^k (1-t)^{m+p-k} dt =$$

$$= \binom{m+p}{k} \beta(k+1, m+p-k+1) = \frac{1}{m+p+1}.$$

ii) We have

$$\left(B_{m,p}^{**}e_1\right)(x) = (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k}x^k(1-x)^{m+p-k}\cdot$$

$$\cdot\,\beta(k+2,m+p-k+1) =$$

$$= (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k}x^k(1-x)^{m+p-k}\frac{k+1}{(m+p+2)(m+p+1)} =$$

$$= \frac{1}{m+p+2} + \frac{m+p}{m+p+2}x\sum_{k=1}^{m+p}\binom{m+p-1}{k-1}x^{k-1}(1-x)^{m+p-k} =$$

$$= \frac{(m+p)x+1}{m+p+2}$$

iii) We find

$$\left(B_{m,p}^{**}e_1\right)(x) = (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k}x^k(1-x)^{m+p-k}\cdot$$

$$\cdot\,\beta(k+3,m+p-k+1) =$$

$$= (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k}x^k(1-x)^{m+p-k}\cdot$$

$$\cdot\frac{(k+2)(k+1)}{(m+p+3)(m+p+2)(m+p+1)} =$$

$$= \frac{(m+p)(m+p-1)x^2+4(m+p)x+2}{(m+p+2)(m+p+3)}.$$

Theorem . The operators defined by (3.1) have the properties :

i) $\lim\limits_{m\to\infty} B_{m,p}^{**}f = f$ uniformly on [0, 1], (□) f 0 C([0,1+p]).

ii) $\left|\left(B_{m,p}^{**}f\right)(x) - f(x)\right| \le 2\omega\left(f;\frac{1}{\sqrt{2(m+p+3)}}\right)$,

(□) f 0 C([0,1+p]), (□) x 0 [0, 1], m ∃ 3, p 0 ⌉ is fixed .

Proof :
i) It results from Bohman-Korovkin theorem .
ii) We used the properties:
If L is a linear positive operator L : C(I) 6 C(I), such that $Le_0 = e_0$ then

$$\left|(Lf)(x) - f(x)\right| \le \left(1 + \delta^{-1}\sqrt{\left(L\varphi_x^2\right)(x)}\right)\omega(f;\delta)$$, (□) f 0 $C_B(I)$

,

(□) x 0 I, ∗ > 0 and $\varphi_x = |t - x|$.
We have

$$\left|\left(B_{m,p}^{**}f\right)(x) - f(x)\right| \le (1 + \delta^{-1}\sqrt{B_{m,p}^{**}\varphi_x^2})\omega(f;\delta),$$

$$\left(B_{m,p}^{**}\varphi_x^2\right)(x) = \left(B_{m,p}^{**}e_2\right)(x) - 2x\left(B_{m,p}^{**}e_1\right)(x) + x^2\left(B_{m,p}^{**}e_0\right)(x) =$$

$$= \frac{2(m + p - 3)x(1 - x) + 2}{(m + p + 2)(m + p + 3)}.$$

If m + p ∃ 3 it is maximal for x = ½ and we find

$$\left(B_{m,p}^{**}\varphi_x^2\right)(x) \le \frac{m + p + 1}{2(m + p + 2)(m + p + 3)}.$$

We get

$$\left|\left(B_{m,p}^{**}f\right)(x) - f(x)\right| \le$$

$$\le \left(1 + \delta^{-1}\sqrt{\frac{m + p + 1}{2(m + p + 2)(m + p + 3)}}\right)\omega(f;\delta) \le$$

$$\le \left(1 + \delta^{-1}\sqrt{\frac{1}{2(m + p + 3)}}\right)\omega(f;\delta).$$

For $\delta = \dfrac{1}{\sqrt{2(m + p + 3)}}$ we obtain the inequalities

$$\left|\left(B_{m,p}^{**}f\right)(x) - f(x)\right| \le 2\omega\left(f;\frac{1}{\sqrt{2(m + p + 3)}}\right).$$

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