THE STARLIKENES PROPERTIES FOR INTEGRAL OPERATORS

by
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Abstract. In this paper we prove a starlikeness property for the Bernardi operator concerning $S(\alpha)$-class and two particular properties for Libera and Alexander operators.

INTRODUCTION.

Let $U=\{z:|z|<1\}$ the unit disc in the complex plane and let A_n the class of functions *f* of the form $f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+...,z\in U$, which are analytic in the unit disc *U* and we denote $A_1=A$.

Let the class of univalent functions $S=\{f:f\in H_n(U),f(0)=f'(0)-1=0\}$ and let

$$S^*(\alpha)=\left\{f:\mathrm{Re}\frac{zf'(z)}{f(z)}>\alpha,z\in U\right\}$$

the class of starlike functions of the order $\alpha, 0\le\alpha<1$, in *U*. For $\alpha=0$, denote $S^*(0)=S^*$ the class of the starlike functions in *U*.

Consider the following integral operator

$$F_c(z)=\frac{c+1}{z^c}\int_0^z f(t)t^{c-1}dt,\ c\ge 0,$$

where $f\in A_n$, where is the Bernardi operator.

For $\gamma=0$ obtained the Alexander operator, and for $\gamma=1$ is the operator of Libera.

PRELIMINARY RESULTS.

Definition [1] For $0<\alpha\le 2$, we define $S(\alpha)$ as the class of functions $f\in A$, which satisfy the conditions $f(z)\ne 0$, for $0<|z|<1$ and $\left|\left(\frac{z}{f(z)}\right)''\right|\le\alpha, z\in U$.

Theorem A. [1] Let $f \in A, f(z) \neq 0,$ for $0 < |z| < 1$. If $\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2, z \in U$, then f is univalent in *U*.

Corollary B. [1] Let

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n} \in A$$

and

$$\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2 .$$

Then *f* is univalent in *U*.

Remark [1] All the functions from $S(\alpha)$, $0 < \alpha \leq 2$ which satisfy theorem A is univalent in *U*.

Theorem C. [1] Let $f(z) = z + a_2 z^2 + ... \in S(\alpha)$, with $0 < \alpha \leq 2$. Then for any $z \in U$ we have:

$$\left|\frac{z}{f(z)} - 1\right| \leq |z|\left(|a_2| + \frac{\alpha}{2}|z|\right);$$

$$1 - |z|\left(|a_2| + \frac{\alpha}{2}|z|\right) \leq \operatorname{Re}\frac{z}{f(z)} \leq 1 + |z|\left(|a_2| + \frac{\alpha}{2}|z|\right);$$

$$|f(z)| \geq \frac{|z|}{1 + |z||a_2| + \frac{\alpha}{2}|z|^2} .$$

The equalities exist if:

$$f(z) = \frac{z}{1 \pm az + \frac{\alpha}{2} z^2} \in S(\alpha), \left(0 \leq a \leq \sqrt{2\alpha}\right).$$

Theorem D. [1] Let $f \in S(\alpha)$ with the following form $f(z) = z + a_3 z^3 + a_4 z^4 + ...$.

a) If $\frac{2}{\sqrt{5}} \leq \alpha \leq 2$, then f is starlike for $|z| < \sqrt{\frac{2}{\alpha}} \cdot \frac{1}{\sqrt[4]{5}}$.

b) If $\sqrt{3}-1 \le \alpha \le 2$, then $\operatorname{Re} f'(z) > 0$, for $|z| < \sqrt{\dfrac{\sqrt{3}-1}{\alpha}}$.

Corollary E. [1] Let $f \in S(\alpha)$ with the following form $f(z) = z + a_3 z^3 + a_4 z^4 + \ldots$.

a)If $0 < \alpha \le \dfrac{2}{\sqrt{5}}$, then f is starlike in U.

b) If $0 < \alpha \le \sqrt{3}-1$, then $\operatorname{Re} f'(z) > 0$, for $z \in U$.

Lemma F [2] Let $\gamma > 0$ and $\beta < 1$. If $f \in A_n$ satisfy inequality

$$\operatorname{Re}\{f'(z) + \gamma z f''(z)\} > \beta,\ z \in U,$$

then

$$\operatorname{Re}\{f'(z)\} > \beta + (1-\beta)\{2\delta(n,\gamma) - 1\},\ z \in U,$$

where $\delta(n,\gamma) = \int_0^1 \dfrac{d\rho}{1+\rho^{n\gamma}}$.

MAIN RESULTS.

Theorem 1. Let $\theta = 0{,}911621907$, $-1 < c \le 4{,}741187$ and

$$\frac{1}{c+1} - \frac{c^2}{3(c+1)} tg^2\theta < \frac{2\delta\left(n, \frac{1}{c+1}\right) - 1}{\left\{1 - \delta\left(2, \frac{1}{c+1}\right)\right\}\{2\delta(n,1) - 1\}}.$$

If $f \in A_n$, satisfies inequality

$$\operatorname{Re} f'(z) > 1 - \frac{2(c+1) + \left(1 - \frac{1}{3}c^2 tg^2\theta\right)}{2(c+1) + 4\{1-\delta(n,1)\}\left\{1 - \delta\left(n, \frac{1}{c+1}\right)\right\}\left(1 - \frac{1}{3}c^2 tg^2\theta\right)} \qquad (1)$$

then

$$F_c \in S^*,$$

where F_c is the Bernardi operator.

Proof. For this proof we need the following result proved by Yong Chan Kim and H.M. Srivastava in [4] and which we are presented below.
We consider $\delta(n,\gamma)$ defined as Lemma F and we have the following conditions satisfied :

$$\theta = 0{,}911621907\,,\ \gamma \geq 0{,}17418$$

$$\gamma - \frac{(1-\gamma)^2}{3\gamma} tg^2\theta < \frac{2\delta(n,\gamma)-1}{\{1-\delta(n,\gamma)\}\{2\delta(n,1)-1\}}.$$

If $f \in A_n$ satisfies inequality:

$$\mathrm{Re}\{f'(z)+\gamma f''(z)\} > 1 - \frac{\frac{2}{\gamma} + \left(1 - \frac{(1-\gamma)^2}{3\gamma^2} tg^2\theta\right)}{\frac{2}{\gamma} + 4\{1-\delta(n,1)\}\{1-\delta(n,\gamma)\}\left(1 - \frac{(1-\gamma)^2}{3\gamma^2} tg^2\theta\right)}\,,\ z \in U\,, \quad (2)$$

then $f \in S^*$.
We go back to our proof.
One easily observe that

$$f'(z) = F_c'(z) + \frac{1}{c+1} z \cdot F_c''(z).$$

It is known the fact that $f \in A_n$ then $F_c \in A_n$.
Concerning the relation (1) and (2) above and consider $\gamma = \frac{1}{c+1}$ we obtain:

$$\mathrm{Re}\, f'(z) = \mathrm{Re}\left\{F_c'(z) + \frac{1}{c+1} z \cdot F_c''(z)\right\} >$$

$$> 1 - \frac{2(c+1) + \left(1 - \frac{1}{3}c^2 tg^2\theta\right)}{2(c+1) + 4\{1-\delta(n,1)\}\left\{1-\delta\left(n, \frac{1}{c+1}\right)\right\}\left(1 - \frac{1}{3}c^2 tg^2\theta\right)} > 0\,.$$

Concerning the result proved by Yong Chan Kim and H.M. Srivastava, this last inequality implies the fact that $F_c \in S^*$.

Corollary 2. Let $\theta = 0{,}911621907$. If $f \in A_1$ satisfies the inequality

$$\operatorname{Re} f'(z) > 1 - \frac{5 - \frac{1}{3} tg^2\theta}{4 + 8(1 - \ln 2)^2 \left(1 - \frac{1}{3} tg^2\theta\right)} \approx -0{,}02531\ldots,\ z \in U\ ,$$

then

$$F_1(z) = \frac{2}{z}\int_0^z f(t)dt \in S^*,$$

where F_1 is the operator of Libera.

Proof. This corollary is obtained from theorem 1, for $c = n = 1$.
So we obtain

$$\operatorname{Re} f'(z) = \operatorname{Re}\left\{F_1{}'(z) + \frac{1}{2} z \cdot F_1{}''(z)\right\} > 1 - \frac{5 - \frac{1}{3} tg^2\theta}{4 + 8(1 - \ln 2)^2 \left(1 - \frac{1}{3} tg^2\theta\right)} \approx -0{,}02531\ldots,$$

condition which implies the starlikenes of Libera's operator.

Corollary 3. Let $\theta = 0{,}911621907$. If $f \in A_1$ satisfy the inequality

$$\operatorname{Re} f'(z) > \frac{3 - 4\ln 2 + 4\ln^2 2}{6 - 4\ln 2 + 4\ln^2 2} \approx 0{,}417352\ldots$$

then

$$F_0(f) = \int_0^z \frac{f(t)}{t} dt \in S^*,$$

where F_0 is Alexander's operator.

Proof. We consider $c = 0, n = 1$ in theorem 1. So we obtain $\delta(1,1) = \ln 2$ and

$$\operatorname{Re} f'(z) = \operatorname{Re}\left\{F_0{}'(z) + z \cdot F_0{}''(z)\right\} > \frac{3 - 4\ln 2 + 4\ln^2 2}{6 - 4\ln 2 + 4\ln^2 2} \approx 0{,}417352\ldots,$$

condition which implies the starlikenes of Alexander's operator.

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